

Preface

Fractional calculus (FC) originated in 1695, nearly at the same time as conventional calculus. However, FC attracted limited attention and remained a pure mathematical exercise in spite of the contributions of important mathematicians, physicists, and engineers. FC has seen a rapid development during the last few decades, both in mathematics and in applied sciences, and today it is recognized as a very effective tool for describing complex phenomena involving long-range memory effects and non-locality. A large number of research papers and books devoted to this subject have been published, and presently several specialized conferences and workshops are organized each year. The popularity of FC across all fields of science is due to its successful application in mathematical models, namely, in the form of FC operators and fractional integrals and especially differential equations. Presently, we are witnessing considerable progress of both the theoretical aspects and the applications of FC in areas such as physics, engineering, biology, medicine, economy, and finance.

The popularity of FC has attracted many researchers from all over the world and there is a demand for works covering all areas of science and engineering in a systematic and rigorous form. In fact, the literature devoted to FC and its applications is very extensive, but readers are often confronted with great heterogeneity and, in some cases, with inaccurate information. The *Handbook of fractional calculus with applications* (HFCA) intends to fill that gap and provides the readers with a solid and systematic treatment of the main aspects and applications of FC. Motivated by these ideas, the editors of the volumes involved a team of internationally recognized experts for a joint publishing project offering a survey of their own and other important results in their fields of research. As a result of these joint efforts, a modern encyclopedia of FC and its applications, reflecting present-day scientific knowledge, is now available with the HFCA. This work is distributed by several distinct volumes, each one developed under the supervision of its editors.

We have organized this volume into different thematic units, starting with time-fractional PDEs, proceeding with space- and space-time fractional PDEs, and concluding with singularities and fast solvers.

Part 1. Time-fractional derivatives

Chapter 1 describes many different numerical methods to compute fractional integrals and fractional derivatives, with emphasis placed on Caputo operators. It also covers several solvers for time-fractional differential equations. Chapter 2 introduces various approximation formulas for time-fractional problems, which can be theoretically proven. It also provides numerical schemes for the time-fractional subdiffusion equation and the diffusion-wave equation along with corresponding theoretical results.

Part 2. Space- and space-time fractional derivatives

Chapter 3 introduces six types of fractional derivatives, both for time- and space-fractional problems, and provides a comprehensive list of high-order discretization of these derivatives. It also provides examples and stability analysis of time- and space-fractional partial differential equations (PDEs). Chapter 4 analyzes two types of spectral methods for fractional differential equations. The first spectral method employs the classical orthogonal polynomials and the Galerkin formulation and is applied to space-time fractional PDEs. The second method employs Muntz polynomials and the weighted Galerkin formulation and is applied to time-fractional PDEs. Chapter 5 presents properties of the generalized Jacobi functions (GJFs) and their application to discretizing accurately space-time fractional PDEs using a Petrov–Galerkin formulation. The use of GJFs enables the resolution of singularities in the solution, which are inherent in fractional PDEs so that high-order numerical accuracy is achieved. Chapter 6 presents both spectral and spectral element methods using Galerkin and collocation formulations. It includes treatments of space-time fractional advection–diffusion as well as fractional Laplacians in multi-dimensions. It also includes hybrid formulations where finite differences are employed in time and spectral or spectral elements for spatial discretization. Chapter 7 focuses on discontinuous Galerkin methods for fractional advection–diffusion equations up to two dimensions. The high-order derivative is decomposed and the equation is recast into a first-order system. Then, a corresponding numerical scheme is presented and its stability and convergence are analyzed. Moreover, some algorithms in two dimensions are presented.

Chapter 8 presents both finite difference and finite element numerical methods for time-space fractional PDEs. In particular, finite difference methods are presented for the one-dimensional time-space fractional advection–dispersion equation and time-space Caputo–Riesz fractional diffusion equation in two dimensions. Also, an unstructured mesh finite element method for the two-dimensional time-space Riesz fractional diffusion equation on an irregular convex domain is presented. Finally, a two-dimensional time-space fractional diffusion equation based on the fractional Laplacian operator is considered. Chapter 9 deals with meshless methods for PDEs with fractional Laplacian in multi-dimensions. It compares two existing radial basis collocation methods on fractional Poisson problems with zero non-local boundary conditions. Numerical results show that the two methods have comparable solution accuracy and same-order time complexity, but different flexibility. Chapter 10 reviews the Lagrangian solvers developed in the last two decades for fractional differential equations using the proper dynamics of random walkers. For the vector fractional differential equations, a multi-scaling compound Poisson process can track the trajectory of particles moving along arbitrary directions with direction-dependent scaling rates. Different random walk particle tracking methods are presented to track particles whose mechanical dispersion follows streamlines.

Part 3. Singularities

Chapter 11 demonstrates by simple examples that singularities are very common in the solutions of fractional differential equations on bounded domains. These singularities reduce substantially the rate of convergence of numerical methods. The chapter reviews four main classes of specific methods that deal with singularities in the solutions of fractional differential equations.

Part 4. Fast solvers

Chapter 12 addresses the issue of computational complexity of the numerical solution of space-fractional PDEs. The non-local nature of fractional differential operators leads to dense or full stiffness matrices with complex structures. This chapter addresses the computational issues of fractional PDEs and outlines some of the recent developments of fast and accurate numerical methods and also discusses possible future directions in the field. Chapter 13 focuses on how to perform fast computation of the Mittag-Leffler function. The series representation is not suitable for computation and hence alternative approaches are presented based on the numerical inversion of the Laplace transform, e. g., a technique known as the optimal parabolic contour. Applications to the evaluation of derivatives of the Mittag-Leffler function and to matrix arguments are also presented.

My special thanks go to the authors of individual chapters that are excellent surveys of selected classical and new results in several important fields of FC. The editors believe that the HFCA will represent a valuable and reliable reference work for all scholars and professionals willing to develop research in the challenging and timely scientific area.

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