Preface

What is 1 + 1 + 1?
John H. Conway, 1973

Individually, each of Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy have received much, rightly deserved, praise. Each made lasting contributions to many areas of mathematics. This volume is dedicated to their work in combinatorial game theory. It is due to their efforts that combinatorial game theory exists as a subject.

Brief History of how *Winning Ways* came to be

Bouton first analyzed NIM [67], little realizing how central NIM was to be. In the next two decades, other researchers contributed the analysis of a few other, specific games. The chess champion Emanuel Lasker came close to a complete theory of impartial games. It was in the 1930s that Grundy [68] and Sprague [72] gave a complete analysis, now known as the Sprague–Grundy theory. Despite being an elegant theory and easy to apply, the subject languished because there was no clear direction in which to develop the theory. In the late 1940s, Richard K. Guy rediscovered the theory and defined the *octal* games. In 1956, Guy and C. A. B. Smith published *The G-values of various games* [42]. This gave the world an infinite number of impartial games and led to many interesting, easy to state, and yet still unsolved conjectures.

The analysis of partizan games looked out of reach. The Fields' medalist John Milnor [70] in 1953 published *Sums of positional games*. This only covered games in which players gained when they played and was not easy to apply. In 1960, John Conway met Michael Guy, Richard's son. Through this friendship, John met Richard and asked about partizan games. This turned out to be a recurring theme in their work in the next two decades. Also in 1960, Elwyn Berlekamp got roped into playing 3×3 DOTS-&-BOXES game against a computer. He lost, but knowing about the Sprague–Grundy theory, he analyzed the game. (Recently, Elwyn claimed that he had never lost a game since.) Elwyn met Richard at the 1967 Chapel Hill conference and suggested that they write a book. Richard agreed, got John and Elwyn together in 1969, and work began. The analysis of each nonimpartial game was well thought out but ad hoc. John, with his training in set theory, started to see a structure emerging when games were decomposed into components. He gave the names of 1 and 1/2 to two abstract games and was delighted (giggled like a baby was the phrase he used) when he discovered that, as games, $1/2 + 1/2 = 1$. He wrote *On Numbers and Games* [28] in a week. This caused some friction among the three, but, eventually, work restarted on *Winning Ways* [3, 4].

R. Austin, S. Devitt, D. Duffus, and myself, as graduate students at Calgary, scoured the early page-proofs. We suggested numerous jokes and puns. Fortunately, the authors rejected all of them.

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One other person deserves to be mentioned, Louise Guy, Richard's wife. A gracious lady made every visitor to their house feel welcome. Some people have asked why the combinatorial game players, Left and Right, are female and male, respectively. The original reasons have been forgotten, but after *Winning Ways* appeared, it became a mark of respect to remember them as **L**ouise and **R**ichard.

Why Elwyn, John, and Richard are important

Many books are written, enjoy a little success, and then are forgotten by all but a few. *On Numbers and Games* but especially *Winning Ways* [3, 4] are still popular today. This popularity is due to the personalities and their approach to mathematics. All were great ambassadors for mathematics, writing explanatory articles and giving many public lectures. More than that, they understood that mathematics needs a human touch. These days, it is easy to get a computer to play a game well, but how do you get a person to play well? This was one of their aims. *Winning Ways* is 800+ pages of puns, humor, easy-to-remember sayings, and verses. These provide great and memorable insights into the games and their structures, and the book is still a rich source of material for researchers. Mathscinet reports that *Winning Ways* is cited by over 300 articles, Google Scholar reports over 3000 citations. Yet, any reader will be hard pressed to find a single mathematical proof in the book. Elwyn, John, and Richard wrote it to entertain, draw in a reader, and give them an intuitive feeling for the games.

After the publication of *Winning Ways*, even though all were well known for their research outside of combinatorial game theory, they remained active in the subject. Each was interested in many parts of the subject, but, very loosely, their main interests were:

- Elwyn Berlekamp considered the problem of how to define and quantify the notion of the "urgency" of a move. He made great strides with his concept of an enriched environment [11, 24, 25]. He was also fascinated by GO [7, 8, 9, 10, 12, 11] and DOTS-&-BOXES [13, 18, 23].
- John Conway remained interested in pushing the theory of surreal numbers, particularly infinite games [30, 37, 41], games from groups and codes [32, 39], and misère games [35].
- Richard K. Guy retained an interest in subtraction and octal games, writing a book for inquisitive youngsters [52]. He continued to present the theory as it was [54, 57, 58, 59, 61] and also summarized the important problems [56, 60, 62, 64].

Standing on their shoulders

Most of the papers in this volume can be traced directly back to *Winning Ways* and *On Numbers and Games*, or to the continuing interests of the three. Several though,

illustrate how far the subject has developed. A general approach of impartial misère games was only started by Plambeck [71]. A. Siegel (a student of Berlekamp), a major figure developing this theory, pushes this further in Chapter 20. The theory of partizan misère games was only started in 2007 [69]. Whilst playing in the context of all misère games, Chapter 10 analyzes a specific game. Chapter 16 contains important results for analyzing misère dead-ending games. In *Winning Ways*, DOTS-&-BOXES and TOP-ENTAILS do not fit into the theory, each in a separate way. They are only partially analyzed and that via ad hoc methods. Chapter 17 finds a normal play extension that covers both types of games. (The authors think this would have intrigued them but are not sure if they would have fully approved.)

Chapters 1, 5–9, 12, 15, 18, and 19 either directly extend the theory or consider a related game to ones given in *Winning Ways*. As is evidenced by Richard K. Guy's early contributions, it is also important to have new sources of games. These are presented in Chapters 2, 3, 11, 13, and 14.

Serendipity gave Chapter 4. This paper is the foundation of Chapters 1 and 5. It gives a simple, effect-for-humans, test for when games are numbers. The authors are sure that Elwyn, John, and Richard would have started it with a rhyming couplet that everyone would then remember.

Elwyn, John, and Richard gave freely of their time. Many people will remember the coffee-time and evenings at the MSRI and BIRS Workshops. Each would be at a large table fully occupied by anyone who wished to be there, discussing and sometimes solving problems. Students were especially welcome. All combinatorial games workshops now follow this inclusive model. A large number of papers originate at these workshops, have several coauthors, and include students. They shared their time outside of conferences and workshops. Many students will remember those offhand moments, with one or more of them, that often stretched to hours. I was a second-year undergraduate student when on meeting John, he immediately asked me what was $1 + 1 + 1$? Even after I answered "3", he still took the time to explain the intricacies of 3-player games. (The question is still unanswered.)

Their wit, wisdom, and willingness to play provided people with pleasure. They will be sorely missed, but their legacy lives on.

Richard J. Nowakowski

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