5 Commonly used discrete distributions

5.1 Introduction

In this chapter, we will deal with some discrete distributions and in the next chapter we will consider continuous distributions. The most commonly appearing discrete distributions are associated with Bernoulli trials. In a random experiment if each outcome consists of only two possibilities, such as in a toss of a coin either head H or tail T can come, only H or T will appear in each trial, only two possibilities are there, then such a random experiment is called *a Bernoulli trial*. If a student is writing an examination and if the final result is to be recorded as either a pass P or a failure F, then only one of the two possibilities can occur. Then attempting each examination is a Bernoulli trial. But if the final result is to be recorded as one of the grades A, B, C, D, then there are four possibilities in each outcome. Then this is not a Bernoulli trial. When a die is rolled once and if we are looking for either an odd number O or an even number E, then there are only two possibilities. It is a Bernoulli trial. But if our aim is to see which number turns up then there are 6 possibilities, that is, one of the numbers $1, 2, 3, 4, 5, 6$ can appear or there are 6 possible items or possibilities in an outcome. It is *a multinomial trial*. It is not a Bernoulli trial.

In a Bernoulli trial, let the possible events in each outcome be denoted by A and B. Then $A \cap B = \phi$ and $A \cup B = S =$ the sure event. Let the occurrence of A be called "a success" and the occurrence of B "a failure". Let the probability of A be p. Then

$$P(A) = p, \quad P(B) = P(A^c) = 1 - P(A) = 1 - p = q$$

where $1 - p$ is denoted by q. If a balanced or unbiased coin is tossed once and if getting a head is a success, then $P(A) = \frac{1}{2}$ and if the coin is not unbiased then $P(A) \neq \frac{1}{2}$. When a balanced die is rolled once and if A is the event of getting the numbers 1 or 2, then B is the event of getting 3 or 4 or 5 or 6. In this case,

$$P(A) = \frac{2}{6} = \frac{1}{3} \quad \text{and} \quad P(B) = \frac{4}{6} = \frac{2}{3}.$$

5.2 Bernoulli probability law

Let x be the number of successes in one Bernoulli trial. Then $x = 1$ means a success with probability p and $x = 0$ means a failure with probability q. These are the only two values x can take with non-zero probabilities here. Then the probability function in this case, denoted by $f_1(x)$, can be written as

$$f_1(x) = \begin{cases} p^x q^{1-x}, & x = 0, 1 \\ 0, & \text{elsewhere.} \end{cases}$$

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Here, p is the only parameter. This is known as the *Bernoulli probability law*. The mean value $E(x)$, variance $\sigma^2 = E[x - E(x)]^2$ and the moment generating function $M(t)$ are the following: Since it is a discrete case, we sum up:

$$E(x) = \sum_x x f_1(x) = 0 + (0)[p^0 q^{1-0}] + (1)[p^1 q^{1-1}] = p. \tag{5.1}$$

$$E(x^2) = \sum_x x^2 f_1(x) = 0 + (0)^2[p^0 q^{1-0}] + (1)^2[p^1 q^{1-1}] = p.$$

$$\mathrm{Var}(x) = E(x^2) - [E(x)]^2 = p - p^2 = p(1-p) = pq. \tag{5.2}$$

$$M(t) = \sum_x \mathrm{e}^{tx} f_1(x) = 0 + \mathrm{e}^{t(0)}[p^0 q^{1-0}] + \mathrm{e}^{t(1)}[p^1 q^{1-1}]$$
$$= q + p\mathrm{e}^t. \tag{5.3}$$

We may note that this $M(t)$ can be expanded in power series and it can be differentiated also. We can obtain the integer moments by expansion or by differentiation:

$$M(t) = q + p\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots\right].$$

Therefore, the coefficient of $\frac{t^1}{1!}$ is $p = E(x)$ and the coefficient of $\frac{t^2}{2!}$ is $p = E(x^2)$. Higher integer moments can also obtained from this series. Now, consider differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[q + p\mathrm{e}^t]\Big|_{t=0} = [p\mathrm{e}^t]|_{t=0} = p.$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}M(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\mathrm{d}}{\mathrm{d}t}M(t)\right\}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[p\mathrm{e}^t]\Big|_{t=0} = p.$$

Example 5.1. A gambler gets Rs 5 if the number 1 or 3 or 6 comes when a balanced die is rolled once and he loses Rs 5 if the number 2 or 4 or 5 comes. How much money can he expect to win in one trial of rolling this die once?

Solution 5.1. This is nothing but the expected value of a Bernoulli random variable with $p = \frac{1}{2}$ since the die is balanced. Hence

$$E(x) = 0 + (5)\left(\frac{1}{2}\right) + (-5)\left(\frac{1}{2}\right) = 0.$$

It is a fair game. Neither the gambler nor the gambling house has an upper hand, the expected gain or win is zero. Suppose that the die is loaded towards the number 2 or 4 or 5 and suppose that the probability of occurrence of any of these numbers is $\frac{2}{3}$ then the expected gain or win of the gambler is

$$E(x) = 0 + (5)\left(\frac{1}{3}\right) + (-5)\left(\frac{2}{3}\right) = -\frac{5}{3}$$

or the gambler is expected to lose Rs $\frac{5}{3}$ in each game or the gambling house has the upper hand.

5.3 Binomial probability law

Suppose that a Bernoulli trial is repeated n times under identical situations or consider n identical independent Bernoulli trials. Let x be the total number of successes in these n trials. Then x can take the values $0, 1, 2, \dots, n$ with non-zero probabilities. In each trial, the probability of success is p. A success or failure in a trial does not depend upon what happened before. If A_2 is the event of getting a success in the second trial and if A_1 is the event of getting a failure in the first trial then

$$P(A_1) = q, \quad P(A_2|A_1) = P(A_2) = p, \quad P(A \cap B) = qp.$$

where $P(A \cap B)$ is the probability of getting the sequence "failure, success". Suppose that the first x trials resulted in successes and the remaining $n - x$ trials resulted in failures. Then the probability of getting the sequence $SS \dots SFF \dots F$, where S denotes a success and F denotes a failure, is

$$pp \cdots pqq \cdots q = p^x q^{n-x}.$$

Suppose that the first three trials were failures, the next x trials were successes and the remaining trials were failures then the probability for this sequence is $qqqp^x q \cdots q = p^x q^{n-x}$. For any given sequence, whichever way S and F appear, the probability is $p^x q^{n-x}$. How many such sequences are possible? It is $\binom{n}{x}$ or $\binom{n}{n-x}$. Hence if the probability of getting exactly x successes in n independent Bernoulli trails is denoted by $f_2(x)$ then

$$f_2(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, \dots, n \\ 0, & \text{elsewhere, } 0 < p < 1, \ q = 1 - p, \ n = 1, 2, \dots. \end{cases}$$

Note that n and p are parameters here. What is the total probability in this case?

$$\sum_x f_2(x) = 0 + \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q + p)^n = 1^n = 1,$$

see equation (3.12) of Section 3.3 for the binomial sum. The total probability is 1 as can be expected when it is a probability law. Since $f_2(x)$ is the general term in a binomial expansion of $(q + p)^n$, this $f_2(x)$ is called a *Binomial probability law*. What are the mean value, variance and the moment generating function in this case?

$$\begin{aligned} E(x) &= \sum_x x f_2(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x} \end{aligned}$$

since at $x = 0$, $xf_2(x) = 0$. For $x \neq 0$, we can cancel x or divide numerator and denominator by x. We can rewrite

$$x \binom{n}{x} = x \left[\frac{n!}{x!(n-x)!} \right] = \frac{n!}{(x-1)!(n-x)!}$$

since for $x \neq 0$ we can cancel x,

$$\frac{n!}{(x-1)!(n-x)!} = n\left[\frac{(n-1)!}{(x-1)!(n-x)!}\right] = n\binom{n-1}{x-1}$$

and

$$p^x q^{n-x} = p[p^{x-1}q^{(n-1)-(x-1)}].$$

Therefore,

$$\begin{aligned}\sum_x x\binom{n}{x}p^x q^{n-x} &= np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{(n-1)-(x-1)} \\ &= np\sum_{y=0}^{N}\binom{N}{y}p^y q^{N-y}, \quad y = x-1,\ N = n-1 \\ &= np(q+p)^{n-1} = np \quad \text{since } q+p=1.\end{aligned}$$

Therefore, the mean value here is

$$E(x) = np. \tag{5.4}$$

For computing the variance, we can use the formula

$$\sigma^2 = E[x - E(x)]^2 = E(x^2) - [E(x)]^2.$$

Let us compute $E(x^2)$ first:

$$E(x^2) = \sum_x x^2 f_2(x) = 0 + \sum_{x=1}^{n} x^2 \frac{n!}{x!(n-x)!}p^x q^{n-x}$$

since at $x = 0$ the right side is zero, and hence the sum starts from $x = 1$. We can cancel one x with $x!$ giving $(x-1)!$ in the denominator. But still there is one more x in the numerator. But we can see that since a factorial is sitting in the denominator it is easier to compute the factorial moments. Hence we may use the identity and write

$$x^2 \equiv x(x-1) + x.$$

Now, we can compute $E[x(x-1)]$, and $E(x)$ which is already computed.

$$\begin{aligned}E[x(x-1)] &= \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x q^{n-x} \\ &= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x q^{n-x}\end{aligned}$$

since at $x = 0$, $x = 1$ the right side is zero. That is,

$$E[x(x-1)] = \sum_{x=2}^{n}\frac{n!}{(x-2)!(n-x)!}p^x q^{n-x}.$$

Now, take out $n(n-1)$ from $n!$, take out p^2 from p^x, rewrite $n-x=(n-2)-(x-2)$, substitute $y=x-2$, $N=n-2$. Then we have

$$E[x(x-1)]=n(n-1)p^2\sum_{y=0}^{N}\binom{N}{y}p^y q^{N-y}=n(n-1)p^2(q+p)^N=n(n-1)p^2$$

since $(q+p)=1$. Therefore,

$$\begin{aligned}\sigma^2&=E[x^2]-[E(x)]^2=n(n-1)p^2+np-(np)^2\\&=np-np^2=np(1-p)=npq.\end{aligned} \tag{5.5}$$

Thus, the mean value in the binomial case $E(x)=np$ and the variance $\sigma^2=npq$. Let us compute the moment generating function:

$$\begin{aligned}M(t)&=\sum_x \mathrm{e}^{tx}f_2(x)=\sum_{x=0}^{n}\binom{n}{x}\mathrm{e}^{tx}p^x q^{n-x}\\&=\sum_{x=0}^{n}\binom{n}{x}(p\mathrm{e}^t)^x q^{n-x}=(q+p\mathrm{e}^t)^n.\end{aligned} \tag{5.6}$$

For a binomial expansion, see Section 3.3. Note that, the integer moments can be easily obtained by differentiation of this moment generating function. Expansion can be done but it is more involved. Deriving the integer moments by differentiation is left to the students.

Before doing some examples, we will introduce two more standard probability functions associated with Bernoulli trials.

5.4 Geometric probability law

Again, let us consider independent Bernoulli trials where the probability of success in every trial is p and $q=1-p$. Let us ask the question: what is the probability that the first success is at the x-th trial? The trial number is the random variable here. Let $f_3(x)$ denote this probability function. If the first success is at the x-th trial, then the first $x-1$ trials resulted in failures with probabilities q each and then a success with probability p. Therefore,

$$f_3(x)=\begin{cases}q^{x-1}p, & x=1,2,\ldots\\ 0, & \text{elsewhere.}\end{cases}$$

Note that p is the only parameter here. The successive terms here are $p, pq, pq^2, \ldots$ which are in geometric progression, and hence the law is called the *geometric probability law*. The graph is shown in Figure 5.1.

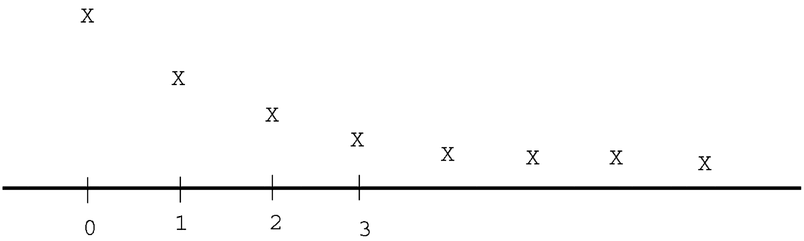

Figure 5.1: Geometric probability law.

Let us see the sum of the probabilities here. The total probability

$$\begin{aligned}\sum_x f_3(x) &= \sum_{x=1}^{\infty} q^{x-1}p = p\{1+q+q^2+\cdots\}\\ &= p(1-q)^{-1} \quad \text{see the binomial expansion in Section 3.3}\\ &= pp^{-1} = 1\end{aligned}$$

as can be expected. Let us compute the mean value $E(x)$, variance σ^2 and the moment generating function $M(t)$ for this geometric probability law:

$$\begin{aligned}E(x) &= \sum_x xf_3(x) = \sum_{x=1}^{\infty} xq^{x-1}p = p\{1+2q+3q^2+\cdots\}\\ &= p(1-q)^{-2}, \quad 0<q<1 \quad \text{see Section 3.3 for the binomial sum}\\ &= pp^{-2} = \frac{1}{p}, \quad 0<p<1. \end{aligned}\tag{5.7}$$

We can also derive this by using the following procedure:

$$\begin{aligned}E(x) &= \sum_{x=1}^{\infty} xq^{x-1}p = p\sum_{x=1}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}q}q^x\right]\\ &= p\frac{\mathrm{d}}{\mathrm{d}q}\sum_{x=1}^{\infty}q^x\\ &= p\frac{\mathrm{d}}{\mathrm{d}q}[q+q^1+\cdots] = p\frac{\mathrm{d}}{\mathrm{d}q}[q(1-q)^{-1}]\\ &= p[(1-q)^{-1}+q(1-q)^{-2}] = 1+\frac{q}{p}\\ &= \frac{q+p}{p} = \frac{1}{p}.\end{aligned}$$

For computing $E(x^2)$, we may observe the following:

$$x^2q^{x-1}p = p\frac{\mathrm{d}}{\mathrm{d}q}\left[q\frac{\mathrm{d}}{\mathrm{d}q}q^x\right].$$

Hence

$$
\begin{aligned}
E(x^2) &= \sum_{x=1}^{\infty} x^2 q^{x-1} p = p \sum_{x=1}^{\infty} \left\{ \frac{\mathrm{d}}{\mathrm{d}q} \left[q \frac{\mathrm{d}}{\mathrm{d}q} q^x \right] \right\} \\
&= \frac{\mathrm{d}}{\mathrm{d}q} q \frac{\mathrm{d}}{\mathrm{d}q} \sum_{x=1}^{\infty} q^x = \frac{\mathrm{d}}{\mathrm{d}q} q \frac{\mathrm{d}}{\mathrm{d}q} [q(1-q)^{-1}] \\
&= p \frac{\mathrm{d}}{\mathrm{d}q} [q(1-q)^{-1} + q^2(1-q)^{-2}] = p(1-q)^{-1} \\
&\quad + p[3q(1-q)^{-2} + 2q^2(1-q)^{-3}] = 1 + \frac{3q}{p} + 2\frac{q^2}{p^2}.
\end{aligned} \tag{5.8}
$$

We can also obtain this from the moment generating function:

$$
\begin{aligned}
M(t) &= \sum_{x=1}^{\infty} \mathrm{e}^{tx} p q^{x-1} = p\{\mathrm{e}^t + q\mathrm{e}^{2t} + q^2 \mathrm{e}^{3t} + \cdots\} \\
&= p\mathrm{e}^t (1 - q\mathrm{e}^t)^{-1} \quad \text{for } q\mathrm{e}^t < 1.
\end{aligned} \tag{5.9}
$$

Differentiating $M(t)$ with respect to t and then evaluating at $t = 0$, we have

$$
\begin{aligned}
E(x) &= \frac{\mathrm{d}}{\mathrm{d}t} M(t) \Big|_{t=0} = p \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^t (1 - q\mathrm{e}^t)^{-1} \Big|_{t=0} \\
&= \{p\mathrm{e}^t (1 - q\mathrm{e}^t)^{-1} + p\mathrm{e}^t (1 - q\mathrm{e}^t)^{-2} q\mathrm{e}^t\}\big|_{t=0} \\
&= pp^{-1} + pp^{-2} q = 1 + \frac{q}{p} = \frac{1}{p}. \\
E(x^2) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} M(t) \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \{p\mathrm{e}^t (1 - q\mathrm{e}^t)^{-1} + pq\mathrm{e}^{2t} (1 - q\mathrm{e}^t)^{-2}\}_{t=0} \\
&= \{p\mathrm{e}^t (1 - q\mathrm{e}^t)^{-1} + p\mathrm{e}^t q\mathrm{e}^t (1 - q\mathrm{e}^t)^{-2} + 2pq\mathrm{e}^{2t} (1 - q\mathrm{e}^t)^{-2}\}_{t=0} \\
&= 1 + \frac{q}{p} + 2\frac{q}{p} + 2\frac{q^2}{p^2} = 1 + 3\frac{q}{p} + 2\frac{q^2}{p^2}.
\end{aligned}
$$

5.5 Negative binomial probability law

Again, let us consider independent Bernoulli trials with the probability of success p remaining the same. Let us ask the question: what is the probability that the x-th trial will result in the k-th success for a fixed k, something like the 10-th trial resulting in the 7-th success? Let this probability be denoted by $f_4(x)$. The k-th success at the x-th trial means that there were $k-1$ successes in the first $x-1$ trials; the successes could have occurred any time in any sequence but a total of $k-1$ of them. This is given by a binomial probability law of $x-1$ trials and $k-1$ successes. The next trial should be a success, then one has the x-th trial resulting in the k-th success. Hence

$$
f_4(x) = \begin{cases} [\binom{x-1}{k-1} p^{k-1} q^{(x-1)-(k-1)}] p = \binom{x-1}{k-1} p^k q^{x-k}, & x = k, k+1, \ldots \\ 0, & \text{elsewhere.} \end{cases}
$$

Note that one has to have at least k trials to get k successes, and hence x varies from $x = k$ onward. Here, p and k are parameters. What is the total probability here?

$$\begin{aligned}\sum_x f_4(x) &= \sum_{x=k}^{\infty}\binom{x-1}{k-1}p^k q^{x-k}\\ &= \sum_{x=k}^{\infty}\binom{x-1}{x-k}p^k q^{x-k} \quad \text{since } \binom{n}{r} = \binom{n}{n-r}\\ &= p^k\left\{\binom{k-1}{0} + \binom{k}{1}q + \binom{k+1}{2}q^2 + \cdots\right\}\\ &= p^k\left\{1 + k\frac{q}{1!} + k(k+1)\frac{q^2}{2!} + \cdots\right\}\\ &= p^k(1-q)^{-k} = p^k p^{-k} = 1\end{aligned}$$

as can be expected. Since $f_4(x)$ is the general term in a binomial expansion with a negative exponent, this probability is known as the *negative binomial probability function*.

Naturally, when $k = 1$ we have the geometric probability law. Thus the geometric probability law is a particular case of the negative binomial probability law. Let us compute $E(x), E(x^2)$ and the moment generating function. The moment generating function:

$$\begin{aligned}M(t) &= p^k\sum_{x=k}^{\infty}\binom{x-1}{k-1}q^{x-k}e^{tx} = p^k\sum_{x=k}^{\infty}\binom{x-1}{x-k}q^{x-k}e^{tx}\\ &= p^k\left\{\binom{k-1}{0}e^{kt} + \binom{k}{1}e^{(k+1)t}q + \cdots\right\}\\ &= p^k e^{kt}\left\{1 + k\frac{qe^t}{1!} + k(k+1)\frac{(qe^t)^2}{2!}q + \cdots\right\}\\ &= p^k e^{kt}(1-qe^t)^{-k} \quad \text{for } qe^t < 1. \end{aligned} \tag{5.10}$$

This is a differentiable function. Hence

$$\begin{aligned}E(x) &= \frac{d}{dt}M(t)\Big|_{t=0} = p^k\{ke^{kt}(1-qe^t)^{-k} + ke^{kt}(1-qe^t)^{-k-1}qe^t\}|_{t=0}\\ &= kp^k\{p^{-k} + qp^{-k-1}\} = k\left\{1 + \frac{q}{p}\right\} = \frac{k}{p}. \end{aligned} \tag{5.11}$$

$$\begin{aligned}E(x^2) &= \frac{d^2}{dt^2}M(t)|_{t=0} = \frac{d}{dt}\left[\frac{d}{dt}M(t)\right]\Big|_{t=0}\\ &= kp^k\{ke^{kt}(1-qe^t)^{-k} + kqe^{(k+1)t}(1-qe^t)^{-(k+1)}\\ &\quad + q(k+1)e^{(k+1)t}(1-qe^t)^{-(k+1)} + (k+1)q^2e^{(k+1)t}(1-qe^t)^{-(k+1)}\}|_{t=0}\\ &= k^2 + \frac{q}{p}(2k^2+k) + \frac{q^2}{p^2}k(k+1). \end{aligned} \tag{5.12}$$

We have given four important probability functions, namely the Bernoulli probability law, the binomial probability law, the geometric probability law and the negative binomial probability law, connected with independent Bernoulli trials. All these are frequently used in statistical literature and probability theory.

Example 5.2. In a multiple choice examination, each question is supplied with 3 possible answers of which one is the correct answer to the question. A student, who does not know any of the correct answers, is doing the examination by selecting answers at random. What is the probability that (a) out of the 5 questions answered the student has (i) exactly 3 correct answers, (ii) at least 3 correct answers; (b) (i) the third question answered is the first correct answer; (ii) at least 3 questions out of the 10 questions to be answered are needed to get the first correct answer; (c) the 5th question answered is the 3rd correct answer; (ii) at least 4 questions, out of the 10 questions answered, are needed to get the 3rd correct answer.

Solution 5.2. Attempting to answer the questions by selecting answers at random can be taken as independent Bernoulli trials with probability of success $\frac{1}{3}$ because out of the 3 possible answers only one is the correct answer to the question. In our notation, $p = \frac{1}{3}, q = \frac{2}{3}$. For (a), it is a binomial situation with $n = 5$. In (i), we need $\Pr\{x = 3\}$. From the binomial probability law,

$$\Pr\{x = 3\} = \binom{5}{3} p^3 q^{5-3} = \binom{5}{2} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2$$
$$= \frac{(5)(4)}{2!} \frac{4}{3^5} = \frac{40}{3^5}.$$

When computing the number of combinations, always use the definition:

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

It will be foolish to use all factorials because it will involve unnecessary computations and often big factorials cannot even be handled by computers. In (a)(ii), we need

$$\Pr\{x = 3\} + \Pr\{x = 4\} + \Pr\{x = 5\} = L \quad \text{say}.$$

Then

$$L = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0.$$

But

$$\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \binom{5}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{(5)(4)}{2!}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{3^5}.$$
$$\binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 = \binom{5}{1}\frac{2}{3^5} = \frac{10}{3^5}.$$
$$\binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 = \frac{1}{3^5}.$$

The total is

$$\frac{51}{3^5} = \frac{17}{81}.$$

For (b), it is a geometric probability law. In (i), $x = 3$ and the answer is

$$q^{3-1}p = \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{27}.$$

For (ii) in (b), we need the sum:

$$\begin{aligned}\sum_{x=3}^{10} q^{x-1}p &= p[q^2 + q^3 + \cdots + q^9] = pq^2[1 + q + \cdots + q^7]\\ &= pq^2\frac{(1-q^8)}{1-q} = q^2(1-q^8) = \left(\frac{2}{3}\right)^2\left[1-\left(\frac{2}{3}\right)^8\right].\end{aligned}$$

For (c), it is a negative binomial situation with $k = 3$. In (i), $x = 5, k = 3, p = \frac{1}{3}, q = \frac{2}{3}$ and the answer is

$$\binom{x-1}{k-1}p^k q^{x-k} = \binom{4}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{24}{3^5}.$$

In (ii), we need the sum:

$$\begin{aligned}&\sum_{x=4}^{10}\binom{x-1}{k-1}p^k q^{x-k}\\ &= \left(\frac{1}{3}\right)^3\left\{\binom{3}{2}\left(\frac{2}{3}\right)^1 + \cdots + \binom{9}{2}\left(\frac{2}{3}\right)^7\right\}\\ &= \frac{1}{27}\left\{3\left(\frac{2}{3}\right) + 6\left(\frac{2}{3}\right)^2 + 10\left(\frac{2}{3}\right)^3\right.\\ &\quad\left. + 15\left(\frac{2}{3}\right)^4 + 21\left(\frac{2}{3}\right)^5 + 28\left(\frac{2}{3}\right)^6 + 36\left(\frac{2}{3}\right)^7\right\}.\end{aligned}$$

Example 5.3. An experiment on rabbits is designed by taking $N = 20$ identical rabbits. But rabbits start dying out before the experiment is completed. Let n be the effective final number. This sample size n has become a random quantity. n could be zero (all rabbits died out), n could be $1, 2, \ldots$ and could be $N = 20$ (no rabbit died). Let 0.1 be the probability of a rabbit dying and suppose that this probability is the same for all rabbits. Construct the probability law for n.

Solution 5.3. Here, n satisfies all the conditions for a binomial random variable with probability of success $p = 0.1$ and the number of trials $N = 20$. Hence the probability law for n is, denoted by P_n,

$$P_n = \binom{N}{n}p^n q^{N-n} = \binom{20}{n}(0.1)^n(0.9)^{20-n}, \quad n = 0, 1, \ldots, 20.$$

Example 5.4. A lunch counter in an office building caters to people working in the building. Apart from regular lunches, the counter operator makes an exotic lunch packet every day. If the exotic packet is not sold on that day, then it is a total loss; it cannot be stored or used again. From past experience, the operator found the daily demand for this exotic packet as follows. It costs Rs 5 to make and she can sell it for Rs 10.

$$(\text{Demand}, \text{probability}) = (0, 0.1), (1, 0.2), (2, 0.2), (3, 0.3), (4, 0.1), (5, 0.1).$$

There is no demand for more than 5. That is, the operator can sell, for example, the 3rd packet if the demand is for 3 or more. How many packets she should make so that her expected profit is a maximum?

Solution 5.4. If she makes 1 packet, the probability that it can be sold is that the demand on that day is for 1 or more packets. The probability for this is $0.1 + 0.2 + 0.2 + 0.3 + 0.1 + 0.1 = 0.9$. It costs Rs 5 and the expected revenue is Rs $10 \times 0.9 =$ Rs 9 and the expected profit is Rs 4.

[As an expected value of a random variable, this is the following: Let y be her gain or loss on a single packet. Then y takes the value +5 (profit) with probability 0.9 [if the demand on that day is for one or more] and y takes the value −5 (loss) with probability 0.1 [if the demand on that day is for less than one or zero]. Then the expected value of y, $E(y) = 5(0.9) - 5(0.1) = 4$. Thus she has an expected profit of Rs 4.]

If she makes 2 packets, then the cost is $2 \times 5 = 10$. She can sell the first packet with probability 0.9 or make the expected revenue of Rs 9. She can sell the second packet if there is demand for 2 or more or with the probability $0.2 + 0.3 + 0.1 + 0.1 = 0.7$ and make the expected revenue $10 \times 0.7 = 7$. Thus the total expected revenue is $9 + 7 = 16$ and the expected profit is $16 - 10 = 6$.

If she makes 3 packets, then she can sell the third packet with probability $0.3 + 0.1 + 0.1 = 0.5$ and the expected revenue is $10 \times 0.5 = 5$. Thus the expected profit is $9 + 7 + 5 = 21 - 15(= 5 \times 3) = 6$.

If she makes 4 packets, then she can sell the 4th packet with probability $0.1 + 0.1 = 0.2$ and the expected revenue is $10 \times 0.2 = 2$ and the expected profit is $9 + 7 + 5 + 2 = 23 - 4 \times 5 = 3$.

If she makes 5 packets, then she can sell the 5th one with probability 0.1 and the expected revenue is Rs 1, the total cost is $5 \times 5 = 25$ and, therefore, there is an expected loss of Rs 1. Hence she should make either 2 or 3 packets to maximize her profit.

5.6 Poisson probability law

We will derive this probability law as a limiting form of the binomial as well as a process satisfying some conditions. This law is named after its inventor, S. Poisson, a French mathematician. Consider a binomial probability law where the number of trials n is very large and the probability of success p is very small but $np = \lambda$ (Greek letter lambda), a finite quantity. This situation can be called a situation of rare events, the number of trials is very large and the probability of success in each trial is very small, something like a lightning strike, earthquake at a particular place, traffic accidents on a certain stretch of a highway and so on. Let us see what happens if $n \to \infty$, $p \to 0$, $np = \lambda$. Since we are assuming λ to be a finite quantity, we may replace one of n or p in terms of λ. Let us substitute $p = \frac{\lambda}{n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\binom{n}{x}p^x &= \frac{1}{x!}\lim_{n\to\infty} n(n-1)\cdots(n-x+1)\left(\frac{\lambda}{n}\right)^x \\ &= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n}{n}\frac{(n-1)}{n}\cdots\frac{(n-x+1)}{n} \\ &= \frac{\lambda^x}{x!}1\times\lim_{n\to\infty}\left[1-\frac{1}{n}\right]\times\cdots\times\lim_{n\to\infty}\left[1-\frac{x-1}{n}\right] \\ &= \frac{\lambda^x}{x!}\end{aligned}$$

since x is finite, there are only a finite number of factors, and we can use the formula that the limit of a finite number of products is the product of the limits and each factor here goes to 1. [If it involved an infinite number of factors, then we could not have taken the limits on each factor.] Now let us examine the factor:

$$q^{n-x} = (1-p)^n(1-p)^{-x} = \left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}.$$

But

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \mathrm{e}^{-\lambda}$$

from the definition of e, and

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

since the exponent $-x$ is finite. Therefore,

$$\begin{aligned}\lim_{n\to\infty,p\to 0,np=\lambda} f_2(x) &= \lim_{n\to\infty,p\to 0,np=\lambda}\binom{n}{x}p^x q^{n-x} \\ &= \begin{cases}\frac{\lambda^x}{x!}\mathrm{e}^{-\lambda}, & x = 0,1,\ldots,\ \lambda > 0\\ 0, & \text{elsewhere.}\end{cases}\end{aligned}$$

Let us call the right side as $f_5(x)$. Let us see whether $f_5(x)$ is a probability function, since we have done a limiting process on a binomial probability function. If you take some sort of limits on a probability function, the limiting form need not remain as a probability function. The total of $f_5(x)$ is given by

$$\sum_x f_5(x) = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}\mathrm{e}^{-\lambda} = \mathrm{e}^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = \mathrm{e}^{-\lambda}\mathrm{e}^{\lambda} = 1.$$

Hence $f_5(x)$ is a probability function and it is known as the Poisson probability law. λ is a parameter here.

Let us evaluate $E(x)$, variance and the moment generating function for the Poisson probability law:

$$E(x) = \sum_{x=0}^{\infty} x\frac{\lambda^x}{x!}e^{-\lambda} = e^{-\lambda}\sum_{x=1}^{\infty} x\frac{\lambda^x}{x!}$$

since at $x = 0$ the right side is zero. Now we take out one lambda, cancel one x, $\frac{x}{x!} = \frac{1}{(x-1)!}$ when $x \neq 0$. Then we have

$$\begin{aligned} E(x) &= \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\ &= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] = \lambda e^{-\lambda}e^{\lambda} = \lambda. \end{aligned} \tag{5.13}$$

Thus, the mean value in the Poisson case is the parameter λ sitting in the probability function. Since $x!$ is sitting in the denominator, for computing $E(x^2)$, we will go through the factorial moments or consider the identity

$$x^2 = x(x-1) + x$$

and proceed to evaluate $E[x(x-1)]$. This procedure has already been done several times before. We cancel $x(x-1)$ from $x!$ since at $x = 0, 1$ the function on the right will be zeros, and thus x only goes from $x = 2$ to infinity in the sum. Then we take out λ^2. That is,

$$\begin{aligned} E[x(x-1)] &= \sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x}{x!}e^{-\lambda} \\ &= \lambda^2 e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} = \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] \\ &= \lambda^2 e^{-\lambda}e^{\lambda} = \lambda^2. \end{aligned}$$

Then the variance

$$\begin{aligned} \sigma^2 &= E[x - E(x)]^2 = E[x(x-1)] + E[x] - [E(x)]^2 \\ &= \lambda^2 + \lambda - [\lambda]^2 = \lambda. \end{aligned} \tag{5.14}$$

Thus it is interesting to see that the mean value and the variance are equal to λ in the Poisson case. But this is not a unique property or a characterizing property of the Poisson distribution. There are also other distributions satisfying this property that the mean value is equal to the variance.

Let us compute the moment generating function in this case

$$M(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!}e^{tx}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t}$$
$$= e^{\lambda[e^t-1]}. \tag{5.15}$$

The last sum is obtained by observing that it is an exponential series with λe^t in the exponent.

Example 5.5. The monthly traffic accidents on a stretch of a particular highway is seen to be Poisson distributed with expected number of accidents 5. Four months are selected at random. What is the probability (i) in all four months the number of accidents is two or more per month; (ii) in at least one of the months the number of accidents is exactly 3; (iii) the first three months had no accidents and the fourth month had two accidents.

Solution 5.5. Let x be the number of monthly accidents on that stretch of the highway. Then the probability function is given as, denoted by $P(x)$,

$$P(x) = \begin{cases} \frac{5^x}{x!}e^{-5}, & x = 0, 1, \dots \\ 0, & \text{elsewhere.} \end{cases}$$

The parameter $\lambda = 5$ because it is given that the mean value is 5. The probability that the number of accidents in a month is 2 or more, denoted by p_1, is given by

$$p_1 = \sum_{x=2}^{\infty}\frac{5^x}{x!}e^{-5} = 1 - \sum_{x=0}^{1}\frac{5^x}{x!}e^{-5}$$

since the total probability is 1. That is,

$$p_1 = 1 - \frac{5^0}{0!}e^{-5} - \frac{5^1}{1!}e^{-5} = 1 - e^{-5}[1+5] = 1 - 6e^{-5}.$$

The answer to (i) is then p_1^4.

The probability for the number of accidents in a month is exactly 3, denoted by p_2, is given by

$$p_2 = \frac{5^3}{3!}e^{-5} = \frac{125}{6}e^{-5}.$$

In (ii), it is a binomial situation with the number of trials $n = 4$ and probability of success is p_2. Hence the answer to (ii) is

$$\sum_{x=1}^{4}\binom{4}{x}p_2^x(1-p_2)^{4-x} = 1 - \binom{4}{0}p_2^0(1-p_2)^{4-0} = 1 - \left(1 - \frac{125}{6}e^{-5}\right)^4.$$

Probability of having no accidents is

$$P(0) = \frac{5^0}{0!}e^{-5} = e^{-5}.$$

Probability of having exactly 2 accidents is $\frac{5^2}{2!}e^{-5} = \frac{25}{2}e^{-5}$. Hence the answer to (iii) is

$$[e^{-5}]^3\left[\frac{25}{2}e^{-5}\right] = \frac{25}{2}e^{-20}.$$

5.6.1 Poisson probability law from a process

Consider an event taking place over time, such as the arrival of persons into a queue at a checkout counter, arrival of cars into a service station for service, arrival of telephone calls into a phone switchboard, floods in a river during rainy season, earthquakes over the years at a particular place, eruption of a certain volcano over time and so on. Let us assume that our event satisfies the following conditions:

(i) The occurrence of the event from time t to $t+\Delta t$, that is in the interval $[t, t+\Delta t]$, where Δt is a small increment in t, is proportional to the length of the interval or it is $\alpha\Delta t$, where α is a constant. Here, Δ (Greek capital letter delta) is not used as a product. Δt is a notation standing for a small increment in t.

(ii) The probability of more than one occurrence of this event in $[t, t+\Delta t]$ is negligibly small and we take it as zero for all practical purposes, or it is assumed that the interval can be subdivided to the extent that probability of more than one occurrence in this small interval is negligible.

(iii) The occurrence or non-occurrence of this event in $[t, t+\Delta t]$ does not depend upon what happened in the interval $[0, t]$ where 0 indicates the start of the observation period. An illustration is given in Figure 5.2. If the event under observation is a flood in a river during the rainy season, then start of the rainy season is taken as zero and time may be counted in days or hours or in any convenient unit.

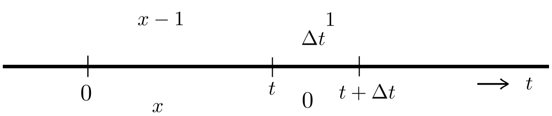

Figure 5.2: An event taking place over time.

Under the conditions (i), (ii), (iii) above, what is the probability of getting exactly x occurrences of this event in time $[0, t]$? This probability function depends upon x and the time t and let us denote it by $f(x, t)$. Then the interpretations are the following:

$f(x, t+\Delta t)$ = the probability of getting exactly x occurrences of the event in time $[0, t+\Delta t]$;

$f(x-1, t)$ = the probability of getting exactly $x-1$ occurrences in time $[0, t]$.

Exactly x occurrences in the interval 0 to $t+\Delta t$ can happen in two mutually exclusive ways of (a) exactly $x-1$ occurrences in time 0 to t or in the interval $[0, t]$, and one occurrence from t to $t+\Delta t$ or in the interval $[t, t+\Delta t]$ [probability for one occurrence is $\alpha\Delta t$], or (b) exactly x occurrences in the interval $[0, t]$ and no occurrence in the interval $[t, t+\Delta t]$ [probability for no occurrence is $1-\alpha\Delta t$]. Therefore, from the total probability law,

$$f(x, t+\Delta t) = f(x-1, t)[\alpha\Delta t] + f(x, t)[1-\alpha\Delta t].$$

We can rearrange the terms and write

$$\frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} = \alpha[f(x-1, t) - f(x, t)].$$

Taking the limit when $\Delta t \to 0$, we get a differential equation in t or a partial differential equation in t. That is,

$$\frac{\partial}{\partial t} f(x, t) = \alpha[f(x-1, t) - f(x, t)]. \tag{5.16}$$

Here, (5.16) is a differential equation in t whereas it is a difference equation in x. We have to solve this difference-differential equation to obtain $f(x, t)$. This can be solved successively by taking values for $x = 0$ solving the equation for t, then $x = 1$ solving the equation for t, and so on. The final result will be the following:

$$f(x, t) = \begin{cases} \frac{(\alpha t)^x}{x!} e^{-\alpha t}, & \alpha > 0,\ 0 \le t < \infty,\ x = 0, 1, \dots \\ 0, & \text{elsewhere,} \end{cases}$$

or it is a Poisson probability law with the parameter $\lambda = \alpha t$.

Example 5.6. Telephone calls are coming to an office switchboard at the rate of 0.5 calls per minute, time being measured in minutes. What is the probability that (a) in a 10-minute interval (i) there are exactly 2 calls; (ii) there is no call; (iii) at least one call; (b) if two 10-minute intervals are taken at random then (i) in both intervals there are no calls; (ii) in one of the intervals there are exactly 2 calls?

Solution 5.6. We will assume that these telephone calls obey the conditions for Poisson arrivals of calls or the Poisson probability law is a good model. We are given $\alpha = 0.5$. In (a) $t = 10$, then $\lambda = 10 \times 0.5 = 5$ and the probability law $P(x)$ is

$$P(x) = \begin{cases} \frac{5^x}{x!} e^{-5}, & x = 0, 1, \dots \\ 0, & \text{elsewhere.} \end{cases}$$

In (a)(i), we need the probability $\Pr\{x = 2\}$.

$$\Pr\{x = 2\} = \frac{5^2}{2!} e^{-5} = \frac{25}{2} e^{-5}.$$

In (a)(ii), we need the probability $\Pr\{x = 0\}$.

$$\Pr\{x = 0\} = \frac{5^0}{0!}e^{-5} = e^{-5}.$$

In (a)(iii), we need $\Pr\{x \geq 1\}$.

$$\Pr\{x \geq 1\} = 1 - \Pr\{x = 0\} = 1 - e^{-5}.$$

In (b)(i), it is a case of two Bernoulli trials where the probability of success p_1 is the probability of having no arrivals in a 10-minute interval or $p_1 = e^{-5}$. We want both trials to result in successes, and hence the answer is

$$p_1^2 = [e^{-5}]^2 = e^{-10}.$$

In (b)(ii), we have two Bernoulli trials and the probability of success in each trial is p_2 where p_2 is the probability of having exactly 2 calls in a 10-minute interval. Then

$$p_2 = \frac{5^2}{2!}e^{-5} = \frac{25}{2}e^{-5}.$$

In (b)(ii), we need the probability of getting exactly one success in two Bernoulli trials. Then it is given by

$$\binom{2}{1}p_2^1(1 - p_2)^{2-1} = 2\left[\frac{25}{2}e^{-5}\right]\left[1 - \frac{25}{2}e^{-5}\right].$$

Another probability law which is frequently used in probability and statistics is the discrete hypergeometric law.

5.7 Discrete hypergeometric probability law

Let us consider a box containing two types of objects: one type is of a in number, which we will call these a-type objects, and the other type is b in number, which we will call these b-type objects. As an example, we can consider a box containing red and green marbles and suppose that there are 10 green and 8 red marbles then we may consider $a = 10$ and $b = 8$ or vice versa. Suppose that a subset of n items is taken at random from this set of $a + b$ objects. When we say "at random" it means that every subset of n has the same chance of being taken or each subset gets a probability of $\frac{1}{\binom{a+b}{n}}$ because there are $\binom{a+b}{n}$ such subsets possible. This can also be done by taking one by one, at random, without replacement. Both will lead to the same probabilities.

In this experiment, what is the probability that the sample of n items contains x of a-type and $n - x$ of b-type objects. Let this probability be denoted by $f_6(x)$. Note that x of a-type can only come from a-type objects and this can be done in $\binom{a}{x}$ ways. For each such selection of a-type objects, we can select $n-x$ b-type objects in $\binom{b}{n-x}$ ways. Therefore, the number of choices favorable to the event is $\binom{a}{x}\binom{b}{n-x}$. Hence

$$f_6(x) = \begin{cases} \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}, & x = 0,1,\dots,n \text{ or } a;\ a,b = 1,2,\dots,\ n = 1,2,\dots \\ 0, & \text{elsewhere.} \end{cases}$$

This is known as the *discrete hypergeometric probability law*. a, b, n are parameters here.

First, let us check to see the sum:

$$\sum_x f_6(x) = \frac{\sum_{x=0}^{n,a}\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}} = 1$$

because, from Section 3.3 we have

$$\sum_{x=0}^{n,a}\binom{a}{x}\binom{b}{n-x} = \binom{a+b}{n}. \tag{5.17}$$

Thus the total probability is 1 as can be expected. What are the mean values and variance in this case?

$$E(x) = \frac{1}{\binom{a+b}{n}}\sum_{x=0}^{n,a} x\binom{a}{x}\binom{b}{n-x}.$$

When $x = 0$, the right side is zero, and hence the sum starts only at $x = 1$. Then one may cancel one x from the $x!$. That is,

$$x\binom{a}{x} = x\frac{a!}{x!(a-x)!} = a\frac{(a-1)!}{(x-1)!((a-1)-(x-1))!} = a\binom{a-1}{x-1}.$$

Hence, taking the sum by putting $y = x - 1$ so that y goes from 0, and by using (5.17), we have

$$a\sum_{y=0}^{n,a}\binom{a-1}{y}\binom{b}{n-1-y} = a\binom{a+b-1}{n-1}.$$

Now, dividing by $\binom{a+b}{n}$ and simplifying we get

$$E(x) = \frac{na}{a+b}. \tag{5.18}$$

By using the same steps, the second factorial moment is given by

$$E[x(x-1)] = \frac{n(n-1)a(a-1)}{(a+b)(a+b-1)}. \tag{5.19}$$

Now, variance is available from the formula

$$\begin{aligned} \mathrm{Var}(x) &= E[x(x-1)] + E(x) - [E(x)]^2 \\ &= \frac{n(n-1)a(a-1)}{(a+b)(a+b-1)} + \frac{na}{(a+b)} - \frac{n^2a^2}{(a+b)^2}. \end{aligned} \tag{5.20}$$

Example 5.7. From a set of 5 women and 8 men a committee of 3 is selected at random [this means all such subsets of 3 are given equal chances of being selected]. What is the probability that the committee consists of (i) no woman; (ii) at least two women; (iii) all women?

Solution 5.7. Let x be the number of women in the committee. Then x is distributed according to a discrete hypergeometric probability law. In (i), we need $\Pr\{x = 0\}$:

$$\Pr\{x=0\} = \frac{\binom{5}{0}\binom{8}{3}}{\binom{13}{3}} = \frac{(8)(7)(6)}{(13)(12)(11)} = \frac{28}{143}.$$

In (ii), we need $\Pr\{x = 2 \text{ or } 3\} = \Pr\{x = 2\} + \Pr\{x = 3\}$. In (iii), we need $\Pr\{x = 3\}$. Let us compute these two probabilities:

$$\Pr\{x=3\} = \frac{\binom{5}{3}\binom{8}{0}}{\binom{13}{3}} = \frac{(5)(4)(3)}{(13)(12)(11)} = \frac{5}{143}.$$

$$\Pr\{x=2\} = \frac{\binom{5}{2}\binom{8}{1}}{\binom{13}{3}} = \frac{(5)(4)}{(1)(2)}(8)\frac{(1)(2)(3)}{(13)(12)(11)} = \frac{40}{143}.$$

Hence the answer in (ii) is $\frac{40}{143} + \frac{5}{143} = \frac{45}{143}$.

5.8 Other commonly used discrete distributions

Here, we list some other commonly used discrete distributions. Only the non-zero part of the probability function is given and it should be understood that the function is zero otherwise. In some of the probability functions, gamma functions, $\Gamma(\cdot)$ (Γ is the Greek capital letter gamma) and beta functions, $B(\cdot,\cdot)$ (B is the Greek capital letter beta) appear. Hence we will list the integral representations of these functions here. Their definitions will be given in the next chapter. Only these integral representations will be sufficient to do problems on the following probability functions where gamma and beta functions appear.

The integral representation for a gamma function, $\Gamma(\alpha)$, is the following:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x, \quad \Re(\alpha) > 0. \tag{5.21}$$

For the integral to converge, we need the condition that the real part of α (alpha) is positive, $\Re(\cdot)$ means the real part of $(\cdot)$.

Note 5.1. Usually in statistical problems the parameters are real but the integrals will exist in the complex domain also, and hence the conditions are written in terms of real parts of the complex parameters.

The beta function, $B(\alpha,\beta)$, can be written in terms of the gamma function. In the following, we give the connection to gamma function and integral representations for beta functions:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, \tag{5.22}$$

$$= \int_0^\infty y^{\alpha-1}(1+y)^{-(\alpha+\beta)}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0. \tag{5.23}$$

For the convergence of the integrals in (5.22) and (5.23), we need the conditions $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, (β is the Greek small letter beta). It may be noted that

$$B(\alpha,\beta) = B(\beta,\alpha). \tag{5.24}$$

That is, the parameters α and β can be interchanged in the integrals:

$$f_7(x) = \binom{n}{x}\frac{\Gamma(\alpha+\beta)\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}$$

for $x = 0,1,\dots,n$; $\alpha > 0$, $\beta > 0$ [Beta-binomial probability function].

$$f_8(x) = \frac{\Gamma(r+s)\Gamma(x+n-r-s)\Gamma(x)\Gamma(n)}{\Gamma(r)\Gamma(s)\Gamma(x-r+1)\Gamma(n-s)\Gamma(x+n)}$$

for $x = r, r+1, \dots$, $s > 0$, $n > s$; r a positive integer [Beta-Pascal probability function].

$$f_9(x) = \sum_{i=1}^{m} w_i \binom{n}{x} p_i^x (1-p_i)^{n-x}$$

for $x = 0,1,\dots,n$; $0 < p_i < 1$, $w_i \geq 0$, $i = 1,\dots,m$; $\sum_{i=1}^m w_i = 1$ [Mixed binomial probability function].

$$f_{10}(x) = \frac{\binom{n}{x}p^x(1-p)^{n-x}}{1-(1-p)^n},$$

for $x = 1,2,\dots,n$; $0 < p < 1$; (truncated below $x = 1$) [Truncated binomial probability function].

$$f_{11}(x) = \frac{(x\beta)^{x-1}}{x!}\mathrm{e}^{-x\beta}, \quad x = 1,2,\dots;\ \beta > 0$$

[Borel probability law].

$$f_{12}(x) = \frac{r}{(x-r)!}x^{x-r-1}\alpha^{x-r}\mathrm{e}^{-\alpha x},$$

for $x = r, r+1, \dots$; $\alpha > 0$, where r is a positive integer [Borel–Tanner probability law].

$$f_{13}(x) = \frac{\Gamma(\nu+x)\Gamma(\lambda)}{\Gamma(\lambda+x)\Gamma(\nu)}\frac{\mu^x}{{}_1F_1(\nu;\lambda;\mu)}$$

for $x = 0,1,2,\dots$; $\nu > 0$, $\lambda > 0$, $\mu > 0$ where ${}_1F_1$ is a confluent hypergeometric function (ν is Greek letter nu; μ is Greek letter mu) [Confluent hypergeometric probability law].

$$f_{14}(x) = w_1 \binom{N}{x} p_1^x (1-p_1)^{N-x} + w_2 \phi(x)$$

for $x = 0, 1, \dots, N$; $w_2 = 1 - w_1$, $0 < w_1 < 1$, $0 < p_1 < 1$ and $\phi(x)$ are some probability functions [Dodge–Romig probability law].

$$f_{15}(x) = \binom{N}{x} p^x \left[1 + \binom{N}{1} p + \dots + \binom{N}{c} p^c\right]^{-1}$$

for $x = 0, 1, \dots, c$; $0 < p < 1$; N, c positive integers [Engset probability law].

$$f_{16}(x) = \frac{a(x)\theta^x}{[\sum_{x \in A} a(x)\theta^x]}, \quad \theta > 0$$

for $a(x) > 0$, $x \in A$ = subset of reals [Generalized power series probability function].

$$f_{17}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x-1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+x)}$$

for $x = 1, 2, \dots$; $\alpha > 0$, $\beta > 0$ [Compound geometric probability law].

$$f_{18}(x) = \mathrm{e}^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \binom{2m}{x} p^x (1-p)^{2m-x}$$

for $x = 0, 1, 2, \dots, 2m$; $\lambda > 0$, $0 < p < 1$ [Hermite probability law].

$$f_{19}(x) = \frac{\Gamma(\lambda)\theta^x}{{}_1F_1(1; \lambda; \theta)\Gamma(\lambda + x)}$$

for $x = 0, 1, 2, \dots$; $\theta > 0$, $\lambda > 0$ [Hyper-Poisson probability function].

$$f_{20}(x) = \frac{\theta^x}{\beta x}, \quad 0 < \theta < 1,$$

for $x = 1, 2, \dots, d$ where $\beta = \sum_{x=1}^{d} \frac{\theta^x}{x}$ [Truncated logarithmic series probability function].

$$f_{21}(x) = \sum_{i=1}^{k} w_i f_i(x), \quad 0 < w_i < 1, \quad \sum_{i=1}^{k} w_i = 1$$

where $f_i(x)$ is a general probability or density function for each $i = 1, \dots, k$ [General mixed probability function].

$$f_{22}(x) = \frac{\Gamma(r+x)}{x!\Gamma(r)} p^r (1-p)^x$$

for $x = 0, 1, \dots$; $0 < p < 1$, $r > 0$ [Negative binomial probability function, model-2].

$$f_{23}(x) = \frac{\alpha^a (\alpha+1)^{p-a} \Gamma(p+x)}{\Gamma(p) x! (\alpha+2)^{p+x}} {}_2F_1\left(a, p+x; p; \frac{1}{\alpha+2}\right)$$

for $x = 0, 1, \dots$; $a > 0$, $p > 0$, $\alpha > 0$ and ${}_2F_1$ is a Gauss' hypergeometric function [Generalized negative binomial probability function].

$$f_{24}(x) = \frac{c^x}{x!} \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{k^x (\lambda \mathrm{e}^{-c})^k}{k!}$$

for $x = 0, 1, \ldots; \lambda, c$ positive constants [Neyman type A probability function].

$$f_{25}(x) = \binom{x + r - 1}{x} p^r (1 - p)^x$$

for $x = 0, 1, \ldots; 0 < p < 1; r$ a positive integer [Pascal probability law].

$$f_{26}(x) = \mathrm{e}^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{nm}{x} p^x (1 - p)^{nm-x}$$

for $x = 0, 1, \ldots, nm; a > 0, 0 < p < 1; n, m$ positive integers [Poisson-binomial probability function].

$$f_{27}(x) = \frac{\mu^x}{x![\exp(\mu) - 1]}$$

for $x = 1, 2, \ldots; \mu > 0$ (truncated below $x = 1$) [Truncated Poisson probability function].

$$f_{28}(x) = \binom{N}{x} \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + x)\Gamma(\beta + N - x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + N)}$$

for $x = 0, 1, \ldots, N; \alpha > 0, \beta > 0$ [Polya probability law or Beta-binomial probability function].

$$f_{29}(x) = \frac{1}{x!} \frac{\Gamma(x + \frac{h}{d})}{\Gamma(\frac{h}{d})} \left(1 + \frac{1}{d}\right)^{-(h/d)} \left(\frac{1}{1 + d}\right)^x$$

for $x = 0, 1, \ldots; d > 0, h > 0$ [Polya–Eggenberger probability function].

$$f_{30}(x) = \frac{1}{n}, \quad x = x_1, \ldots, x_n$$

[Discrete uniform probability function].

$$f_{31}(x) = \frac{\exp[-(\lambda + \phi)]}{x!} \sum_{k=0}^{\infty} (k\theta + \phi)^x \frac{[\lambda \exp(-\theta)]^k}{k!}$$

for $x = 0, 1, \ldots; \phi > 0, \theta > 0, \lambda > 0$ [Short's probability law].

$$f_{32}(x) = \frac{1}{[1 - P(0)]} \sum_{t=x+1}^{\infty} \frac{P(t)}{t}$$

for $x = 0, 1, \ldots$; where $P(t)$ is any discrete probability function over the range $t = 0, 1, 2, \ldots$ [Ster's probability function].

$$f_{33}(x) = [\zeta(k) x^k]^{-1}$$

for $x = 1, 2, \ldots$ where $\zeta(k) = \sum_{t=1}^{\infty} t^{-k}$, $k > 1$ (ζ is the Greek letter zeta) [Zeta probability function].

Exercises 5

5.1. Compute $E(x^2)$ for the geometric probability law by summing up or by using the definition, that is, by evaluating

$$E(x^2) = \sum_{x=1}^{\infty} x^2 q^{x-1} p.$$

5.2. Compute (i) $E(x)$; (ii) $E(x^2)$; for the negative binomial probability law by using the definition (by summing up).

5.3. Compute (i) $E(x)$; (ii) $E(x^2)$; by using the technique used in the geometric probability law by differentiating the negative binomial probability law.

5.4. Compute $E(x)$ and $E(x^2)$ by differentiating the moment generating function in the Poisson probability case.

5.5. Compute $E(x)$ and variance of x by using the moment generating function in the binomial probability law.

5.6. Construct two examples of discrete probability functions where $E(x) = \mathrm{Var}(x)$.

5.7. Solve the difference-differential equation in (5.16) and show that the solution is the probability function given therein.

5.8. Show that the functions $f_7(x)$ to $f_{33}(x)$ in Section 5.8 are all probability functions, that is, the functions are non-negative and the sum in each case is 1.

5.9. For the probability functions in Exercise 5.8, evaluate the first two moments about the origin, that is, $E(x)$ and $E(x^2)$, whenever they exist.

Truncation. In some practical problems, the general behavior of the discrete random variable x may be according to a probability function $f(x)$ but certain values may not be admissible. In that case, we remove the total probability masses on the non-admissible points, then re-weigh the remaining points to create a new probability function. For example, in a binomial case suppose that the event of getting zero success is not admissible. In this case, we remove the point $x = 0$. At $x = 0$, the probability is $\binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$. Therefore, the remaining mass is $c_0 = 1 - (1-p)^n$. Hence if we divide the remaining probabilities by c_0 then the remaining points can produce a truncated binomial probability law, which is

$$g(x) = \begin{cases} \frac{1}{c_0}\binom{n}{x} p^x (1-p)^{n-x}, & x = 1, 2, \dots, n,\ 0 < p < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Here, $g(x)$ is called the truncated binomial probability function, truncated below $x = 1$ or at $x = 0$. Thus truncation is achieved by multiplying the probability function by an appropriate constant c. In the above case, it is $c = \frac{1}{c_0}$.

5.10. Compute the truncation constant c so that $cf(x)$ is a truncated probability function of $f(x)$ in the following cases:

(i) Binomial probability function, truncated below $x = 1$ (Here, $c = \frac{1}{c_0}$ where c_0 is given above);
(ii) Binomial probability, truncated at $x = n$;
(iii) Poisson probability function, truncated below $x = 1$;
(iv) Poisson probability function, truncated below $x = 2$;
(v) Geometric probability function, truncated below $x = 2$;
(vi) Geometric probability function, truncated above $x = 10$.

Probability Generating Function. Consider a discrete random variable taking non-zero probabilities at the points $x = 0, 1, \dots$ and let $f(x)$ be the probability function. Consider the expected value of t^x for some parameter t. Let us denote it by $P(t)$. Then we have

$$P(t) = E(t^x) = \sum_{x=0}^{\infty} t^x f(x) \tag{5.25}$$

where, for example, the probability that x takes the value 5 is $\Pr\{x = 5\} = f(5)$ or it is the coefficient of t^5 on the right side of (5.25). Thus the various probabilities, such as $\Pr\{x = 0\}, \Pr\{x = 1\}, \dots$ are generated by $P(t)$ or they are available from the right side series in (5.25), provided the right side series is convergent. In the case when $x = 0, 1, 2, \dots$ with non-zero probabilities then $P(t)$ in (5.25) is called the generating function for the probability function $f(x)$ of this random variable x. We can also notice further properties of this generating function. Suppose that the series on the right side in (5.25) or $P(t)$ is differentiable, then differentiate with respect to t and evaluate at $t = 1$, then we get $E(x)$. For example,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P(t)\Big|_{t=1} &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{x=0}^{\infty} t^x f(x)\Big|_{t=1} \\ &= \sum_{x=0}^{\infty} x t^{x-1} f(x)\Big|_{t=1} = \sum_{x=0}^{\infty} x f(x) = E(x). \end{aligned}$$

Successive derivatives evaluated at $t = 1$ will produce $E(x), E[x(x-1)], E[x(x-1)(x-2)]$ and so on, when $P(t)$ series is uniformly convergent and differentiable term by term.

5.11. Compute the (a) the probability generating function $P(t)$, (b) $E(x)$ by using $P(t)$, (c) $E(x^2)$ by using $P(t)$ for the following cases: (i) Geometric probability law; (ii) Negative binomial probability law.

5.12. A gambler is betting on a dice game. Two dice will be rolled once. The gambler puts in Rs 5 (His bet is Rs 5). If the same numbers turn up on the two dice, then the gambler wins double his bet, that is, Rs 10, otherwise he loses his bet (Rs 5). Assuming that the dice are balanced

(i) What is the gambling house's expected return per game from this gambler?
(ii) What is the probability of the gambler winning exactly five out of 10 such games?
(iii) What is the gambler's expected return in 10 such games?

5.13. Cars are arriving at a service station at the rate of 0.1 per minute, time being measured in minutes. Assuming a Poisson arrival of cars to this service station, what is the probability that

(a) in a randomly selected twenty minute interval there are
  (i) exactly 3 arrivals;
  (ii) at least 2 arrivals;
  (iii) no arrivals;
(b) if 5 such 20-minute intervals are selected at random then what is the probability that in at least one of these intervals
  (i) (a)(i) happens;
  (ii) (a)(ii) happens;
  (iii) (a)(iii) happens.

5.14. The number of floods in a local river during rainy season is known to follow a Poisson distribution with the expected number of floods 3. What is the probability that

(a) during one rainy season
  (i) there are exactly 5 floods;
  (ii) there is no flood;
  (iii) at least one flood;
(b) if 3 rainy seasons are selected at random, then none of the seasons has
  (i) (a)(i) happening;
  (ii) (a)(ii) happening;
  (iii) (a)(iii) happening;
(c) (i) (a)(i) happens for the first time at the 3rd season;
  (ii) (a)(iii) happens for the second time at the 3rd season.

5.15. From a well-shuffled deck of 52 playing cards (13 spades, 13 clubs, 13 hearts, 13 diamonds) a hand of 8 cards is selected at random. What is the probability that the hand contains (i) 5 spades? (ii) no spades? (iii) 5 spades and 3 hearts? (iv) 3 spades 2 clubs, 2 hearts, 1 diamond?