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Language and mathematics: How children learn arithmetic through specifying their lexical concepts of natural numbers

1 Introduction

When children are about 18 months old their speech output rapidly increases. It's like an explosion where about 10 new words are learned every day. It seems as if children suddenly understand how they can use language to interact with their surroundings. At 21 months of age the 100-word milestone in productive vocabularies is reached (Pine, 2005). Words are still mostly content words, used to refer to concrete objects and to describe the relationship between objects with expressions such as "car there," "mommy's mug," or "doggy sleep" being common. Around their second birthday they start to use words to describe the relationship of singular and plural. After being able to say and point out "car there" and doing so for all the cars seen at that moment, all of the sudden they say "car there, many car," pointing out all the cars observed (Barner et al., 2007). With the usage of natural quantifiers infants engage verbally with the world of numerical relationships. Soon after this development, children are able to describe objects as being "two." What seems like simply naming a group of things needs in fact the development of deep lexical concepts, which rely, on the one hand, on innate structures, and which, on the other hand, is learned from conversational interactions (Carey, 2009).

Being able to name the number of things seen in their surroundings means that infants refer to lexical concepts which are concrete and abstract at the same time. The twoness of something is concrete because of being unique and distinct from being "three" or "one"; on the other hand, it is abstract because it names and highlights just one feature of the objects seen. At the same time the word "two" has a whole bundle of different significations. We are, for example, referring to two cars meaning the magnitude, to the second car meaning the numerical order, and to two gallons of water describing a continuous substance. So, while "two" always has the same numerical value, it differs in shape, color, form, and size (Wiese, 2007). To integrate all these and even more information into one lexical concept requires about six years to develop as we will elaborate in the chapter.

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Our aim in this chapter is to describe how lexical concepts for natural numbers develop. In order to address the complexity of the topic, we present interlinking sections, each dealing with a distinct though connected topic.

In Section 2, we provide a brief overview of the innate knowledge of number and magnitude, as it manifests in the child's numerical development. We focus on the innate core systems being the *approximate number system (ANS)* and the *object tracking system (OTS)*. In parallel, drawing on Carey (2009), we introduce "language" as the third core system and point out which linguistic structures seem to be fundamental for the numerical development.

In Section 3, we refer to the construct "bootstrapping," and explain this most important learning tool that is needed to integrate all numerical knowledge we presume is stored separately in each of the three core systems. To do so we introduce the *knower-level theory* (Le Corre & Carey, 2007), which explains how children gain the specific lexical knowledge, constituting the vocabulary of natural numbers up to "four."

In Section 7, we describe cognitive constraints as a key learning tool. Cognitive constraints function as a parallel working process, which helps children to organize their surroundings and order all global knowledge. This process, the functioning of cognitive constraints, may be likened to children building a mental closet with drawers for all different categories.

In Section 4, we present a model of early arithmetic development, where we focus on hierarchy in the development of numerical knowledge and align this hierarchy with age.

Considering the number of theories introduced and intertwined with each other each section ends with an interim conclusion summing up what has been stated so far. We close the chapter with an all-embracing conclusion bringing all knowledge components together.

2 Innate knowledge of number and magnitude

Over the last 20 years of research a lot has been discovered about innate structures of number and their magnitude. Whereas arithmetic, dealing with the properties and operations of number, is a culturally dependent and learned tool, knowledge of magnitude is an evolutionally old structure securing, for example, survival. Dehaene (1999) calls this knowledge of magnitude, the number sense. The two systems we rely on, as well as most of the animals, are called core systems. The first core system holds the knowledge of magnitude and is called the *approximate number system*. Here the approximately cardinal value of perceived sets is represented by a physical magnitude that is roughly proportional to the number of objects, or individuals, in the set being enumerated. What is currently known from the literature is that analog magnitude representations of number are available as early as six months of age. Number represented as analog magnitude is not unusual; dimensions like brightness, loudness, and temporal duration are also represented this way. In all cases the absolute distance of two entities of greater magnitudes is increasingly harder to discriminate and underlies a function of their ratio described by Weber`s psychological law of quantifying change (Sarnecka & Carey, 2006). With this system it is possible for infants and children to process the number of big sets and compare or distinguish these if the ratio between the sets is as big as 1:2, slowly decreasing to a 2:3 ratio. Xu and Spelke (2000) could show this ability to distinguish starting with seven-month-old infants. In their experiment the children were shown displays of eight dots. After a while the children lost their interest in displays showing eight or even more than eight dots. They recovered interest only when a novel display held at least 16 dots.

The second core system is called the *object tracking system*. This system processes mental representations of individual object-files as in "one," "one-one," and "one-one-one." The symbols in this system explicitly represent discrete objects. This innate input analyzer represents implicit numbers; each object-file corresponds one-to-one with its match in the world. Being nonverbal this system does not hold any numerical information and the models are just compared as being either equal or unequal.

Wynn (1992) found that children even form expectations about how these models should interact. Five-month-old children are able to perform transformations on small sets. In her experiment children observed a small set of one or two discrete objects. After familiarization a screen hid the objects from view. The children could then only see how one entity was added or removed. In both events the child saw the action without seeing the outcome. After removing the screen children reacted with more attention to the unexpected event, which led Wynn to the supposition that they had mentally also performed the adding or subtracting task and expected the situation to match their "new" working memory model. The argument is that children are able to mentally hold and match sets like this, and they therefore can use this information to distinguish entities according to quantity.

Feigenson et al. (2002) showed that children always chose the greater quantity, if numbers ranged between one and four. In their experiment they placed up to four cookies in two opaque cookie jars of the same size using different pairings. Being just 12 months of age children always chose the cookie jar having more cookies in it, if the ratio was 1 to 2 or 2 to 3. If the pairing was 1 to 4, 2 to 4, or 3 to 4 children chose by random, showing no differentiation. Here the limit to store up to three object-files simultaneously is reached, after which performance declines. While observing how the cookies are placed the system creates a Working Memory Model, which contains one object-file for each cookie. Afterward the system must keep track of whether the object seen at one point in time, is the same *one* as that object seen at the previous point in time. The decision the system makes dictates whether an additional individual file is established, and this guarantees that a mental model of a set of three crackers will contain three cracker symbols.

This outcome leads to the interim conclusion that pre-numerical set sizes are supported by *iconic mental representations*. Each object-file held in the short-term memory is an icon for its match in the real world (Wiese, 2007). Nonetheless children have no awareness of the fact that three object-files are equivalent to the numerical quantity of three. In fact, neither of the core systems, although storing numerical content, has the power to represent natural number (Carey, 2009). Both core systems share constraints that do not allow the child to distinguish between a single object and multiple objects precisely. There are no symbols, in these core systems for plural. There are no concepts in either of these systems of core concepts with the content, some, all, or the indefinite article "a" meaning "one" (Carey, 1999: 194). Based on this argument that the complex cognitive system, holding these two systems, is incomplete, a third system, that of language, is introduced as playing a pivotal role in the development of number knowledge.

The starting point for children is learning and recognizing number words in speech. Toddlers do this very early in their development, and while observing the surrounding speech patterns they quickly understand that number words are linguistically different from other adjectives. The usage of number words differs very much from descriptive adjectives; they are used not to denote something, but to describe a relation, for example, a set size or a sequential position (Wiese, 2007). The class of number words soon forms surface concepts, meaning that toddlers know that these words are used to somehow describe the quantity and relationship of quantities to each other, and in addition, that you need to point to entities while using these words. Number words are now the "placeholder." As expressed by Negen and Sarnecka (2012), "The placeholder symbols are the memorized count list and associated counting routine – without those, the number concepts themselves would not be created. In that sense, number-concept creation may depend heavily on language" (Negen & Sarnecka, 2012).

Moving forward in their development children sequentially integrate over the years the following critical pieces of information. Around their second birthday they integrate knowledge about the stable order of the number sequence. They understand singular and plural relationships and can describe them verbally with quantifiers such as *more* or *less*. In the year following, by the third birthday, they start integrating the knowledge that a number word matched to a point sequentially further along the number line is the same number used for a larger quantity. The vocabularies of spatial (e.g., between) and temporal prepositions like “before” or “after” are crucial in the early development of number comprehension. “Before” and “after” used in context of the number line refer to changing magnitudes and can be synonym to “less” and “more.”

Children gradually improve their counting by applying the five counting principles introduced by Gelman and Gallistel (1978). The first principle is the one-to-one relation; the second, the stable order of the number line; and third, the cardinal principle, or the *last word rule*, referring to the number word of the last object counted, which is the answer to the question, “How many objects are there altogether?” As we will see, children knowing how to answer this question are not necessarily aware of the fact that this answer describes all the counted objects (Fritz et al., 2013). The fourth and fifth principle, respectively, acknowledge that counting can take place without a real counterpart *(abstraction principle)*, and that the starting point of counting is flexible *(order irrelevance principle)*.

In summary, it may be asserted that to learn the natural number system children start off with three separate systems: the ANS, the OTS, and language. None of the systems independently, as we have seen, holds enough numerical information to form stable concepts of natural number. The ANS has the power to form approximate representations of large magnitudes, the OTS stores iconic representations as object-files up to a number of three. Language has the power to discriminate between individual objects and sets. Chierchia (1998) formulates this to be a semi-lattice structure (Fig. 1) which is linguistically universal. This is the basis of all *language quantification systems,* which will be used by children to link core knowledge together into numerical concepts.

{a,b,c,d,...}

{a,b,c} {a,b,d} {b,c,d} {a,c,d} ...

{a,b} {a,c} {a,d} {b,c} {b,d} {c,d} ...

a b c d = At

Fig. 1: Semi-lattice structure (Chierchia, 1998).

Thus, for children there is the need to combine all three systems to construct new stable concepts representing the various features. Therefore, what starts as a surface lexical concept increases and at approximately four years of age holds information of a stable order, that each number word is used for exactly one corresponding magnitude; it is also the last word voiced while counting a set of objects, or the spaces on a number line, and it represents the quantity counted.

3 Bootstrapping

To combine all three systems, toddlers use what Carey calls bootstrapping. This process is used to not simply match information together, but to actually construct completely new mental concepts building on the foundation of innate structures.

One could think that the knowledge that the number word "three" refers to the sum of three individual objects is simply a problem of matching the number word and the iconic representation of three object-files correctly together. One might liken this process to grown-ups learning the number line in French and matching each number word to the corresponding magnitude. If this were the case, the development of numeracy knowledge would be a relatively quick process once the three core systems were in place as they hold all the information that is needed (Sarnecka & Carey, 2006). But in fact, the development is a slow and extended process, in which each child constructs new mental concepts. We will see this development in the *knower-level theory*.

Building new concepts on the foundation of innate structures is a unique human resource. We use this resource every time we learn a cultural tool like, for example, arithmetic. To do so Carey (2009) coined the term *conceptual-role bootstrapping* – bootstrapping as a metaphor, which means to get oneself out of a situation using existing tools. In the case of a toddler confronted with the task to differentiate, describe, and work with number and magnitude, it needs to find an answer to this dilemma from within its own set of innate resources. Early on children are confronted with number in speech. They soon know that number words refer to something special, since this group of adjectives does not work like "normal" adjectives, such as adjectives of comparison; for example, number words cannot be increased. To give an example you can talk about a small dog and then about a smaller one, but you cannot talk about two dogs and then about "twoer" dogs. Only number words and color words share this feature.

Although rooted in the same class of words and being linguistically more alike, talking of "three bricks" and "red bricks" still has different reference objects. Red builds a feature belonging to the brick; three does not belong to the brick in itself, but rather describes the relationship the bricks have to each other. A motherly demand, "Hand me those three bricks," challenges the child on a very high level. The child first of all needs to understand brick and the task of "giving something." But the real work starts with analyzing the adjective "three" and its grammatical and semantic features.

To truly understand the cognitive demand on the child, let us consider the situation the child shares with its mother and what information the child gets from its innate knowledge about the phrase “Hand me those three bricks.” The child and its mother play with bricks; there is a group of three bricks lying on the floor next to the child. The mother then asks the child to give them to her.

1. The word “those” indicates that mother and child have a common focus while the mother points out which group of bricks she means, making sure her child pays attention to the right group.
2. When focusing on the group of bricks, the child’s innate core structures respond and the OTS builds – in this particular case – an abstract mental representation of “one–one–one.” The model matches the real situation (iconic representation).
3. The iconic representation of three is held securely in the working memory during the whole analyzing process.
4. In the word “bricks” the plural “s” is grammatically decoded.
5. The number word “three” has – dependent on the depth of the lexical concept of “three” – to be matched to the iconic representation.
6. The child might count out the number; or hand three bricks over directly.

The cognitive information up to step four comes from innate structures and emerges without much effort from the child. But actually, being able to hand over any exact number of things is a learning process that children need to work on for over a year. Earlier we stated that number words form placeholders, meaning children know the words and use them, but they have not assigned deep lexical meaning to them. These surface concepts will now be enriched step by step. Le Corre and Carey (2007) describe the process of gaining knowledge of the cardinal number by evoking the *knower-level theory*.

Toddlers understanding only a “giving task” will grab any amount and hand it over. They might insist that the number matches the magnitude asked for, thereby showing they understood that a quantity was asked for but may not be able to count out the correct number. These children are referred to as “grabbers.” Afterward they start working out the cardinal number of “one.” Being a *One-knower* means toddlers are capable of handing over exactly one object but would grab an arbitrary amount of bricks when asked for more than one. Once the child knows what *one* means, the child will then start working out cardinal values of *two* to *four*. This development means that in the next step the number word “two” and sets containing two objects intersect. Children are then capable of naming sets of two, plus after being asked for any set of two, they can respond with the correct amount. Having conceptualized the set number of “two,” they start the process on “three” in the same manner. It takes about one year to develop cardinal knowledge up to a magnitude of “four.”

Referring to our example this means that to meet the mother's request our child has to be at least a three-knower since in this case the iconic representation shows three and the number word is "three."

Sarnecka (2014) affirmed that the attainment of deep lexical concepts occurs via mapping number words to the corresponding object-files. She compared the amount of time children required to step up to conceptualize the next cardinal number in different languages. Doing this she could even show that children acquiring the next knower level took a longer time when reaching the boundary to the undefined plural. If languages have a singular/plural marking system like English or in fact German, children stay *one knower* for a longer time than children growing up with languages marking singular/dual/plural as used in Slovenian or Saudi Arabic. Here the children step up to know the meaning of *two* faster but take a longer time to step up to knowing the undefined plural starting with three. If languages do not mark singular/plural like, for example, Japanese, children need more time to even become a *one knower*.

So far, the two systems of need were *quantificational language* and the *OTS*. The natural limit of the OTS is the cardinal value of four. This limit means there are no more exact mental models as a reference. From this point on the process of counting forms the basis for attainment of exact cardinal number. Children having worked out the meaning of *one* to *three* have understood how quantities can be measured by counting them out. Yet not all higher magnitudes can be detected easily. It seems as if now the ANS needs to sharpen. Le Corre and Carey (2007) tested children on magnitudes raging between 6 and 10 and could show that even though children could give any exact number asked for through counting, the capability to semantically map higher magnitudes firmly to their corresponding number word still needs, on average, about six more months to develop. They differentiated children linguistically in groups of *non-mappers* and *mappers*. When presented with magnitudes raging between 6 and 10, non-mappers answer at random while mappers estimate closely to the correct cardinal number. Le Corre and Carey (2007) interpret this to mean that even though children have understood how to count out any number asked for, they still need about six months to actually map the approximate representation held by the ANS to the corresponding number word.

Drawing the evidence together, it can be concluded that to learn natural number children rely on three innate structures. These structures being the *analog magnitude system* for cardinal number higher than 4, the *object tracking system* with its parallel individuation models for cardinal number up to three at the most four and the *system of quantificational language*. To grasp the full meaning, they have to override the limitations of the two core systems representing numerical content and map all numerical features to the number words.

What seems most important is that not one system alone has the power to represent all these pieces of information, but that all three must be intertwined together to actually build new stable and enriched lexical concepts. Therefore, although it seems as if number words form the foundation for all this development, Sarnecka (2016) states that they, the number words, are only the scaffolding. Once the new concept is attained, the scaffolding is no longer necessary, and the number word becomes just one feature among all others in the deep conceptualization of natural number.

In addition to matching the initial cardinal knowledge to number words, children need to deepen their understanding of the relationship numbers hold to each other and the relationship with regard to the number line construct. Additionally, children are constrained by the kinds of hypotheses (referred to as cognitive constraints) that they use to work out not only cardinal aspects but also ordinal dimensions of number. How these constraints work and the extent of their influence on the deep lexical concepts of natural number will be discussed in the following section.

4 Cognitive constraints

The previous section dealt with the children's ability to work out initial aspects of quantity. This means, in essence, that they learned how number words can be used to count out small quantities by applying a one-to-one correspondence of number words to matching objects. They learned that the last counted number word answers the question "how many?" Having grasped these aspects of cardinality they still do not know how each number stands for a whole set representing the magnitude of a number. This aspect is gained through counting out quantities and responding to the demand to give an exact amount seen in the "give a number" task.

They have also learned so far that the number word sequence has a fixed order that has to be maintained. This preservation of order is the linguistic feature crucial to attributing ordinal number assignment. Ordinal assignment describes through language the position or rank of a number (Wiese, 2007) and requires the conceptual development of a mental number line.

Even before children are verbally well grounded in their knowledge of the order of numbers up to ten, and before they have built up cardinal knowledge about magnitude, they are capable of arranging quantities according to their amount. The innate system used here is the ANS, which leads children to estimate quantities. With this knowledge children can through a one-to-one-correspondence work comprehensively with quantities. This knowledge means that they can put them into a sequential arrangement following their magnitude or even equally split magnitudes. Linguistically they do not use number words to do this but rather quantificational language such as *more*, *less*, or the *same* (even) (Fritz & Ricken, 2008).

Keeping these aspects of innate knowledge derived from the ANS and the OTS (see section one) in mind, the children´s task is to integrate those concepts. They do this using cognitive constraints in organizing learning that are an innate foundation in itself. Sophian (2019) states that cognitive constraints underlie the children's task to learn very complex bodies of knowledge by assuming that they start with some expectations that simplify the learning task. With respect to vocabulary learning, one might understand the challenging problem children experience here. Each unknown word has a large number of possible interpretations. It can describe focal objects, parts of an object, characteristics, or actions of an object. To narrow down their chances Markman (1990) describes the whole-object constraint to be the first one that children rely on. This means that hearing any new word children tend to search for a referential object. Using the whole-object constraint they pick out objects that have not been labeled yet and guess the new word to name the whole object. With this nominal strategy children form distinct objects. The taxonomic constraint then leads children to the assumption that single words can be used to label a whole class of objects that are taxonomically related. A third constraint, the mutual exclusivity constraint, leads the children to assume that each object has only one name, so hearing a new word they search for an unfamiliar reference, which will then be labeled. If there is no unfamiliar object this constraint is the one that helps children to override the others. If all objects are named, children seek new referents for the new word, which may be one part of the object in question. It is in this way that constraints are used heuristically to get a learning process started (Sophian, 2019).

Congruent with the counting principles described in section one, Gelman (1991) suggests these principles function as constraints. She proposes that they also need to be overridden. Shipley and Shepperson's (1990) ideas of how constraints help children's development of natural number resemble the whole object constraint of Markman (1990). The tendency to focus on whole, distinct objects leads children to interpret numbers used in counting as tags in the counting sequence (Sophian, 2019). Wiese (2007) refers to this as the nominal dimension numbers hold. This means that each tag (number word) is used like a fixed label or name for the objects counted (iconic counting).

A central task in numerical development is that of reasoning about relations. Resnick (1992) describes four levels of numerical reasoning. Roughly speaking these four levels can be divided into two levels, “protoquantitative reasoning” and “quantitative reasoning” in the preschool years, and “mathematics of numbers” and “mathematics of operations” in the school years (Sophian, 2019). Since this chapter focuses on the preschool years levels III and IV will be omitted.

The protoquantitative schemata basically describe prenumercial relations of equality and inequality as well as less-than and greater-than. As seen earlier in this development, AMS and quantificational language work together.

When children have gained knowledge of the number word sequence and have developed counting abilities, they enter the world of quantitative reasoning and are able to think about ordinal relations. With matching quantity to number words “less-than” and “greater-than,” relations hold numerical content that can in the following years be used to order magnitudes due to their exact quantity. Starting off with the development of a mental number line, ordering numbers describes the relations numbers have with each other.

5 The mental number line

The mental number line comes into existence as soon as global knowledge of magnitude and counting combine. It allows children to understand the hierarchical arrangement of number words to represent a sequence of increasing numbers and, based on this understanding, the subsequent sequences of increasing quantities (Resnick, 1983), meaning that each number has a fixed position in the hierarchy of numbers. Integrating more and more number words with their magnitudes means therefore that more positions on the number word line become “numerical.” What has to be kept in mind here is that children order magnitudes due to their fixed position coming from the number words so there is as yet no numerical knowledge of distance relations between numbers. Children at this developmental stage do not know about the successor function, meaning that all neighboring numbers/magnitudes differ by exactly *one* (Fritz et al., 2013). Tasks asking children to write out a number line show that bigger numbers are positioned at narrower intervals (Fig. 2).

The placeholder structure which the line of number words had is thereby transformed into the mental number line, where each number word will by the end of kindergarten hold knowledge about set size and position.

Children now have an abstract representation of the number line where they can locate any given number, know the neighboring numbers, and use sections of the number line to solve arithmetic problems encountered in the early years. Fuson (1992) showed that children solved contextually rich arithmetic tasks by

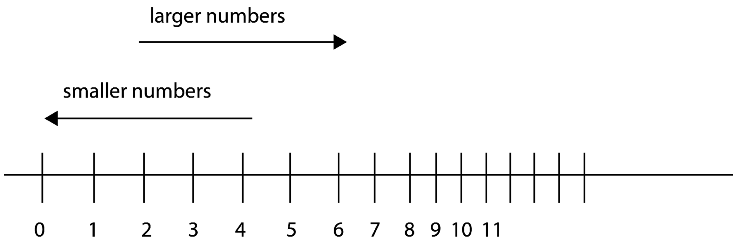

Fig. 2: Number line (Fritz et al., 2013).

“wandering up and down” the mental number line. Using the counting-all strategy, they first count out both subsets and then count the whole starting from one.

6 Second interim conclusion

The argument up to now shows that children construct cardinal and ordinal knowledge using innate structures. On the one hand they use the core systems described in section one; on the other hand, they use cognitive constraints that help them organize their surroundings and order the development of concepts.

In the preceding sections we have described two levels of learning that children consolidate in their first five years of life. The first level defines initial cardinal knowledge. Children have integrated magnitude and number word, and with this development are able to enumerate small sets of numbers. They answer questions concerning the magnitude by counting the whole set. The second step describes how ordinal aspects of numerical knowledge are formed and a mental number line is constructed. Third, children deepen their cardinal knowledge. By the age of around five they are full cardinal principle knowers, knowing about the magnitude of any number.

But having come this far in their numerical development children still lack deeper understanding of numerical relations. At this point they do not know about the part-part-whole relation or the complex ordinal relation that numbers on the mental number line always differ by a magnitude of one.

During the next two years, about ages six and seven, most children develop firm concepts of part-part-whole relations and concepts based on the relation of congruent intervals that exist between successive numbers on the number line. Sophian (2019) considers these firm concepts, part-part-whole relations and congruent intervals, to be the step from quantitative meaning to the mathematics of numbers that in her account seems to be closely related to success in schooling, since the disassociation of number from any physical referent is typically called on in school exercises.

7 Model of early arithmetic development

Taking these results into account, Fritz and Ricken (2008) (see also Ricken et al., 2013, 2011) framed a model of early arithmetic knowledge and its development. Theoretically confirmed by substantial research they could show empirically how numerical knowledge develops following six levels of arithmetic understanding. Each layer is thereby acquired separately and represents its own distinct numerical conceptual innovations. Though being hierarchical, the successive levels of understanding are not discrete but rather develop in "waves"; for example new knowledge is already present, while old knowledge is still in use (Fig. 3).

A model describing the competences that children acquire has the chance to become a didactically powerful tool, where children at the end of kindergarten and in grammar school can be tested for their actual conceptual knowledge. Since the model represents the conceptual ability that a child has, by mapping the child's competence against the model, help can be structured more easily, and children can then be guided on to acquire incrementally hierarchically higher mathematical concepts.

Each level thereby formulates the key competences children gain and will now be described in brief.

Level I: Count number and level II: The mental number line

These two levels form the foundation of any numerical learning and are basically described in the earlier sections of the chapter. they describe how children gain the fundamental knowledge and how these levels are intertwined in cognitive learning and linguistic processes, they take up most of the time of children entering the world of numerical relations.

From level III onward formal instruction is known to take over as the main influence in elaborating arithmetic knowledge and the chance to perform higher mathematics. Sophian (2019) calls this the mathematics of numbers, where "children move beyond quantitative reasoning" (p.162), where numbers no longer need references to physical amounts and where relations between numbers form the focus of learning.

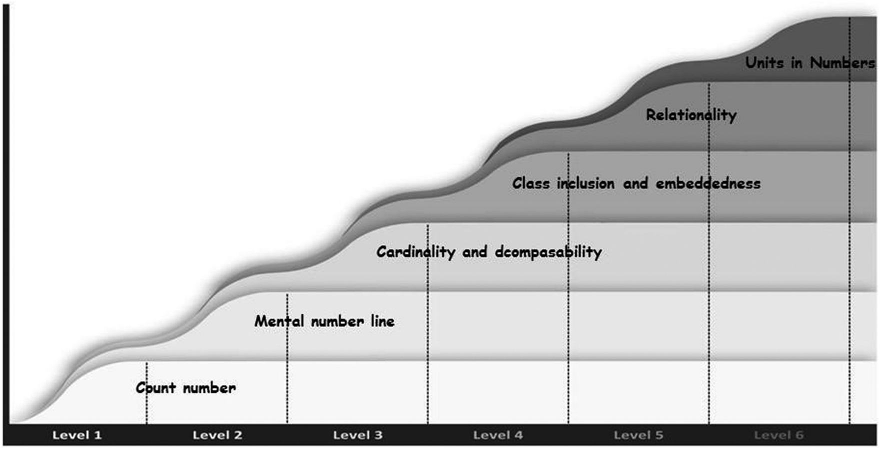

Fig. 3: Developmental model of early arithmetic learning.

Level III: Cardinality and decomposability

At the start of primary school (1st grade) children should be at level III or be "on the jump" from level II to level III. At level III the principle of *cardinal number* is developed. This basically means that a number word does not only stand for the ordinal position but that each number word unites all its counted elements. This does not happen automatically; instruction is required. To conceptualize the cardinal principle, children need to mentally integrate all counted elements into a whole. So, the question "How many are there?" is no longer being answered relying on the *last word rule* – the last number word counted indicates how many there are – but is grasped on a deeper level. The reasoning is as follows. If the quantity being counted holds eight elements, each single element is assigned a number word, and all together are assigned the mightiness, in this case "the eightness." With this mental integration, linguistic-numerical concepts now hold, alongside the feature *ordinal,* also the feature *cardinal.* Both features exist side by side, meaning that the task, which is more, seven or eight? can be answered and extended in two ways. Seven is less than eight, firstly because of its position on the number line, and then secondly, the amount seven holds fewer elements than the amount eight. Having mastered the step that number is a composite unit that consists of single elements, children also understand that numbers can be decomposed again. This very first understanding of part and whole forms is not only the basis for the next level but is also the key competence required to access effective calculating skills. While up to now all adding and subtracting tasks had to be worked out using the *counting all strategy*, children now have the ability to switch to the *counting on strategy* (Fuson, 1992). This strategy builds on the newly added knowledge that each partial quantity stands for a cardinal unit which is embedded in a whole (Fritz & Ricken, 2008). In order to solve an addition task, the first sum is counted out and the second sum is counted onward. The same strategy can now be used for a subtraction task. A subtrahend can be taken away from the whole as a unit. Also tasks where the second sum is to be completed, for example, "I need to go 9 steps to win this board game, my cube shows five. How many steps are missing?" can now be solved by counting on from the first partial quantity.

Level IV: Class inclusion and embeddedness

At level IV the most important concept is learned. They are able to work with the scheme of part-part-whole when given three partial quantities. This means they know that whatever the task is, the relationship of these three quantities is fixed in a triangular relationship and will not ever change. Now any task can be solved when two quantities are given, and in addition, children are able to create new triangles by shifting elements from one partial quantity to the other. Knowing this means children have grasped the concept of class inclusion which states that the connection of partial quantities and the whole can be expressed by the child. They now work at a more formalized cognitive level. With the understanding of the part-part-whole concept they have understood that numbers include other numbers and can be decomposed flexibly. The relation between parts and whole is determined. Thus, addition and subtraction problems, asking for the starting quantity, the exchange, or final quantity, are of equivalent difficulty.

Level V: Relationality

When children reach level V, they include all the knowledge gained on the previous levels resulting in a deep understanding of the concept of natural number. This means in essence that children have integrated ordinal and cardinal knowledge plus knowledge of their relationship in the part-part-whole composition. Numbers are now in position on the mental number line; they are representatives for both the mightiness and countable units in themselves. A newly added feature on this level is that of equidistance. This means that coming from the right of the mental number line on level II, children now know that the distance between any two neighboring numbers is exactly one. With this they can define by how many two quantities differ. This principle holds equally for distances of the same size that are equidistant independent of their position on the number line.

Level VI: Units in numbers

With reaching the final rung in the ladder in the developmental model of arithmetic learning children specify their part-part-whole concept that they gained on level IV. They already know that two partial quantities and their sum form a fixed triangle, where the partial quantities can be changed by shifting elements from one to the other partial quantity. What they now understand is that one whole can be divided into bundles of the same mightiness – 18 can, for example, be decomposed into three bundles each holding six elements. These bundles in themselves form now abstract units and are therefore countable as used in multiplication tasks. At the same time children know that each whole includes different sets of bundles of differing size (18: 3×6 or 2×9). The ability to bundle and unbundle numbers flexibly into sets of the same magnitude therefore shows that children have finally reached a stable, deep, and complex linguistic-numerical concept of natural number.

8 Composite conclusion

It can be concluded, based on the reasoning thus far, that learning arithmetic and gaining deep mathematical understanding is a process that not only takes about seven years to develop but is also dependent on different inputs.

There are, firstly, the innate structures that help children to construct cardinal and ordinal knowledge. The core systems from section one (levels l and ll) and the cognitive constraints from section two are fundamentally responsible for the childlike capability to construct concepts obtaining knowledge about "how many" and a ranking of the number line. The number word hierarchical order builds on the scaffolding that will finally be integrated into the concept itself.

This development takes place throughout the preschool years. With the transition from kindergarten to school, children leave the notion of "quantitative meaning" and enter the world of the "mathematics of number" (both Sophian, 2019).

Children's numerical concepts are now enriched due to formal instruction and become progressively more abstract. They no longer rely on a physical referent but start to work with numbers as abstract units that hold relations to each other. These relations reveal a successive order and describe firstly the establishment of cardinal knowledge and with this understanding the mightiness of a number. Numbers will no longer stand for the position in a string alone, but also for the whole set of all counted objects. Secondly, the expanded knowledge for the structure of the part-part-whole concept, describing the ability to compose and decompose quantities, is gained. Children now understand that quantities are flexible units. Thirdly, children acquire the concept of congruent intervals between all numbers on the number line and finally they gain the competence to bundle numbers into quantities of the same mightiness. These bundles then become new abstract composite units and are therefore countable.

All these different steps have been corroborated empirically and can be described in a model of early arithmetic development (Fritz & Ricken, 2008).

Although research on numerical development is still in early stages and is an ongoing process, it can be confidently stated that numerical reasoning in children underlies fundamental changes during childhood. All empirical data including those of competence and limitation suggest that numerical cognition is based on the *relational* character of numerical reasoning. During the developmental process it is the changes of what and how children relate "kinds of entities (unmeasured quantities, measured quantities, or abstract numbers), and in the kinds of relation among those entities they consider (equivalent relations, additive relations, or multiplicative relations)" (Sophian, 2019: 168).

Thinking about numbers as lexically complex concepts built via bootstrapping processes it can be emphasized that it needs the impact of both innate structures and culturally transmitted knowledge to build up firm numerical cognition.

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