Preface

In the title of this book is the word “Modern”, but—really—we should use the expression “Modern Classical”. In fact, within the past years, different forms of umbral calculi have begun to be studied.

The Umbral Calculus was described for the first time by John Blissard in the 1850’s ([23]) in a form that we call “Classical”. After a short phase of success, the Umbral Calculus was largely rejected by the mathematics community due to the “lack of rigor” ([166]).

In the late 1960s the theory, worked out by Gian Carlo Rota and his co-workers, gave a completely rigorous formulation to the Umbral Calculus, which greatly rehabilitated it. The work [167] and the book [166] give an extensive and lucid presentation of the Umbral Calculus, whereas a shorter introduction can be found in [93]. The “Modern Classical” Umbral Calculus is now the systematic study of Sheffer polynomial sequences, including Binomial and Appell sequences. In fact, the Umbral Calculus, in Rota’s acceptation, allows an algebraic treatment of classical polynomials and numbers beginning from generating functions, recursive and reciprocity formulas, expansion theorems et cetera, depending on the choice of the formal power series as the “Umbra” (Latin for Shadow) of linear functionals and polynomials. Therefore, the Umbral Calculus is a mix of linear algebra, theory of formal power series, and classical analysis.

The Modern Umbral Calculus has been approached from different points of view. For example, by formal power series ([165, 166, 179]), algebraic ([170, 193]), or operator theoretic ([146, 168]). Each of these approaches has been followed by many authors in different applications (see, for example, [5, 10, 38, 78, 81–83, 104, 146]).

In recent times, it has been observed that there is an isomorphism between the Riordan matrices (see, for example, [5, 104, 177]) and the Sheffer polynomials ([179]) (and hence also the Appell polynomials [17] and Binomial sequences [93]). At the same time, the possibility to define the Sheffer polynomials through determinantal forms has been proved (see [60, 62–66, 213, 215]).

Based on these dernier papers, in this book there is an attempt to present a theory of Modern Umbral Calculus in one variable, that is, known and also unknown results, using essentially elementary matrix calculus: lower triangular, infinite matrices, Hessemberg, Toeplitz, Riordan-type matrix, determinant and Cramer’s rule, recurrence relations, and few more. Hence, this book is not a complete and updated survey, but a new approach to the classic umbral calculus. In truth, the use of matrices in the theory of umbral calculus goes back to Vein’s papers ([197, 198]). In particular, in ([198]) it is written: “The referee printed out that this work is an explicit matrix version of umbral calculus as presented in Rota et al. ([167, 168, 170])”. This work shows that Vein’s procedures are really different from ours. Our procedures are simpler and for a

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wider audience. In Section 1.2 of Chapter 1, Vein's approach will be sketched and the differences will be clarified.

The motivation of our choice is to target the largest number of readers: from undergraduate students to young researchers, even those in disciplines other than mathematics. We also stress the importance of umbral calculus in the training of young students in mathematics.

The modern umbral calculus has more applications and in various disciplines: probability theory (for example, [38, 74, 78, 82, 83, 169]), number theory (for example, [87, 96]), linear recurrence (for example, [146]), et cetera. In the sequel, we consider the applications to general linear interpolation (for example, [61, 62, 67, 128, 200]) and operators approximation theory (for example, [10, 112, 152, 188, 189]). Moreover, we point out that a sufficiently comprehensive bibliography up to 2000 is in [79].

In the Chapter 1 of Part I, there is a more detailed presentation of the contents of the book.

In closing this preface, I would like to quote H. J. Stetter ([187]) and G. Walz ([203]): "*I ventured to write this book in English because it will be more easily read in poor English than in good Italian by* 90 % *of my intended readers.*"

Rende, May 2018 *Francesco Aldo Costabile*