Preface

The theory of optimal transport was born towards the end of the eighteenth century, its founding father being Gaspard Monge [35].

Optimal transport theory has connections with partial differential equations (PDEs), kinetic theory, fluid dynamics, geometric inequalities, probability, and many other mathematical fields as well as with computer science and economics. As such, it has attracted many leading mathematicians in the last decades.

There are several very good textbooks and monographs on the subject. For the novice we recommend, as an appetizer, the first book of C. Villani [48], titled "Topics in optimal Transport." This book describes, in a vivid way, most of what was known on this subject on its publication date (2003). For a dynamical approach we recommend the book of Ambrosio, Gigli, and Savare [2], dealing with paths of probability measures and the vector-field generating them. This fits well with the thesis of Alessio Figalli on optimal transport and action minimizing measures [16]. The main treat is, undoubtedly, the second monster book [49] of Villani, published in 2008. This book emphasizes the geometric point of view and contains a lot more. For the reader interested in application to economics we recommend the recent book [18] of A. Galichon, while for those interested in connections with probability theory and random processes we recommend the book of Rachev and Rüschendorf [38]. As a dessert we recommend the recent book of F. Santambrogio [41], which provides an overview of the main landmarks from the point of view of applied mathematics, and includes also a description of several up-to-date numerical methods.

In between these courses the reader may browse through a countless number of review papers and monographs, written by leading experts in this fast growing field.

In the current book I suggest an off-road path to the subject. I tried to avoid prior knowledge of analysis, PDE theory, and functional analysis as much as possible. Thus I concentrate on discrete and semi-discrete cases, and always assume compactness for the underlying spaces. However, some fundamental knowledge of measure theory and convexity is unavoidable. In order to make it as self-contained as possible I included an appendix with some basic definitions and results. I believe that any graduate student in mathematics, as well as advanced undergraduate students, can read and understand this book. Some chapters can also be of interest for experts.

It is important to emphasize that this book cannot replace any of the books mentioned above. For example, the very relevant subject of elliptic and parabolic PDEs (the Monge–Ampere and the Fokker–Planck equations, among others) is missing, along with regularity issues and many other subjects. It provides, however, an alternative way to the understanding of some of the basic ideas behind optimal transport and its applications and, in addition, presents some extensions which cannot be found elsewhere. In particular, the subject of vector transport, playing a major role in Part II

https://doi.org/10.1515/9783110635485-201

of this book is, to the best of my knowledge, new. The same can be said about some applications discussed in Chapter 8 and Part III.

Starting with the most fundamental, fully discrete problem I attempted to place optimal transport as a particular case of the celebrated stable marriage problem. From there we proceed to the partition problem, which can be formulated as a transport from a continuous space to a discrete one. Applications to information theory and game theory (cooperative and non-cooperative) are introduced as well. Finally, the general case of transport between two compact measure spaces is introduced as a coupling between two semi-discrete transports.