1 Vectors

1.0 Introduction

We start with vectors as ordered sets in order to introduce various aspects of these objects called vectors and the different properties enjoyed by them. After having discussed the basic ideas, a formal definition, as objects satisfying some general conditions, will be introduced later on. Several examples from various disciplines will be introduced to indicate the relevance of the concepts in various areas of study. As the students may be familiar, a collection of well-defined objects is called a *set*. For example $\{2, \alpha, B\}$ is a set of 3 objects, the objects being a number 2, a Greek letter α and the capital letter B. Sets are usually denoted by curly brackets {list of objects}. Each object in the set is called an *element* of the set. Let the above set be denoted by S, then $S = \{2, \alpha, B\}$. Then 2 is an element of S. It is usually written as $2 \in S$ (2 in S or 2 is an element of S). Thus we have

$$S = \{2, \alpha, B\}, \quad 2 \in S,\ \alpha \in S,\ B \in S,\ 7 \notin S,\ -\gamma \notin S \tag{1.0.1}$$

where $\notin$ indicates "not in". That is, 7 is not in S and $-\gamma$ (gamma) is not an element of S.

For a set, the order in which the elements are written is unimportant. We could have represented S equivalently as follows:

$$\begin{aligned} S &= \{2, \alpha, B\} = \{2, B, \alpha\} = \{\alpha, 2, B\} \\ &= \{\alpha, B, 2\} = \{B, 2, \alpha\} = \{B, \alpha, 2\} \end{aligned} \tag{1.0.2}$$

because all of these sets contain the same objects and hence they represent the same set. Now, we consider ordered sets. In (1.0.2) there are 6 ordered arrangements of the 3 elements. Each permutation (rearrangement) of the objects gives a different ordered set. With a set of n distinct objects we can have a total of $n! = (1)(2)\dots(n)$ ordered sets.

1.1 Vectors as ordered sets

For the time being we will define a vector as an ordered set of objects. More rigorous definitions will be given later on in our discussions. Vectors or these ordered sets will be denoted by ordinary brackets (*ordered list of elements*) or by square brackets [*ordered list of elements*]. For example, if the ordered sequences are taken from (1.0.2) then we have six vectors. If these are denoted by $V_1, V_2, \dots, V_6$ respectively, then we have

$$\begin{aligned} &V_1 = (2, \alpha, B), \quad V_2 = (2, B, \alpha), \quad V_3 = (\alpha, 2, B), \\ &V_4 = (\alpha, B, 2), \quad V_5 = (B, 2, \alpha), \quad V_6 = (B, \alpha, 2). \end{aligned}$$

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We could have also represented these by square brackets, that is,

$$V_1 = [2, \alpha, B], \quad \dots, \quad V_6 = [B, \alpha, 2]. \tag{1.1.1}$$

As a convention, we will use either all ordinary brackets $(\cdot)$ or all square brackets $[\cdot]$ when we discuss a given collection of vectors. The two notations will not be mixed up in the same collection. We could have also written the ordered sequences as columns, rather than as rows. For example,

$$U_1 = \begin{bmatrix} 2 \\ \alpha \\ B \end{bmatrix}, \quad \dots, \quad U_6 = \begin{bmatrix} B \\ \alpha \\ 2 \end{bmatrix} \quad \text{or} \quad U_1 = \begin{pmatrix} 2 \\ \alpha \\ B \end{pmatrix}, \quad \dots, \quad U_6 = \begin{pmatrix} B \\ \alpha \\ 2 \end{pmatrix} \tag{1.1.2}$$

also represent the same collection or ordered sets or vectors. In (1.1.1) they are written as row vectors whereas in (1.1.2) they are written as column vectors.

Definition 1.1.1 (An n-vector)**.** It is an ordered set of n objects written either as a row (a row n-vector) or as a column (a column n-vector).

Example 1.1.1 (Stock market gains)**.** A person has invested in 4 different stocks. Taking the January 1, 1998 as the base the person is watching the gain/loss, from this base value, at the end of each week.

	Stock 1	Stock 2	Stock 3	Stock 4
Week 1	100	150	−50	50
Week 2	50	−50	70	−50
Week 3	−150	−100	−20	0

The performance vector at the end of week 1 is then $(100, 150, -50, 50)$, a negative number indicating the loss and a positive number denoting a gain. The performance vector of stock 1 over the three weeks is $\left[\begin{smallmatrix} 100 \\ 50 \\ -150 \end{smallmatrix}\right]$. Observe that we could have also written weeks as columns and stocks as rows instead of the above format. Note also that for each element the position where it appears is relevant, in other words, the elements above are ordered.

Example 1.1.2 (Consumption profile)**.** Suppose the following are the data on the food consumption of a family in a certain week, where q denotes quantity (in kilograms) and p denotes price per unit (per kilogram).

	Beef	Pork	Chicken	Vegetables	cereals
q	10	15	20	10	5
p	\$2.00	\$1.50	\$0.50	\$1.00	\$3.45

The vector of quantities consumed is $[10, 15, 20, 10, 5]$ and the price vector is $[2.00, 1.50, 0.50, 1.00, 3.45]$.

Example 1.1.3 (Discrete statistical distributions). If a discrete random variable takes values $x_1, x_2, \dots, x_n$ with probabilities $p_1, \dots, p_n$ respectively where $p_i > 0$, $i = 1, \dots, n$, $p_1 + \cdots + p_n = 1$ then this distribution can be represented as follows:

x-values	x_1	x_2	$\dots$	x_n
probabilities	p_1	p_2	$\dots$	p_n

As an example, if x takes the values $0, 1, -1$, (such as a gambler gains nothing, gains one dollar, loses one dollar) with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively then the distribution can be written as

x-values	0	1	–1
probabilities	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Here the observation vector is $(0, 1, -1)$ and the corresponding probability vector is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Note that when writing the elements of a vector, the elements may be separated by sufficient spaces, or by commas if there is possibility of confusion. Any vector $(p_1, \dots, p_n)$ such that $p_i > 0$, $i = 1, \dots, n$, $p_1 + \cdots + p_n = 1$ is called a *discrete probability distribution*.

Example 1.1.4 (Transition probability vector). Suppose at El Paso, Texas, there are only two possibilities for a September day. It can be either sunny and hot or cloudy and hot. Let these be denoted by S (sunny) and C (cloudy). A sunny day can be followed by either a sunny day or a cloudy day and similarly a cloudy day can follow either a sunny or a cloudy day. Suppose that the chances (transition probabilities) are the following:

	S	C
S	0.95	0.05
C	0.90	0.10

Then for a sunny day the transition probability vector is $(0.95, 0.05)$ to be followed by a sunny and a cloudy day respectively. For a cloudy day the corresponding vector is $(0.90, 0.10)$.

Example 1.1.5 (Error vector). Suppose that an automatic machine is filling 5 kg bag of potatoes. The machine is not allowed to cut or chop to make the weight exactly 5 kg. Naturally, if one such bag is taken then the actual weight can be less than or greater than or equal to 5 kg. Let ϵ denote the error = observed weight minus the expected weight(5 kg). [One could have defined "error" as expected value minus the observed value]. Suppose 4 such bags are selected and weighed. Suppose the observation vector, denoted by X, is

$$X = (5.01, 5.10, 4.98, 4.92).$$

Then the error vector, denoted by ϵ, is

$$\begin{aligned}\epsilon &= (0.01, 0.10, -0.02, -0.08)\\ &= (5.01 - 5.00, 5.10 - 5.00, 4.98 - 5.00, 4.92 - 5.00).\end{aligned}$$

Note that we could have written both X and ϵ as column vectors as well.

Example 1.1.6 (Position vector)**.** Suppose a person walks on a straight path (horizontal) for 4 miles and then along a perpendicular path to the left for another 6 miles. If these distances are denoted by x and y respectively then her position vector is, taking the starting points as the origin,

$$(x, y) = (4, 6).$$

Example 1.1.7 (Vector of partial derivatives)**.** Consider $f(x_1, \dots, x_n)$, a scalar function of n real variables $x_1, \dots, x_n$. As an example,

$$f(x_1, x_2, x_3) = 3x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 5x_1x_3 - 2x_1 + 7.$$

Here $n = 3$ and there are 3 variables in f. Consider the partial derivative operators $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$, that is, $\frac{\partial}{\partial x_1}$ operating on f means to differentiate f with respect to x_1 treating x_2 and x_3 as constants. For example, $\frac{\partial}{\partial x_1}$ operating on the above f gives

$$\frac{\partial f}{\partial x_1} = 6x_1 - 2x_2 + 5x_3 - 2.$$

Consider the partial differential operator

$$\frac{\partial}{\partial X} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

Then $\frac{\partial}{\partial X}$ operating on f gives the vector

$$\frac{\partial f}{\partial X} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

For the above example,

$$\begin{aligned}\frac{\partial f}{\partial X} &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)\\ &= (6x_1 - 2x_2 + 5x_3 - 2, 2x_2 - 2x_1, 2x_3 + 5x_1).\end{aligned}$$

Example 1.1.8 (Students' grades)**.** Suppose that Miss Gomez, a first year student at UTEP, is taking 5 courses, Calculus I (course 1), Linear Algebra (course 2),..., (course 5). Suppose that each course requires 2 class tests, a set of assignments to be submitted and a final exam. Suppose that Miss Gomez' performance profile is the following (all grades in percentages):

	course 1	course 2	course 3	course 4	course 5
test 1	80	85	80	90	95
test 2	85	85	85	95	100
assignments	100	100	100	100	100
final exam	90	95	90	92	95

Then for example, her performance profiles on courses 1 and 4 are

$$\begin{bmatrix} 80 \\ 85 \\ 100 \\ 90 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 90 \\ 95 \\ 100 \\ 92 \end{bmatrix}$$

respectively. Her performances on all courses is the vector $(80, 85, 80, 90, 95)$ for test 1.

Example 1.1.9 (Fertility data)**.** Fertility of women is often measured in terms of the number of children produced. Suppose that the following data represent the average number of children in a particular State according to age and racial groups:

	group 1	group 2	group 3	group 4
≤ 16	1	0.8	1.5	0.5
16 to ≤ 18	1	1	0.8	0.9
18 to ≤ 35	4	2	3	2
35 to ≤ 50	1	0	2	0
> 50	0	0	1	0

The first row vector in the above table represents the performance of girls 16 years or younger over the 4 racial groups. Column 2 represents the performance of racial group 2 over the age groups, and so on.

Example 1.1.10 (Geometric probability law)**.** Suppose that a person is playing a game of chance in a casino. Suppose that the chance of winning at each trial is 0.2 and that of losing is 0.8. Suppose that the trials are independent of each other. Then the person can win at the first trial, or lose at the first trial and win at the second trial, or lose at the first two trials and win at the third trial, and so on. Then the chance of winning at the x-th trial, $x = 1, 2, 3, \dots$ is given by the vector

$$[0.2, (0.8)(0.2), (0.8)^2(0.2), (0.8)^3(0.2), \dots].$$

It is an n-vector with $n = +\infty$. Note that the number of ordered objects, representing a vector, could be finite or infinitely many (countable, that is one can draw a one-to-one correspondence to the natural numbers $1, 2, 3, \dots$).

In Example 1.1.1 suppose that the gains/loses were in US dollars and suppose that the investor was a Canadian and she would like to convert the first week's gain/loss into Canadian dollar equivalent. Suppose that the exchange rate is US$ 1=CA$ 1.60.

Then the first week's performance is available by multiplying each element in the vector by 1.6. That is,

$$\begin{aligned} 1.6(100, 150, -50, 50) &= \big((1.6)(100), (1.6)(150), (1.6)(-50), (1.6)(50)\big) \\ &= (160, 240, -80, 80). \end{aligned}$$

Another example of this type is that someone has a measurement vector in feet and that is to be converted into inches, then each element is multiplied by 12 (one foot = 12 inches), and so on.

Definition 1.1.2 (Scalar multiplication of a vector)**.** Let c be a scalar, a 1-vector, and $U = (u_1, \dots, u_n)$ an n-vector. Then the scalar multiple of U, namely cU, is defined as

$$cU = (cu_1, \dots, cu_n). \tag{1.1.3}$$

As a convention the scalar quantity c is written on the left of U and not on the right, that is, not as Uc but as cU. As numerical illustrations we have

$$-3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}; \quad 0\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix};$$
$$4(2, -1) = (8, -4).$$

In Example 1.1.1 if the total (combined) gain/loss at the end of the second week is needed then the combined performance vector is given by

$$(100 + 50, 150 - 50, -50 + 70, 50 - 50) = (150, 100, 20, 0).$$

If the combined performance of the first three weeks is required then it is the above vector added to the third week's vector, that is,

$$(150, 100, 20, 0) + (-150, -100, -20, 0) = (0, 0, 0, 0).$$

Definition 1.1.3 (Addition of vectors)**.** Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two n-vectors. Then the sum is defined as

$$a + b = (a_1 + b_1, \dots, a_n + b_n), \tag{1.1.4}$$

that is, the vector obtained by adding the corresponding elements.

Note that vector addition is defined only for vectors of the same category and order. Either both are row vectors of the same order or both are column vectors of the same order. In other words, if U is an n-vector and V is an m-vector then $U + V$ is not defined unless $m = n$, and further, both are either row vectors or column vectors.

Definition 1.1.4 (A null vector)**.** A vector with all its elements zeros is called a null vector and it is usually denoted by a big O.

In Example 1.1.1 the combined performance of the first 3 weeks is a null vector. In other words, after the first 3 weeks the performance is back to the base level. From the above definitions the following properties are evident. If U, V, W are three n-vectors (either all row vectors or all column vectors) and if a, b, c are scalars then

$$\begin{aligned} &U + V = V + U; \quad U + (V + W) = (U + V) + W \\ &U - V = U + (-1)V; \quad U + O = O + U = U; \quad U - U = O; \\ &a[bU + cV] = abU + acV = b(aU) + c(aV). \end{aligned} \tag{1.1.5}$$

Some numerical illustrations are the following:

$$\begin{aligned} 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2+0+0 \\ 0-3+0 \\ -2+6+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}; \\ (1,-7) + 6(0,-1) + (0,0) &= (1,-7) + (0,-6) + (0,0) \\ &= (1+0+0, -7-6+0) = (1,-13); \\ (1,1,2) - (1,1,2) &= (1,1,2) + (-1,-1,-2) \\ &= (1-1, 1-1, 2-2) = (0,0,0). \end{aligned}$$

Definition 1.1.5 (Transpose of a vector)**.** [Standard notations: U' = transpose of U, U^T = transpose of U.] If U is a row n-vector then U' is the same written as a column and vice versa.

Some numerical illustrations are the following, where "⇒" means "implies":

$$U = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow U' = [-3, 0, 1]$$

$$V = [1, 5, -1] \Rightarrow V' = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = V^T.$$

Note that in the above illustration $U + V$ is not defined but $U + V'$ is defined. Similarly $U' + V$ is defined but $U' + V'$ is not defined. Also observe that if z is a 1-vector (a scalar quantity) then $z' = z$, that is, the transpose is itself.

In Example 1.1.6 the position vector is $(x, y) = (4, 6)$. Then the distance of this position from the starting point is obtained from Pythagoras' rule as,

$$\sqrt{x^2 + y^2} = \sqrt{4^2 + 6^2} = \sqrt{52}.$$

This then is the straight distance from the starting point $(0, 0)$ to the final position $(4, 6)$. We will formally define the length of a vector as follows, the idea will be clearer when we consider the geometry of vectors later on:

Definition 1.1.6 (Length of a vector)**.** Let U be a real n-vector (either a column vector or a row vector). If the elements of U are $u_1, \ldots, u_n$ then the length of U, denoted by $\|U\|$, is defined as

$$\|U\| = \sqrt{u_1^2 + \cdots + u_n^2}, \tag{1.1.6}$$

when the elements are real numbers. When the elements are not real then the length will be redefined later on. Some numerical illustrations are the following:

$$U = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \|U\| = \sqrt{(1)^2 + (-1)^2 + (0)^2} = \sqrt{2};$$

$$V = (1, 1, -2) \Rightarrow \|V\| = \sqrt{(1)^2 + (1)^2 + (-2)^2} = \sqrt{6};$$

$$O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \|O\| = 0; \quad \mathrm{e}_1 = (1, 0, 0, 0) \Rightarrow \|\mathrm{e}_1\| = 1;$$

$$Z = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \Rightarrow \|Z\| = 1.$$

Note that the "length", by definition, is a non-negative quantity. It is either zero or a positive quantity and it cannot be negative. For a null vector the length is zero. The length of a vector is zero iff (if and only if) the vector is a null vector.

Definition 1.1.7 (A unit vector)**.** A vector whose length is unity is called a unit vector.

Some numerical illustrations are the following:

$$\mathrm{e}_4 = (0, 0, 0, 1) \Rightarrow \|\mathrm{e}_4\| = 1.$$

But $U = (1, -2, 1) \Rightarrow \|U\| = \sqrt{6}$, U is not a unit vector whereas

$$V = \frac{1}{\|U\|} U = \frac{1}{\sqrt{6}}(1, -2, 1) = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \Rightarrow \|V\| = 1,$$

that is, V is a unit vector. Observe the following: A null vector is not a unit vector. If the length of any vector is non-zero (the only vector with length zero is the null vector) then taking a scalar multiple, where the scalar is the reciprocal of the length, a unit vector can be created out of the given non-null vector. In general, if $U = (u_1, \dots, u_n)$, where $u_1, \dots, u_n$ are real, then

$$\|U\| = \|U'\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

and

$$V = \frac{1}{\|U\|} U \Rightarrow \|V\| = 1 \tag{1.1.7}$$

when $\|U\| \neq 0$.

From the definition of length itself the following properties are obvious. If U and V are n-vectors of the same type and if a, b, c are scalars, then

$$\begin{aligned} &\|cU\| = |c|\,\|U\|; \quad \|cU + cV\| = |c|\,\|U + V\| \\ &\|U + V\| \le \|U\| + \|V\|; \\ &\|aU + bV\| \le |a|\,\|U\| + |b|\,\|V\| \end{aligned} \tag{1.1.8}$$

where, for example, $|c|$ means the absolute value of c, that is, the magnitude of c, ignoring the sign. For example,

$$\|-2(1, -1, 1)\| = |-2|\,\sqrt{(1)^2 + (-1)^2 + (1)^2} = 2\sqrt{3};$$

$$\|2(1, -1, 1)\| = 2\sqrt{3};$$

$$U = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \Rightarrow U + V = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix};$$

$$\|U\| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3};$$

$$\|U + V\| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = \sqrt{9} = 3 < \|U\| + \|V\| = \sqrt{3} + \sqrt{14}.$$

Now, we will look at another concept. In Example 1.1.2 the family's total expense of the week on those food items is available by multiplying the quantities with unit prices and then adding up. That is, if the quantity vector is denoted by Q and the per unit price vector is denoted by P then

$$Q = (10, 15, 20, 10, 5)$$

and

$$P = (2.00, 1.50, 0.50, 1.00, 3.45).$$

Thus the total expense of that family for that week on these 5 items is obtained by multiplying and adding the corresponding elements in P and Q. That is,

$$(10)(2.00) + (15)(1.50) + (20)(0.50) + (10)(1.00) + (5)(3.45) = \$79.75.$$

It is a scalar quantity (1-vector) and not a 5-vector, even though the vectors Q and P are 5-vectors. For computing quantities such as the one above we define a concept called the *dot product or the inner product* between two vectors.

Definition 1.1.8 (Dot product or inner product). Let U and V be two real n-vectors (either both row vectors or both column vectors or one row vector and the other column vector). Then the dot product between U and V, denoted by $U.V$ is defined as

$$U.V = u_1v_1 + \cdots + u_nv_n$$

that is, the corresponding elements are multiplied and added, where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the elements (real) in U and V respectively. (Vectors in the complex field will be considered in a later chapter.)

Some numerical illustrations are the following: In the above example, the family's consumption for the week is $Q.P = P.Q = 79.75$.

$$U_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow$$

$$U_1.U_2 = (0)(1) + (1)(-1) + (2)(1) = 1.$$

$$V_1 = (3, 1, -1, 5), \quad V_2 = (-1, 0, 0, 1) \Rightarrow$$

$$V_1.V_2 = (3)(-1) + (1)(0) + (-1)(0) + (5)(1) = 2.$$

From the definition itself the following properties are evident:

$$U.O = 0, \quad aU.V = (aU).V = U.(aV)$$

where a is a scalar.

$$U.V = V.U, \quad (aU).(bV) = ab(U.V)$$

where a and b are scalars.

$$U.(V + W) = U.V + U.W = (W + V).U. \tag{1.1.9}$$

The notation with a dot, $U.V$, is an awkward one. But unfortunately this is a widely used notation. A proper notation in terms of transposes and matrix multiplication will be introduced later. Also, further properties of dot products will be considered later, after looking at the geometry of vectors as ordered sets.

Exercises 1.1

1.1.1. Are the following defined? Whichever is defined compute the answers.

(a) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; (b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$;

(c) $(3, -1, 4) - (2, 1)$; (d) $5(1, 0) - 3(-2, -1)$.

1.1.2. Compute the lengths of the following vectors. Normalize the vectors (create a vector with unit length from the given vector) if possible:

$$\text{(a)}\quad (0,0,0);\qquad \text{(b)}\quad (1,1,-1);$$

$$\text{(c)}\quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix};\qquad \text{(d)}\quad \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix};\qquad \text{(e)}\quad 3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

1.1.3. Convert the stock market performance vectors in Example 1.1.1 to the following: First week's performance into pound sterling (1 \$ = 0.5 pounds sterling); the second week's performance into Italian lira (1 \$ = 2 000 lira).

1.1.4. In Example 1.1.3 compute the expected value of the random variable. [The expected value of a discrete random variable is denoted as $E(x)$ and defined as $E(x) = x_1p_1 + \cdots + x_np_n$ if x takes the values $x_1, \dots, x_n$ with probabilities $p_1, \dots, p_n$ respectively.] If it is a game of chance where the person wins \$0, \$1, \$(–1) (loses a dollar) with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively how much money can the person expect to win in a given trial of the game?

1.1.5. In Example 1.1.3 if the expected value is denoted by $\mu = X.P$ (μ the Greek letter mu), where $X = (x_1, \dots, x_n)$ and $P = (p_1, \dots, p_n)$ then the variance of the random variable is defined as the dot product between $((x_1 - \mu)^2, \dots, (x_n - \mu)^2)$ and P. Compute the variance of the random variable in Example 1.1.3. [Variance is the square of a measure of scatter or spread in the random variable.]

1.1.6. In Example 1.1.5 compute the sum of squares of the errors [Hint: If ϵ is the error vector then the sum of squares of the errors is available by taking the dot product $\epsilon.\epsilon$.]

1.1.7. In Example 1.1.8 suppose that for each course the distribution of the final grade is the following: 20 points each for each test, 10 points for assignments and 50 points for the final exam. Compute the vector of final grades of the student for the 5 courses by using the various vectors and using scalar multiplications and sums.

1.1.8. From the chance vector in Example 1.1.10 compute the chance of ever winning (sum of the elements) and the expected number of trials for the first win, $E(x)$ (note that x takes the values $1, 2, \dots$ with the corresponding probabilities).

1.1.9. Consider an n-vector of unities denoted by $J = (1, 1, \dots, 1)$. If $X = (x_1, \dots, x_n)$ is any n-vector then compute (a) $X.J$; (b) $\frac{1}{n}X.J$.

1.1.10. For the quantities in Exercise 1.1.9 establish the following:

$$\text{(a)}\qquad (X - \tilde{\mu}).J = 0 \quad \text{where } \tilde{\mu} = \left(\frac{1}{n}X.J, \dots, \frac{1}{n}X.J\right).$$

[This holds whatever be the values of $x_1, \dots, x_n$. Verify by taking some numerical values.]

$$\text{(b)} \qquad (X-\tilde{\mu}).(X-\tilde{\mu}) = X.X - n\left(\frac{1}{n}X.J\right)^2$$
$$= X.X - \frac{1}{n}(X.J)(X.J)$$

whatever be the values of $x_1, \dots, x_n$.

(c) Show that the statement in (a) above is equivalent to the statement $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$ where $\bar{x} = \sum_i^n \frac{x_i}{n}$ with $\sum$ denoting a sum.

(d) Show that the statement in (b) is equivalent to the statements

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$
$$= \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2.$$

A note on $\sum$ notation. This is a convenient notation to denote a sum.

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n,$$

that is, i is replaced by $1, 2, \dots, n$ and the elements are added up.

$$\sum_{i=1}^{n} 4 = 4 + 4 + \cdots + 4 = 4n;$$
$$\sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n = a.b$$

where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$.

$$\sum_{i=1}^{n} a_i^2 = a_1^2 + \cdots + a_n^2 = a.a;$$
$$\sum_{i=1}^{n}(5a_i) = 5a_1 + \cdots + 5a_n = 5(a_1 + \cdots + a_n) = 5\sum_{i=1}^{n} a_i;$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j = \sum_{i=1}^{n} a_i\left[\sum_{j=1}^{m} b_j\right]$$
$$= \sum_{i=1}^{n} a_i(b_1 + \cdots + b_m) = (a_1 + \cdots + a_n)(b_1 + \cdots + b_m)$$
$$= \sum_{j=1}^{m}\sum_{i=1}^{n} a_i b_j;$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = a_{11} + \cdots + a_{1n}$$
$$+ a_{21} + \cdots + a_{2n}$$
$$\vdots$$

$$+a_{m1} + \cdots + a_{mn} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij};$$

$$\sum_{i=1}^{n}(a_i + 3b_i) = \sum_{i=1}^{n} a_i + 3\sum_{i=1}^{n} b_i;$$

$$\bar{x} = \sum_{j=1}^{n} \frac{x_j}{n} = \frac{1}{n}(x_1 + \cdots + x_n) = \frac{1}{n} X.J$$

where

$$X = (x_1, \ldots, x_n) \quad \text{and} \quad J = (1, 1, \ldots, 1);$$

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \bar{x}) &= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} \\ &= \sum_{i=1}^{n} x_i - n\bar{x} = \sum_{i=1}^{n} x_i - n\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) \\ &= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0\end{aligned}$$

whatever be $x_1, \ldots, x_n$.

1.1.11. When searching for maxima/minima of a scalar function f of many real scalar variables the critical points (the points where one may find a maximum or a minimum or a saddle point) are available by operating with $\frac{\partial}{\partial X}$, equating to a null vector and then solving the resulting equations. For the function

$$f(x_1, x_2) = 3x_1^2 + x_2^2 - 2x_1 + x_2 + 5$$

evaluate the following: (a) the operator $\frac{\partial}{\partial X}$, (b) $\frac{\partial f}{\partial X}$, (c) $\frac{\partial f}{\partial X} = O$, (d) the critical points.

1.1.12. For the following vectors U, V, W compute the dot products $U.V$, $U.W$, $V.W$ where

$$U = (1, 1, 1), \quad V = (1, -2, 1), \quad W = (1, 0, -1).$$

1.1.13. If V_1, V_2, V_3 are n vectors, either $n \times 1$ column vectors or $1 \times n$ row vectors and if $\|V_j\|$ denotes the length of the vector V_j then show that the following results hold in general:

(i) $\|V_1 - V_2\| > 0$ and $\|V_1 - V_2\| = 0$ iff $V_1 = V_2$;
(ii) $\|cV_1\| = |c|\,\|V_1\|$ where c is a scalar;
(iii) $\|V_1 - V_2\| + \|V_2 - V_3\| \geq \|V_1 - V_3\|$.

1.1.14. Verify (i), (ii), (iii) of Exercise 1.1.13 for

$$V_1 = (1, 0, -1), \quad V_2 = (0, 0, 2), \quad V_3 = (2, 1, -1).$$

1.1.15. Let $U = (1, -1, 1, -1)$. Construct three non-null vectors V_1, V_2, V_3 such that $U.V_1 = 0$, $U.V_2 = 0$, $U.V_3 = 0$, $V_1.V_2 = 0$, $V_1.V_3 = 0$, $V_2.V_3 = 0$.

1.2 Geometry of vectors

From the position vector in Example 1.1.6 it is evident that $(x, y) = (4, 6)$ can be denoted as a point in a 2-space (plane) with a rectangular coordinate system. In general, since an n-vector of real numbers is an ordered set of real numbers it can be represented as a point in a Euclidean n-space.

1.2.1 Geometry of scalar multiplication

If the position $(4, 6)$, which could also be written as $\binom{x}{y} = \binom{4}{6}$, is marked in a 2-space then we have the following Figure 1.2.1. One can also think of this as an arrowhead starting at $(0, 0)$ and going to $(4, 6)$. In this representation the vector has a length and a direction. In general, if U is an arrowhead from the origin $(0, 0, \dots, 0)$ in n-space to the point $U = (u_1, \dots, u_n)$ then $-U$ will represent an arrowhead with the same length but going in the opposite direction. Then cU will be an arrowhead in the same direction with length $c\|U\|$ if $c > 0$ and in the opposite direction with length $|c|\,\|U\|$ if $c < 0$, where $|c|$ denotes the absolute value or the magnitude of c, and it is the origin itself if $c = 0$. In physics, chemistry and engineering areas it is customary to denote a vector with an arrow on top such as $\vec{U}$, meaning the vector $\vec{U}$.

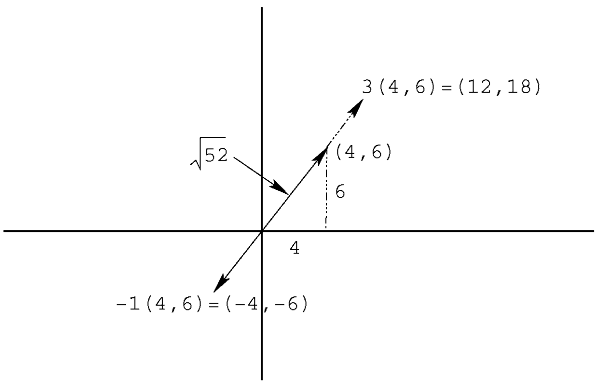

Figure 1.2.1: Geometry of vectors.

1.2.2 Geometry of addition of vectors

Scalar multiplication is interpreted geometrically as above. Then, what will be the geometrical interpretation for a sum of two vectors? For simplicity, let us consider a 2-space. If $\vec{U} = \binom{u_1}{u_2}$ and $\vec{V} = \binom{v_1}{v_2}$ then algebraically

$$\vec{U} + \vec{V} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

which is the arrowhead representing the diagonal of the parallelogram as shown in Figure 1.2.2. From the geometry of vectors one can notice that a vector, as an ordered set of real numbers, possesses two properties basically, namely, a length and a direction. Hence we can give a coordinate-free definition as an arrowhead with a length and a direction.

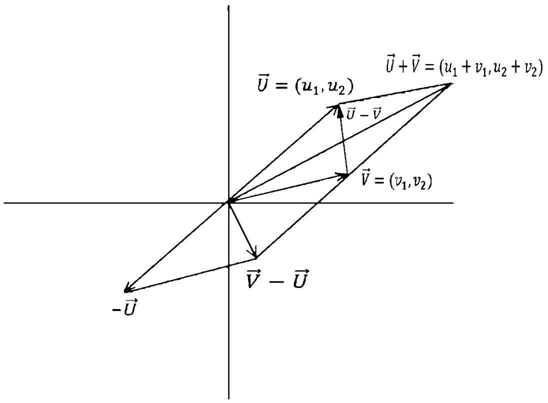

Figure 1.2.2: Sum of two vectors.

1.2.3 A coordinate-free definition of vectors

Definition 1.2.1 (A coordinate-free definition for a vector)**.** It is defined as an arrowhead with a given length and a given direction.

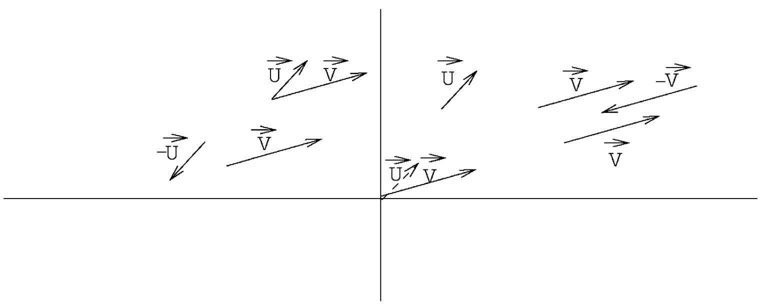

Figure 1.2.3: Coordinate-free definition of vectors.

In this definition, observe that all arrowheads with the same length and same direction are taken to be one and the same vector as shown in Figure 1.2.3. We can move an arrowhead parallel to itself. All such arrowheads obtained by such displacements are taken as one and the same vector. If one has a coordinate system then move the vector parallel to itself so that the tail-end (the other end to the arrow tip) coincides with the origin of the coordinate system. Thus the position vectors are also included in this general definition. In a coordinate-free definition one can construct $\vec{U}+\vec{V}$ and $\vec{U}-\vec{V}$ as follows: Move $\vec{U}$ or $\vec{V}$ parallel to itself until the tail-ends coincide. Complete the parallelogram. The leading diagonal gives $\vec{U}+\vec{V}$ and the diagonal going from the head of $\vec{U}$ to the head of $\vec{V}$ gives $\vec{V}-\vec{U}$ and the one the other way around is $-(\vec{V}-\vec{U})=\vec{U}-\vec{V}$.

1.2.4 Geometry of dot products

Consider a Euclidean 2-space and represent the vectors $\vec{U}=(u_1,u_2)$ and $\vec{V}=(v_1,v_2)$ as points in a rectangular coordinate system. Let the angles, the vectors $\vec{U}$ and $\vec{V}$ make with the x-axis be denoted by θ_1 and θ_2 respectively. Let

$$\theta=\theta_1-\theta_2.$$

Then

$$\cos\theta_1=\frac{u_1}{\sqrt{u_1^2+u_2^2}},\quad \cos\theta_2=\frac{v_1}{\sqrt{v_1^2+v_2^2}},$$
$$\sin\theta_1=\frac{u_2}{\sqrt{u_1^2+u_2^2}},\quad \sin\theta_2=\frac{v_2}{\sqrt{v_1^2+v_2^2}}.$$

But

$$\begin{aligned}\cos\theta=\cos(\theta_1-\theta_2)&=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\\&=\frac{u_1v_1+u_2v_2}{\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}}=\frac{\vec{U}.\vec{V}}{\|\vec{U}\|\,\|\vec{V}\|}\end{aligned}\tag{1.2.1}$$

whenever $\|\vec{U}\|\neq 0$ and $\|\vec{V}\|\neq 0$. Thus

$$\vec{U}.\vec{V}=\|\vec{U}\|\,\|\vec{V}\|\cos\theta,\quad \|\vec{U}\|\neq 0,\ \|\vec{V}\|\neq 0.\tag{1.2.2}$$

The dot product is the product of the lengths multiplied by the cosine of the angle between the vectors. This result remains the same whatever be the space. That is, it holds in 2-space, 3-space, 4-space and so on. The Figure 1.2.4 shows the situation when $0\le\theta_1\le\pi/2,\ 0\le\theta_2\le\pi/2,\ \theta_1>\theta_2$. The student may verify the result for all possible cases of θ_1 and θ_2, as an exercise. From (1.2.1) we can obtain an interesting result. Since $\cos\theta$, in absolute value, is less than or equal to 1 we have a result known as Cauchy–Schwartz inequality:

$$|\cos\theta|=\left|\frac{\vec{U}.\vec{V}}{\|\vec{U}\|\,\|\vec{V}\|}\right|\le 1\ \Rightarrow\ |\vec{U}.\vec{V}|\le\|\vec{U}\|\,\|\vec{V}\|.$$

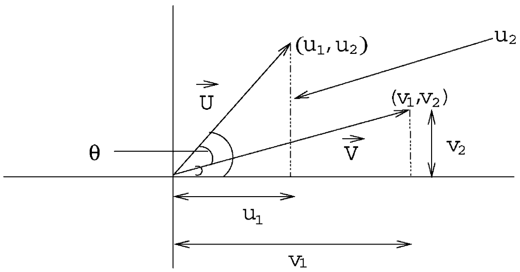

Figure 1.2.4: Geometry of the dot product.

1.2.5 Cauchy–Schwartz inequality

$$|\vec{U}.\vec{V}| \leq \|\vec{U}\|\,\|\vec{V}\|.$$

In other words, if $\vec{U} = (u_1, \dots, u_n)$ and $\vec{V} = (v_1, \dots, v_n)$ then for real $u_1, \dots, u_n$ and $v_1, \dots, v_n$,

$$|u_1v_1 + \cdots + u_nv_n| \leq \sqrt{u_1^2 + \cdots + u_n^2}\sqrt{v_1^2 + \cdots + v_n^2}. \tag{1.2.3}$$

When the angle θ between the vectors $\vec{U}$ and $\vec{V}$ is zero or $2n\pi$, $n = 0, 1, \dots$ then $\cos\theta = 1$ which means that the two vectors are scalar multiples of each other. Thus we have an interesting result:

(i) When equality in the Cauchy–Schwartz inequality holds the two vectors are scalar multiples of each other, that is, $\vec{U} = c\vec{V}$ where c is a scalar quantity.

When $\theta = \pi/2$ then $\cos\theta = 0$ which means $\vec{U}.\vec{V} = 0$. When the angle between the vectors $\vec{U}$ and $\vec{V}$ is $\pi/2$, we may say that the vectors are orthogonal to each other, then the dot product is zero. Orthogonality will be taken up later.

Example 1.2.1. A girl is standing in a park and looking at a bird sitting on a tree. Taking one corner of the park as the origin and the rectangular border roads as the (x, y)-axes the positions of the girl and the tree are $(1, 2)$ and $(10, 15)$ respectively, all measurements in feet. The girl is 5 feet tall to her eye level and the bird's position from the ground is 20 feet up. Compute the following items: (a) The vector from the girl's eyes to the bird and its length; (b) The vector from the foot of the tree to the girl's feet and its length; (c) When the girl is looking at the bird the angle this path makes with the horizontal direction; (d) The angle this path makes with the vertical direction.

Solution 1.2.1. The positions of the girl's eyes and the bird are respectively $\vec{U} = (1, 2, 5)$ and $\vec{V} = (10, 15, 20)$.

(a) The vector from the girl's eyes to the bird is then

$$\vec{V} - \vec{U} = (10 - 1, 15 - 2, 20 - 5) = (9, 13, 15)$$

and its length is then

$$\|\vec{V} - \vec{U}\| = \sqrt{(9)^2 + (13)^2 + (15)^2} = \sqrt{475}.$$

(b) The foot of the tree is $\vec{V}_1 = (10, 15, 0)$ and the position of the girl's feet is $\vec{U}_1 = (1, 2, 0)$. The vector from the foot of the tree to the girl's feet is then

$$\vec{U}_1 - \vec{V}_1 = (1, 2, 0) - (10, 15, 0) = (-9, -13, 0)$$

and its length is

$$\|\vec{U}_1 - \vec{V}_1\| = \sqrt{(-9)^2 + (-13)^2 + (0)^2} = \sqrt{250}.$$

(c) From the girl's eyes the vector in the horizontal direction to the tree is

$$\vec{V}_2 - \vec{U}_2 = (10, 15, 5) - (1, 2, 5) = (10 - 1, 15 - 2, 5 - 5) = (9, 13, 0)$$

and its length is

$$\|\vec{V}_2 - \vec{U}_2\| = \sqrt{(9)^2 + (13)^2 + (0)^2} = \sqrt{250}.$$

Let θ be the angle between the vectors $\vec{V} - \vec{U}$ and $\vec{V}_2 - \vec{U}_2$. Then

$$\begin{aligned}\cos\theta &= \frac{(\vec{V} - \vec{U}).(\vec{V}_2 - \vec{U}_2)}{\|\vec{V} - \vec{U}\|\,\|\vec{V}_2 - \vec{U}_2\|}\\ &= \frac{(9, 13, 15).(9, 13, 0)}{\sqrt{475}\sqrt{250}} = \frac{\sqrt{250}}{\sqrt{475}} = \sqrt{\frac{10}{19}}.\end{aligned}$$

Then the angle θ is given by

$$\theta = \cos^{-1}\sqrt{\frac{10}{19}}.$$

(d) The angle in the vertical direction is

$$\frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}\sqrt{\frac{10}{19}}.$$

1.2.6 Orthogonal and orthonormal vectors

Definition 1.2.2 (Orthogonal vectors)**.** Two real vectors $\vec{U}$ and $\vec{V}$ are said to be orthogonal to each other if the angle between them is $\frac{\pi}{2} = 90^\circ$ or equivalently, if $\cos\theta = 0$ or equivalently, if $\vec{U}.\vec{V} = 0$.

It follows, trivially, that every vector is orthogonal to a null vector since the dot product is zero.

Definition 1.2.3 (Orthonormal system of vectors)**.** A system of real vectors $\vec{U}_1, \dots, \vec{U}_k$ is said to be an orthonormal system if $\vec{U}_i.\vec{U}_j = 0$ for all i and j, $i \neq j$ (all different vectors are orthogonal to each other or they form an *orthogonal system*) and in addition, $\|\vec{U}_j\| = 1$, $j = 1, 2, \dots, k$ (all vectors have unit length).

As an illustrative example, consider the vectors

$$\vec{U}_1 = (1,1,1), \quad \vec{U}_2 = (1,0,-1), \quad \vec{U}_3 = (1,-2,1).$$

Then

$$\begin{aligned}
\vec{U}_1.\vec{U}_2 &= (1)(1) + (1)(0) + (1)(-1) = 0;\\
\vec{U}_1.\vec{U}_3 &= (1)(1) + (1)(-2) + (1)(1) = 0;\\
\vec{U}_2.\vec{U}_3 &= (1)(1) + (0)(-2) + (-1)(1) = 0.
\end{aligned}$$

Thus $\vec{U}_1, \vec{U}_2, \vec{U}_3$ form an orthogonal system. Let us normalize the vectors in order to create an orthonormal system. Let us compute the lengths

$$\|\vec{U}_1\| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}, \quad \|\vec{U}_2\| = \sqrt{2}, \quad \|\vec{U}_3\| = \sqrt{6}.$$

Consider the vectors

$$\begin{aligned}
\vec{V}_1 &= \frac{1}{\|\vec{U}_1\|}\vec{U}_1 = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\\
\vec{V}_2 &= \frac{1}{\|\vec{U}_2\|}\vec{U}_2 = \left(\frac{1}{\sqrt{2}}(1,0,-1)\right)\\
\vec{V}_3 &= \frac{1}{\|\vec{U}_3\|}\vec{U}_3 = \left(\frac{1}{\sqrt{6}}(1,-2,1)\right).
\end{aligned}$$

Then $\vec{V}_1, \vec{V}_2, \vec{V}_3$ form an orthonormal system.

As another example, consider the vectors,

$$\mathrm{e}_1 = (1,0,\dots,0), \quad \mathrm{e}_2 = (0,1,0,\dots,0), \quad \dots, \quad \mathrm{e}_n = (0,\dots,0,1).$$

Then evidently

$$\mathrm{e}_i.\mathrm{e}_j = 0, \quad i \neq j, \quad \|\mathrm{e}_i\| = 1, \quad i,j = 1,\dots,n.$$

Hence $\mathrm{e}_1, \dots, \mathrm{e}_n$ is an orthonormal system.

Definition 1.2.4 (Basic unit vectors)**.** The above vectors $e_1, \dots, e_n$ are called the basic unit vectors in n-space. [One could have written them as column vectors as well.]

Engineers often use the notation

$$\vec{i} = (1,0), \quad \vec{j} = (0,1) \quad \text{or} \quad \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.4}$$

to denote the basic unit vectors in 2-space and

$$\vec{i} = (1,0,0), \quad \vec{j} = (0,1,0), \quad \vec{k} = (0,0,1) \quad \text{or}$$
$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.2.5}$$

to denote the basic unit vectors in 3-space. One interesting property is the following:

(ii) Any n-vector can be written as a linear combination of the basic unit vectors $e_1, \dots, e_n$.

For example, consider a general 2-vector $\vec{U} = (a,b)$. Then

$$a\vec{i} + b\vec{j} = a(1,0) + b(0,1) = (a,0) + (0,b) = (a,b) = \vec{U}. \tag{1.2.6}$$

If $\vec{V} = (a,b,c)$ is a general 3-vector then

$$\begin{aligned} a\vec{i} + b\vec{j} + c\vec{k} &= a(1,0,0) + b(0,1,0) + c(0,0,1) \\ &= (a,0,0) + (0,b,0) + (0,0,c) = (a,b,c) = \vec{V}. \end{aligned} \tag{1.2.7}$$

Note that the same notation $\vec{i}$ and $\vec{j}$ are used for the unit vectors in 2-space as well as in 3-space. There is no room for confusion since we will not be mixing 2-vectors and 3-vectors at any stage when these are used. In general, we can state a general result. Let $\vec{U}$ be an n-vector with the elements $(u_1, \dots, u_n)$ then

$$\vec{U} = u_1 e_1 + \cdots + u_n e_n. \tag{1.2.8}$$

[Either all row vectors or all column vectors.]

The geometry of the above result can be illustrated as follows: We take a 2-space for convenience.

The vector $\vec{i}$ is in the horizontal direction with unit length. Then $a\vec{i}$ will be of length $|a|$ and in the same direction if $a > 0$ and in the opposite direction if $a < 0$. Similarly $\vec{j}$ is a unit vector in the vertical direction and $b\vec{j}$ is of length $|b|$ and in the same direction

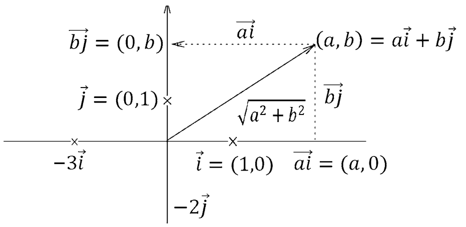

Figure 1.2.5: Geometry of linear combinations.

if $b > 0$ and in the opposite direction if $b < 0$ as shown in Figure 1.2.5. Then the point (a, b), as an arrowhead, is $a\vec{i} + b\vec{j}$. If the angle the vector

$$\vec{U} = (a, b) = a\vec{i} + b\vec{j}$$

makes with the x-axis is θ then

$$\begin{aligned}\cos\theta &= \frac{(a\vec{i}+b\vec{j}).(a\vec{i})}{\|a\vec{i}+b\vec{j}\|\,\|a\vec{i}\|} = \frac{(a)(a)+(b)(0)}{\sqrt{a^2+b^2}\sqrt{a^2}} \\ &= \frac{a}{\sqrt{a^2+b^2}}\end{aligned} \tag{1.2.9}$$

and

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{b}{\sqrt{a^2+b^2}}. \tag{1.2.10}$$

Observe that (1.2.9) and (1.2.10) are consistent with the notions in ordinary trigonometrical calculations as well.

1.2.7 Projections

If $\vec{U} = (a, b)$ then the projection of $\vec{U}$ in the horizontal direction is

$$a = \sqrt{a^2+b^2}\cos\theta = \|\vec{U}\|\cos\theta$$

which is the shadow on the x-axis if light beams come parallel to the y-axis and hit the vector (arrowhead), and the projection in the vertical direction is

$$b = \sqrt{a^2+b^2}\sin\theta = \|\vec{U}\|\sin\theta$$

which is the shadow on the y-axis if light beams come parallel to the x-axis and hit the vector. These results hold in n-space also. Consider a plane on which the vector $\vec{V}$ in n-space lies. Consider a horizontal and a vertical direction in this plane with the tail-end of the vector at the origin and let θ be the angle $\vec{V}$ makes with the horizontal direction. Then

$$\|\vec{V}\|\cos\theta = \text{projection of } \vec{V} \text{ in the horizontal direction} \tag{1.2.11}$$

and

$$\|\vec{V}\| \sin\theta = \text{projection of } \vec{V} \text{ in the vertical direction.} \qquad (1.2.12)$$

In practical terms one can explain the horizontal and vertical components of a vector as follows: Suppose that a particle is sitting at the position $(0, 0)$. A wind with a speed of $5\cos 45^\circ = \frac{5}{\sqrt{2}}$ units is blowing in the horizontal direction and a wind with a speed of $5\sin 45^\circ = \frac{5}{\sqrt{2}}$ units is blowing in the vertical direction. Then the particle will move at 45° angle to the x-axis and move at a speed of 5 units.

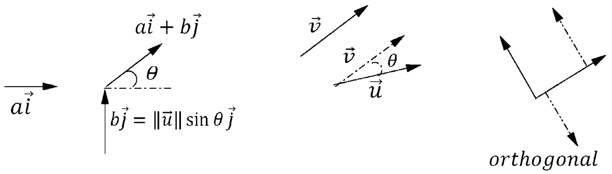

Figure 1.2.6: Movement of a particle.

Consider two arbitrary vectors $\vec{U}$ and $\vec{V}$ (coordinate-free definitions). What is the projection of $\vec{V}$ in the direction of $\vec{U}$? We can move $\vec{V}$ parallel to itself so that the tail-end of $\vec{V}$ coincides with the tail-end of $\vec{U}$. Consider the plane where these two vectors lie and let θ be the angle this displaced $\vec{V}$ makes with $\vec{U}$. Then the projection of $\vec{V}$ onto $\vec{U}$ is $\|\vec{V}\| \cos\theta$ as shown in Figure 1.2.6 (b). But

$$\cos\theta = \frac{\vec{U}.\vec{V}}{\|\vec{U}\|\,\|\vec{V}\|} \quad \Rightarrow$$

$$\|\vec{V}\| \cos\theta = \frac{\vec{U}.\vec{V}}{\|\vec{U}\|} = \text{projection of } \vec{V} \text{ onto } \vec{U}. \qquad (1.2.13)$$

If $\vec{U}$ is a unit vector then $\|\vec{U}\| = 1$ and then the projection of $\vec{V}$ in the direction of $\vec{U}$ is the dot product between $\vec{U}$ and $\vec{V}$.

Definition 1.2.5 (Projection vector of $\vec{V}$ in the direction of a unit vector $\vec{U}$)**.** A vector in the direction of $\vec{U}$ with a length equal to $\|\vec{V}\| \cos\theta$, the projection of $\vec{V}$ onto $\vec{U}$, is called the projection vector of $\vec{V}$ in the direction of $\vec{U}$.

Then the projection vector $\vec{V}$ in the direction of $\vec{U}$ is given by

$$(\vec{U}.\vec{V})\vec{U} \quad \text{if } \vec{U} \text{ is a unit vector}$$

and

$$(\vec{U}.\vec{V})\frac{\vec{U}}{\|\vec{U}\|^2} \quad \text{if } \vec{U} \text{ is any non-null vector.} \qquad (1.2.14)$$

Example 1.2.2. Evaluate the projection vector $\vec{V}$ in the direction of $\vec{U}$ if

$$\begin{aligned}&\text{(a)}\quad \vec{V} = 2\vec{i} + \vec{j} - \vec{k}, \quad \vec{U} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k});\\ &\text{(b)}\quad \vec{V} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{U} = 2\vec{i} + \vec{j} + \vec{k};\\ &\text{(c)}\quad \vec{V} = \vec{i} + \vec{j} + \vec{k}, \quad \vec{U} = \vec{i} - \vec{k}.\end{aligned}$$

Solution 1.2.2. (a) Here $\vec{U}$ is a unit vector and hence the required vector is

$$\begin{aligned}(\vec{U}.\vec{V})\vec{U} &= \Big[(2, 1, -1).\Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big)\Big]\Big(\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}}\Big)\\ &= \frac{2}{\sqrt{3}}\Big(\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}}\Big) = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k}).\end{aligned}$$

(b) Here $\vec{U}$ is not a unit vector. Let us create a unit vector in the direction of $\vec{U}$, namely

$$\vec{U}_1 = \frac{\vec{U}}{\|\vec{U}\|} = \frac{1}{\sqrt{6}}(2\vec{i} + \vec{j} + \vec{k}).$$

Now apply the formula on $\vec{V}$ and $\vec{U}_1$. The required vector is the following:

$$\begin{aligned}(\vec{V}.\vec{U}_1)\vec{U}_1 &= \Big[(1, -1, 1).\Big(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\Big)\Big]\Big(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\Big)\\ &= \frac{2}{\sqrt{6}}\Big(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\Big)\\ &= \frac{2}{6}(2\vec{i} + \vec{j} + \vec{k}).\end{aligned}$$

(c) Here $\vec{V}.\vec{U} = (1, 1, 1).(1, 0, -1) = 0$. Hence the projection vector is the null vector.

Definition 1.2.6 (Velocity vector)**.** In the language of engineers and physicists, the velocity is a vector with a certain direction and magnitude (length of the vector) and speed is the magnitude of the velocity vector.

For example, if $\vec{V} = (a, b)$ is the velocity vector as in Figure 1.2.6 then the direction of the vector is shown by the arrowhead there and the speed in this case is $\sqrt{a^2 + b^2} = \|\vec{V}\|$. If the velocity vector of a wind is $\vec{V} = 2\vec{i} + \vec{j} + \vec{k}$ in a 3-space then its speed is $\|\vec{V}\| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$.

Example 1.2.3. A plane is flying straight East horizontally at a speed of 200 km/hour and another plane is flying horizontally North-East at a speed of 600 km/hour. Draw the velocity vectors for both the planes.

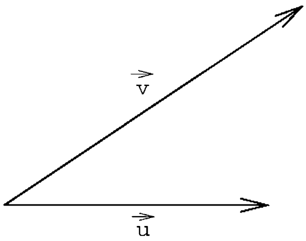

Figure 1.2.7: Velocity vectors.

Solution 1.2.3. If the velocity vectors for the two planes are denoted by $\vec{U}$ and $\vec{V}$ respectively, as shown in Figure 1.2.7 then the given information is that

$$\|\vec{U}\| = 200 \quad \text{and} \quad \|\vec{V}\| = 600.$$

If the direction of $\vec{U}$ is taken as the x-axis on the plane where the two vectors lie (displaced if necessary so that the tail-ends meet at $(0,0)$) then on this plane

$$\vec{U} = 200\vec{i} \quad \text{and} \quad \vec{V} = (600\cos 45°)\vec{i} + (600\sin 45°)\vec{j}$$
$$= \frac{600}{\sqrt{2}}\vec{i} + \frac{600}{\sqrt{2}}\vec{j}.$$

Example 1.2.4. A sail boat is steered to move straight East. There is a wind with a velocity in the North-East direction and with a speed of 50 km/hour. What is the speed of the boat if (a) the only force acting on the boat is the wind, (b) in addition to the wind the sail boat has a motor which is set for a speed of 20 km/hour.

Solution 1.2.4. (a) The only component here is the component of the wind velocity vector in the direction of the boat which is $\|\vec{V}\|\cos\theta$ if $\vec{V}$ is the velocity vector and θ is the angle $\vec{V}$ makes with the East direction (East direction is taken as the x-axis direction). We are given $\|\vec{V}\| = 50$ and $\theta = 45°$. Then the speed of the boat is $\|\vec{V}\|\cos\theta = \frac{50}{\sqrt{2}}$ and the velocity is $\vec{U} = \frac{50}{\sqrt{2}}\vec{i}$.

(b) In this case the above component plus the speed set by the engine are there. Then the combined speed is $\frac{50}{\sqrt{2}} + 20$ and the velocity vector is

$$\vec{U} = \left(\frac{50}{\sqrt{2}} + 20\right)\vec{i}.$$

1.2.8 Work done

When a force of magnitude F is applied on an object and the object is moved in the same direction of the force for a distance d then we say that the work done is Fd (F multiplied by d). For example if the force vector has the magnitude 20 units and the distance moved in the same direction of the force is 10 units then the work done is 200 units (force, distance and work are measured in different units such as force in *new-*

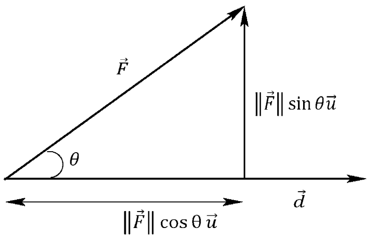

Figure 1.2.8: Work done.

tons, distance in kilometers and work in *joules*). Suppose that the force vector is in a certain direction and the distance moved is in another direction then what will be the work done? Let $\vec{F}$ be the force vector and $\vec{d}$ the displacement vector as shown in Figure 1.2.8.

Let the force vector $\vec{F}$ make an angle θ with the displacement vector $\vec{d}$. Then the projection of $\vec{F}$ in the direction of $\vec{d}$ is $\|\vec{F}\| \cos\theta$ [and the projection vector is $\|\vec{F}\|(\cos\theta)\vec{U}$ where $\vec{U}$ is a unit vector in the direction of $\vec{d}$. The component vector of $\vec{F}$ in the perpendicular direction to $\vec{d}$ is

$$\|\vec{F}\| \sin\theta = \vec{F} - \|\vec{F}\|(\cos\theta)\vec{U}.$$

This is not required in our computations]. Then the work done, denoted by w, is

$$\begin{aligned} w &= \|\vec{F}\| \cos\theta \|\vec{d}\| \\ &= \|\vec{F}\| \frac{(\vec{F}.\vec{d})}{\|\vec{F}\| \, \|\vec{d}\|} \|\vec{d}\| = \vec{F}.\vec{d}. \end{aligned} \tag{1.2.15}$$

Example 1.2.5. The ground force $\vec{F} = 5\vec{i} + 2\vec{j}$ of a wind moved a stone in the direction of the displacement $\vec{d} = \vec{i} + 3\vec{j}$. What is the work done by this wind in moving the stone?

Solution 1.2.5. According to (1.2.15) the work done is

$$w = \vec{F}.\vec{d} = (5,2).(1,3) = (5)(1) + (2)(6) = 11.$$

Example 1.2.6. Consider a triangle ABC with the angles denoted by A, B, C and the lengths of the sides opposite to these angles by a, b, c, as shown in Figure 1.2.9. Then show that

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Solution 1.2.6. Consider the vectors $\vec{AB}$ and $\vec{AC}$, starting from A and going to B and C respectively.

Then the vector $\vec{BC} = \vec{AC} - \vec{AB}$. Therefore

$$\begin{aligned} \|\vec{BC}\|^2 &= \|\vec{AC} - \vec{AB}\|^2 \\ &= \|\vec{AC}\|^2 + \|\vec{AB}\|^2 - 2\|\vec{AC}\| \, \|\vec{AB}\| \cos A. \end{aligned}$$

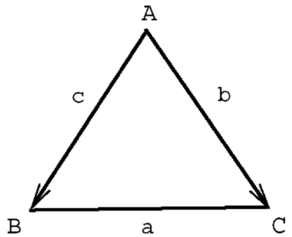

Figure 1.2.9: A triangle.

That is, $a^2 = b^2 + c^2 - 2bc\cos A$. Here we have used the fact that the square of the length is the dot product with itself:

$$\begin{aligned} a^2 &= \|\vec{AC} - \vec{AB}\|^2 \\ &= (\vec{AC} - \vec{AB}).(\vec{AC} - \vec{AB}) \\ &= (\vec{AC}.\vec{AC}) - (\vec{AC}).(\vec{AB}) - (\vec{AB}.\vec{AC}) + (\vec{AB}).(\vec{AB}) \\ &= \|\vec{AC}\|^2 + \|\vec{AB}\|^2 - 2(\vec{AC}.\vec{AB}) \\ &= \|\vec{AC}\|^2 + \|\vec{AB}\|^2 - 2\|\vec{AC}\|\,\|\vec{AB}\|\cos A \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

Exercises 1.2

1.2.1. Give geometric representation to the following vectors:

(a) $\vec{U} = 2\vec{i} - 3\vec{j}$, (b) $2\vec{U}$, (c) $-2\vec{U}$,
(d) $\vec{V} = \vec{i} + \vec{j}$, (e) $\vec{U} + \vec{V}$,
(f) $\vec{U} - \vec{V}$, (g) $\vec{V} - 2\vec{U}$, (h) $2\vec{U} + 3\vec{V}$.

1.2.2. Compute the angle between the following vectors:

(a) $\vec{U} = \vec{i} + \vec{j} - \vec{k}$, $\vec{V} = 2\vec{i} - \vec{j} + 3\vec{k}$;
(b) $\vec{U} = \vec{i} + \vec{j} + \vec{k}$, $\vec{V} = \vec{i} - 2\vec{j} + \vec{k}$;
(c) $\vec{U} = (1, -1, 2, 3, 5, -1)$, $\vec{V} = (2, 0, 0, -1, 1, 2)$.

1.2.3. Verify Cauchy–Schwartz inequality for $\vec{U}$ and $\vec{V}$ in the three cases in Exercise 1.2.2.

1.2.4. Normalize the following vector $\vec{U}$, then construct two vectors which are orthogonal among themselves as well as both are orthonormal to $\vec{U}$, where $\vec{U} = (1, 1, 1, 1)$.

1.2.5. Given the two vectors $\vec{U}_1 = (1, 1, 1, 1)$ and $\vec{U}_2 = (1, 2, -1, 1)$ construct two vectors $\vec{V}_1$ and $\vec{V}_2$ such that $\vec{V}_1$ is the normalized $\vec{U}_1$, $\vec{V}_2$ is a normalized vector orthogonal to $\vec{V}_1$ and both $\vec{V}_1$ and $\vec{V}_2$ are linear functions of $\vec{U}_1$ and $\vec{U}_2$.

1.2.6. Let $P = (x_0, y_0, z_0)$ a fixed point in 3-space, $Q = (x, y, z)$ an arbitrary point in 3-space. Construct the vector going from P to Q. Derive the equation to the plane where the vector $\vec{PQ}$ lies on the plane as well as another vector $\vec{N} = (a, b, c)$ is normal to this plane (Normal to a plane means orthogonal to every vector lying on the plane).

1.2.7. If $x - y + z = 7$ is a plane, (i) is the point $(1, 1, 1)$ on this plane? (ii) construct a normal to this plane with length 5, (iii) construct a plane parallel to the given plane and passing through the point $(1, 1, 2)$, (iv) construct a plane orthogonal to the given plane and passing through the point $(1, -1, 4)$.

1.2.8. Derive the equation to the plane passing through the points

$$(1, 1, -1), \quad (2, 1, 2), \quad (2, 1, 0).$$

1.2.9. Find the area of the parallelogram formed by the vectors (by completing it as in Figure 1.2.2 on the plane determined by the two vectors),

$$\vec{U} = 2\vec{i} + \vec{j} - \vec{k} \quad \text{and} \quad \vec{V} = \vec{i} - \vec{j} + 3\vec{k}.$$

1.2.10. Find the work done by the force $\vec{F} = 2\vec{i} - \vec{j} + 3\vec{k}$ for the displacement $\vec{d} = 3\vec{i} + \vec{j} - \vec{k}$.

1.2.11. A boat is trying to cross a river at a speed of 20 miles/hour straight across. The river flow downstream is 10 miles/hour. Evaluate the eventual direction and speed of the boat.

1.2.12. In Exercise 1.2.11 if the river flow speed is the same what should be the direction and speed of the boat so that it can travel straight across the river?

1.2.13. Evaluate the area of the triangle whose vertices are $(1, 0, 1)$, $(2, 1, 5)$, $(1, -1, 2)$ by using vector method.

1.2.14. Find the angle between the planes (angle between the normals to the planes)

$$x + y - z = 7 \quad \text{and} \quad 2x + y - 3z = 5.$$

1.2.15. In some engineering problems of signal processing a concept called *convolution of two vectors* is defined. Let $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ be two row vectors of the same order. Then the convolution, denoted by $X * Y$, is defined as follows: It is again a $1 \times n$ vector where the i-th element in $X * Y$ is given by

$$\begin{aligned} &x_1 y_i + x_2 y_{i-1} + \cdots + x_i y_1 \\ &\quad + x_{i+1} y_n + x_{i+2} y_{n-1} + \cdots + x_n y_{i+1}. \end{aligned}$$

For example, for $n = 2$

$$X * Y = (x_1, x_2) * (y_1, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1)$$

(a) Write down the explicit expression for $(x_1, x_2, x_3) * (y_1, y_2, y_3)$.

(b) Show that the operator $*$ is commutative as well as associative for a general n.
(c) Evaluate $(1, 0, -1, 2) * (3, 4, 5, -2)$.

1.2.16. Find the angle between the planes

$$x - y + z = 2 \quad \text{and} \quad 2x + 3y - 4z = 8.$$

1.2.17. Evaluate the area of the triangle whose vertices are $(1, 1, 1)$, $(2, 5, 3)$, $(1, -1, -1)$.

1.2.18. Evaluate the area of the parallelogram determined by the vectors $U = (1, -1, 2, 5)$ and $V = (1, 1, -1, -1)$.

1.3 Linear dependence and linear independence of vectors

Consider the vectors $U_1 = (1, 0, -1)$ and $U_2 = (1, 1, 1)$. For arbitrary scalars a_1 and a_2 let us try to solve the equation

$$a_1 U_1 + a_2 U_2 = O \tag{1.3.1}$$

to see whether there exist nonzero a_1 and a_2 such that (1.3.1) is satisfied.

$$a_1 U_1 + a_2 U_2 = O \Rightarrow$$
$$a_1(1, 0, -1) + a_2(1, 1, 1) = O = (0, 0, 0).$$

That is,

$$(a_1 + a_2, a_2, -a_1 + a_2) = (0, 0, 0) \Rightarrow$$
$$a_1 + a_2 = 0, \quad a_2 = 0, \quad -a_1 + a_2 = 0.$$

The only values of a_1 and a_2 satisfying the three equations $a_1 + a_2 = 0$, $a_2 = 0$ and $-a_1 + a_2 = 0$ are $a_1 = 0$ and $a_2 = 0$. This means that the only solution for a_1 and a_2 in (1.3.1) is $a_1 = 0$ and $a_2 = 0$. Observe that $a_1 = 0$, $a_2 = 0$ is always a solution to the equation (1.3.1). But here we have seen that $a_1 = 0$, $a_2 = 0$ is the only solution. Now, let us look at another situation. Consider the vectors

$$U_1 = (1, 1, 1), \quad U_2 = (1, -1, 2), \quad U_3 = (2, 0, 3).$$

Solve the equation

$$a_1 U_1 + a_2 U_2 + a_3 U_3 = O \tag{1.3.2}$$

for a_1, a_2, a_3. Then

$$a_1 U_1 + a_2 U_2 + a_3 U_3 = O \Rightarrow$$
$$a_1(1, 1, 1) + a_2(1, -1, 2) + a_3(2, 0, 3) = (0, 0, 0).$$

That is,

$$(a_1 + a_2 + 2a_3, a_1 - a_2, a_1 + 2a_2 + 3a_3) = (0, 0, 0).$$

This means,

$$a_1 + a_2 + 2a_3 = 0, \tag{i}$$

$$a_1 - a_2 = 0, \tag{ii}$$

$$a_1 + 2a_2 + 3a_3 = 0. \tag{iii}$$

From (ii), $a_1 = a_2$; substituting in (i), $a_1 = a_2 = -a_3$; substituting in (iii) the equation is satisfied. Then there are infinitely many non-zero a_1, a_2, a_3 for which (1.3.2) is satisfied. For example, $a_1 = 1 = a_2, a_3 = -1$ will satisfy (1.3.2). In the above considerations we have two systems of vectors. In one system the only possibility for the coefficient vector is the null vector which means that no vector can be written as a linear function of the other vectors. In the other case the coefficient vector is not null which means that at least one of the vectors there can be written as a linear combination of others.

Definition 1.3.1 (Linear independence)**.** Let $U_1, U_2, \dots, U_k$ be k given non-null n-vectors, where k is finite. Consider the equation

$$a_1U_1 + a_2U_2 + \cdots + a_kU_k = O \tag{1.3.3}$$

where $a_1, \dots, a_k$ are scalars. If the only possibility for (1.3.3) to hold is when $a_1 = 0, \dots, a_k = 0$ then the vectors $U_1, \dots, U_k$ are called *linearly independent*. If there exists at least one non-null vector $(a_1, \dots, a_k)$ such that (1.3.3) is satisfied then the system of vectors $U_1, \dots, U_k$ are *linearly dependent*.

If a non-null vector $(a_1, \dots, a_k)$ exists then at least one of the elements is nonzero. Let $a_1 \neq 0$. Then from (1.3.3)

$$U_1 = -\frac{a_2}{a_1}U_2 - \cdots - \frac{a_k}{a_1}U_k. \tag{1.3.4}$$

That is, U_1 can be written as a linear function of $U_2, \dots, U_k$. Note that not all $a_2, \dots, a_k$ can be zeros. If they are all zeros then from (1.3.4) U_1 is a null vector. But a null vector is not included in our definition. Thus at least one of them can be written as a linear function of the others if $U_1, \dots, U_k$ are linearly dependent. If they are linearly independent then none can be written as a linear function of the others.

(i) A null vector is counted among dependent vectors. A set consisting of one non-null vector is counted as an independent system of vectors.

Example 1.3.1. Show that the basic unit vectors $e_1, \dots, e_n$ are linearly independent.

Solution 1.3.1. Consider the equation

$$a_1e_1 + \cdots + a_ne_n = O \Rightarrow$$

$$\begin{aligned} a_1(1,0,\dots,0)+\dots+a_n(0,\dots,0,1)=(0,\dots,0) &\Rightarrow \\ (a_1,\dots,a_n)=(0,\dots,0) &\Rightarrow \\ a_1=0,\dots,a_n=0 & \end{aligned}$$

is the only solution, which means that $\mathrm{e}_1,\dots,\mathrm{e}_n$ are linearly independent.

Example 1.3.2. Show that a system of non-null mutually orthogonal vectors are linearly independent.

Solution 1.3.2. Let $V_1,\dots,V_k$ be a system of mutually orthogonal vectors. Consider the equation

$$a_1V_1+\dots+a_kV_k=O.$$

Take the dot product on both sides with respect to V_1. Then we have

$$a_1V_1.V_1+a_2V_2.V_1+\dots+a_kV_k.V_1=O.V_1=O.$$

But $V_j.V_1=0$ for $j\neq 1$ and $V_1.V_1=\|V_1\|^2\neq 0$. This means that $a_1=0$. Similarly $a_2=0,\dots,a_k=0$ which means that $V_1,\dots,V_k$ are linearly independent. This is a very important result.

(ii) Every set of mutually orthogonal non-null vectors are linearly independent.
(iii) Any finite collection of vectors containing the null vector is counted as a linearly dependent system of vectors. If S_1 and S are two finite collections of vectors where S_1 is a subset of S, that is, $S_1\subset S$, then the following hold: If S_1 is a linearly dependent system then S is also a linearly dependent system. If S is a linearly independent system then S_1 is also a linearly independent system.

Example 1.3.3. Check the linear dependence of the following sets of vectors:

(a) $U_1=(1,2,1),\quad U_2=(1,1,1);$
(b) $U_1=(1,-1,2),\quad U_2=(1,1,0);$
(c) $U_1=(1,2,1),\quad U_2=(1,-1,1),\quad U_3=(3,3,3).$

Solution 1.3.3. (a) For two vectors to be dependent one has to be a non-zero scalar multiple of the other. Hence U_1 and U_2 here are linearly independent.

(b) By inspection $U_1.U_2=0$ and hence they are orthogonal thereby linearly independent.

(c) U_1 and U_2 are evidently linearly independent, being not multiples of each other. By inspection $U_3=2U_1+U_2$ and hence the set $\{U_1,U_2,U_3\}$ is a linearly dependent system.

(iv) Linear dependence or independence in a system of vectors is not altered by scalar multiplication of the vectors by non-zero scalars.

This result can be easily seen from the definition itself. Let the n-vectors $U_1, \dots, U_k$ be linearly independent. Then

$$a_1 U_1 + \cdots + a_k U_k = O \Rightarrow a_1 = 0, \dots, a_k = 0.$$

Let $c_1, \dots, c_k$ be non-zero scalars. If $a_i = 0$ then $a_i c_i = 0$ and vice versa since $c_i \neq 0$, $i = 1, 2, \dots, k$. Thus

$$a_1(c_1 U_1) + \cdots + a_k(c_k U_k) = O \Rightarrow a_1 = 0, \dots, a_k = 0.$$

On the other hand, if $U_1, \dots, U_k$ are linearly dependent then at least one of them can be written as a linear function of the others. Let

$$U_1 = b_2 U_2 + \cdots + b_k U_k$$

where $b_2, \dots, b_k$ are some constants, at least one of them nonzero. Then for $c_1 \neq 0, \dots, c_k \neq 0$

$$c_1 U_1 = \frac{c_1 b_2}{c_2}(c_2 U_2) + \cdots + \frac{c_1 b_k}{c_k}(c_k U_k).$$

Thus $c_1 U_1, \dots, c_k U_k$ are linearly dependent.

We have another important result on linear independence.

(v) Linear independence or dependence in a system of vectors is not altered by adding a scalar multiple of any vector in the system to any other vector in the system.

This result is easy to establish. Let the system $U_1, \dots, U_k$ of n-vectors be linearly independent. Then

$$a_1 U_1 + \cdots + a_k U_k = O \Rightarrow a_1 = 0, \dots, a_k = 0.$$

Now, consider a new system $U_1, c\,U_1 + U_2, \dots, U_k$. [That is, U_2 is replaced by $c\,U_1 + U_2$, $c \neq 0$. In other words, $c\,U_1$ is added to U_2.] Consider the equation

$$a_1 U_1 + a_2(c\,U_1 + U_2) + a_3 U_3 + \cdots + a_k U_k = O.$$

That is,

$$(a_1 + c a_2) U_1 + a_2 U_2 + \cdots + a_k U_k = O.$$

Then since $U_1, \dots, U_k$ are linearly independent $a_1 + c a_2 = 0$, $a_2 = 0$, …, $a_k = 0$ which means $a_1 = 0$ also which establishes that the system of vectors $U_1, cU_1 + U_2, U_3, \dots, U_k$ is linearly independent. A similar procedure establishes that if the original system is linearly dependent then the new system is also linearly dependent.

By combining the results (iii) and (iv) above we can have the following result:

(vi) Consider a finite collection of n-vectors. If any number of vectors in this collection are multiplied by nonzero scalars or a linear function of any number of them is added to any member in the set, linear independence or dependence in the system is preserved. That is, if the original system is linearly independent then the new system is also linearly independent and if the original system is dependent then the new system is also linearly dependent.

Example 1.3.4. Check to see whether the following system of vectors is linearly independent or dependent:

$$\begin{aligned} U_1 &= (1, 0, 2, -1, 5) \\ U_2 &= (-1, 1, 1, -1, 2) \\ U_3 &= (2, 1, 7, -4, 17) \end{aligned}$$

Solution 1.3.4. Since nonzero scalar multiplication and addition do not alter independence or dependence let us create new systems of vectors. In what follows the following standard notations will be used:

A few standard notations

$$\text{“}\alpha(i) \Rightarrow \text{” means the } i\text{-th vector multiplied by } \alpha \tag{1.3.5}$$

In this operation the i-th vector in the set is replaced by α (Greek letter alpha) times the original i-th vector. For example “$-3(1) \Rightarrow$” means that “the first vector multiplied by -3 gives”, that is, the new first vector is the original first vector multiplied by -3.

$$\text{“}\alpha(i) + (j) \Rightarrow \text{” means } \alpha \text{ times the } i\text{-th vector added to the } j\text{-th vector} \tag{1.3.6}$$

In this operation the original i-th vector remains the same whereas the new j-th vector is the original j-th vector plus α times the original i-th vector. Let us apply these types of operations on U_1, U_2, U_3, remembering that linear independence or dependence is preserved.

$$\begin{aligned} (1) + (2) \ \Rightarrow \ U_1 &= (1, 0, 2, -1, 5) \\ V_2 &= (0, 1, 3, -2, 7) \\ U_3 &= (2, 1, 7, -4, 17) \end{aligned}$$

In the above operation the second vector U_2 is replaced by $U_2 + U_1 = V_2$. Let us continue the operations.

$$-2(1) + (3) \ \Rightarrow \ U_1 = (1, 0, 2, -1, 5)$$

$$V_2 = (0, 1, 3, -2, 7)$$
$$V_3 = (0, 1, 3, -2, 7)$$

In the set U_1, V_2, V_3 we will do the next operation.

$$-(2) + (3) \Rightarrow U_1 = (1, 0, 2, -1, 5)$$
$$V_2 = (0, 1, 3, -2, 7)$$
$$W_3 = (0, 0, 0, 0, 0)$$

Here W_3 is obtained by adding (−1) times V_2 to V_3 or replacing V_3 by $V_3 - V_2 = W_3$. By the above sequence of operations W_3 has become a null vector which by definition is dependent. Hence the original system U_1, U_2, U_3 is a linearly dependent system.

Example 1.3.5. Check the linear dependence or independence of the following system of vectors:

$$U_1 = (2, -1, 1, 1, 3, 4)$$
$$U_2 = (5, 2, 1, -1, 2, 1)$$
$$U_3 = (1, -1, 1, 1, 1, 4)$$

Solution 1.3.5. Since linear dependence or independence is not altered by the order in which the vectors are selected we will write U_3 first and write only the elements in 3 rows and 6 columns as follows, rather than naming them as U_3, U_1, U_2:

$$\begin{array}{cccccc} 1 & -1 & 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & 1 & 3 & 4 \\ 5 & 2 & 1 & -1 & 2 & 1 \end{array}$$

We have written them in the order U_3, U_1, U_2 to bring a convenient number, namely 1, at the first row first column position. This does not alter linear independence or dependence in the system. Now, we will carry out more than one operations at a time. [We add (−2) times the first row to the second row and (−5) times the first row to the third row. The first row remains the same. The result is the following:]

$$\begin{array}{rcccccc} & 1 & -1 & 1 & 1 & 1 & 4 \\ -2(1) + (2); -5(1) + (3) \Rightarrow & 0 & 1 & -1 & -1 & 1 & -4 \\ & 0 & 7 & -4 & -6 & -3 & -19 \end{array}$$

[On the new configuration we add the second row to the first row and (−7) times the second row to the third row. The second row remains the same. The net result is the following:]

$$\begin{array}{rcccccc} & 1 & 0 & 0 & 0 & 2 & 0 \\ (2) + (1); -7(2) + (3) \Rightarrow & 0 & 1 & -1 & -1 & 1 & -4 \\ & 0 & 0 & 3 & 1 & -10 & 9 \end{array}$$

[The third row is divided by 3. The third row changes.]

$$\frac{1}{3}(3) \Rightarrow \begin{matrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 & -4 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{10}{3} & 3 \end{matrix}$$

[This operation is done to bring a convenient number at the third column position on the third row. Now we add the new third row to the second row. The new third row remains the same.]

$$(3) + (2) \Rightarrow \begin{matrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{7}{3} & -1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{10}{3} & 3 \end{matrix}$$

The aim in the above sequences of operations is to bring a unity at all leading diagonal (the diagonal from the upper left-end corner down) positions, if possible. Interchanges of rows can be done if necessary to achieve the above aim, because interchanges do not alter the linear independence or dependence. During such a process if any row becomes null then automatically the original system, represented by the starting rows, is dependent. If no row becomes null during the process then at the end of the process look at the final first, second, etc columns. In our example above look at the first column. No non-zero linear combination of the second and third rows can create a 1 at the first position. Hence the first row cannot be written as a linear function of the second and third rows. Now look at the second column. By the same argument above the second row cannot be written as a linear function of the first and third rows. Now look at the third column. By the same argument the third row cannot be written as a linear combination of the first and second rows. Hence all the three rows are linearly independent or the original system $\{U_1, U_2, U_3\}$ is a linearly independent system.

The above procedure is called a *sweep-out procedure*. Then the principles to remember in a sweep-out procedure are the following: Assume that the system consists of m vectors, each is an n-vector.

Principles in a sweep-out procedure

(1) *Write the given vectors as rows, interchange if necessary to bring a convenient nonzero number,* 1 *if available, at the first row first column position. Do not interchange columns, the vectors will be altered.*
(2) *Add suitable multiples of the first row to the second, third, …, m-th row to make the first column elements, except the first element, zeros.*
(3) *Start with the second row. Interchange* 2nd*, …, m-th rows if necessary to bring a convenient nonzero number at the second position on the second row.*
(4) *Add suitable multiples of the second row to the first row, third row, …, m-th row to make all elements in the second column, except the second element, zeros.*

(5) *Repeat the process with the third, fourth etc rows until all the leading diagonal elements are non-zeros, unities if possible.*
(6) *During the process if any row becomes null then shift it to the bottom position. If at any stage a vector has become null then the system is dependent. If all the leading diagonal elements are non-zeros when all other elements in the corresponding columns are wiped out (made zeros) by the above process then the system is linearly independent.*
(7) *If the first r, for some r, leading diagonal elements are non-zeros, none of the rows has become null so far and the* $(r+1)$*th elements in all the remaining rows are zeros then continue the process with the* $(r+2)$*th element on the* $(r+1)$*th row and so on. If no row has become null by the end of the whole process then all the rows are linearly independent.*
(8) *Division of a row by a non-zero scalar usually brings in fractions. Hence multiply the rows with appropriate numbers to avoid fractions and to achieve the* sweep-out *process.*

The leading diagonal elements need not be brought to unities to check for linear dependence or independence. Only nonzero elements are to be brought to the diagonal positions, if possible. When doing the operations, try to bring the system to a triangular format by reducing all elements below the leading diagonal to zeros, if possible. When the system is in a triangular format all elements above nonzero diagonal elements can be simply put as zeros because this can always be achieved by operating with the last row first, wiping out all last column elements except the last column last row element, then last but one column elements and so on. Thus all elements above nonzero diagonal elements can be simply put as zeros once the matrix is in a triangular format.
(9) *If the vectors to be checked for linear independence or dependence are column vectors then write them as rows before executing a sweep-out process. This is done only for convenience because operations on rows are easier to visualize.*
(10) *When doing a sweep-out process always write first the row that you are operating with because this row is not changing and others can change as a result of the operations.*

Example 1.3.6. Check for linear independence or dependence in the following system of vectors:

$$U_1 = (2, 0, 1, 5)$$
$$U_2 = (1, -1, 1, 1)$$
$$U_3 = (4, 2, 2, 8)$$

Solution 1.3.6. For convenience write in the order U_2, U_1, U_3 and write only the elements and continue with the sweep-out process.

$$\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 2 & 0 & 1 & 5 \\ 4 & 2 & 2 & 8 \end{array} \quad -2(1)+(2); \quad -4(1)+(3) \Rightarrow \begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 6 & -2 & 4 \end{array}$$

$$2(1) \Rightarrow \begin{array}{cccc} 2 & -2 & 2 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 6 & -2 & 4 \end{array} \quad (2)+(1); \quad -3(2)+(3) \Rightarrow \begin{array}{cccc} 2 & 0 & 1 & 5 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 1 & -5 \end{array}$$

$$(3)+(2); \quad -(3)+(1) \Rightarrow \begin{array}{cccc} 2 & 0 & 0 & 10 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}$$

The leading diagonal elements are $2, 2, 1$ which are non-zeros and hence the system is linearly independent. [During the process above the first row is multiplied by 2 in order to avoid fractions in the rest of the operations.]

Note that in the above operations the row that you are operating with remains the same and the other rows, to which constant multiples are added, change. In the last form above, are all the four columns linearly independent? Evidently not. The last column = 5 (column 1)–(column 2)–5(column 3).

If our aim is only to check for linear independence or dependence then we need to bring the original set to a triangular type format. In the second step above the operation 2(1) and in the third stage the operation (2) + (1) need not be done. That is,

$$\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 6 & -2 & 4 \end{array} \quad -3(2)+(3) \Rightarrow \begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 1 & -5 \end{array}$$

Now we have the triangular type format with nonzero diagonal elements. Note that the first row cannot be written as a linear function of the second and third rows. Similarly no row can be written as a linear function of the other two. At this stage if we wish to create a diagonal format for the first three columns then by using the third row one can wipe out all other elements in the third column, then by using the second row we can wipe out all other elements in the second column. In other words, we can simply replace all those elements by zeros, then only the last column will change.

$$\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 1 & -5 \end{array} \rightarrow \begin{array}{cccc} 1 & -1 & 0 & 6 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array} \quad \text{(operating with the third row)}$$

$$\rightarrow \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array} \quad \text{(operating with the second row)} \rightarrow \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -5 \end{array}$$

dividing the second row by 2. Thus, the first three columns are made basic unit vectors, the same procedure if we wish to create unit vectors in the first r columns and if there are r linearly independent rows.

At a certain stage, say the rth stage, suppose that all elements in the $(r+1)$th column below the rth row are zeros. Then start with a nonzero element in the remaining configuration of the columns in the remaining row and proceed to create a triangular format. For example, consider the following situation:

$$\begin{matrix} 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

The first two rows are evidently linearly independent. Our procedure of triangularization cannot proceed. Write the 5th row in the 3rd row position to get the matrix on the right above. Now we see that the new 3rd, 4th and 5th rows form a triangular type format. This shows that all the five rows are linearly independent. Note that by using the last row one can wipe out all other elements in the 7th column. Then by using the 4th row we can wipe out all other elements in the 5th column. Then by using the 3rd row we can wipe out all other elements in the 4th column. Then by using the second row we can wipe out all other elements in the second column. Now, one can see linear independence of all the five rows clearly. In the light of the above examples and discussions we can state the following result:

(vii) There cannot be more than n mutually orthogonal n-vectors and there cannot be more than n linearly independent n-vectors.

It is not difficult to establish this result. Consider the n-vectors $U_1, \dots, U_n, U_{n+1}$, that is, $n+1$ vectors of n elements each. Write the $n+1$ vectors as $n+1$ rows and apply the above sweep-out process. If the first n vectors are linearly independent then all the n leading diagonal spots have nonzero entries with all elements in the corresponding columns zeros. Thus automatically the $(n+1)$th row becomes null. Hence the $(n+1)$th row depends on the other n rows or the maximum number of linearly independent n-vectors possible is n.

If possible, let $V_1, \dots, V_{n+1}$ be mutually orthogonal n-vectors. From what we proved just above not all these $n+1$ vectors can be linearly independent. Then the $(n+1)$th can be written as a linear function of the other n vectors. Then there exists a non-null vector $b = (b_1, \dots, b_n)$ such that

$$V_{n+1} = b_1 V_1 + \dots + b_n V_n.$$

Take the dot product on both sides with respect to V_i. If all $V_1, \dots, V_{n+1}$ are mutually orthogonal then we have

$$0 = 0 + b_i \|V_i\|^2 + 0 \Rightarrow b_i = 0, \quad i = 1, \dots, n$$

since $\|V_i\| \neq 0,\ i = 1, \dots, n$. This then contradicts the fact that b is a non-null vector. Thus they cannot be all mutually orthogonal. Since the orthogonal vectors are linearly independent, proved earlier, the maximum number of n-vectors which are mutually orthogonal is n.

1.3.1 A vector subspace

The vectors in our discussion so far are ordered n-tuples of real numbers. The notions of vector spaces, dimension etc will be introduced for such vectors. Then later we will generalize these ideas to cover some general objects called vectors satisfying some general postulates. Consider, for example, two given vectors

$$U_1 = (1, 0, -1) \quad \text{and} \quad U_2 = (2, 3, 1).$$

Evidently U_1 and U_2 are linearly independent. Two vectors being dependent means one is a multiple of the other. Consider a collection S_1 of vectors which are spanned by U_1 and U_2 by the following process. Every scalar multiple of U_1 as well as of U_2 is in S_1. For example

$$\begin{aligned} 3U_1 &= 3(1, 0, -1) = (3, 0, -3) \in S_1 \\ -2U_2 &= -2(2, 3, 1) = (-4, -6, -1) \in S_1 \\ 0U_1 &= (0, 0, 0) \in S_1. \end{aligned}$$

Every linear combination of U_1 and U_2 is also in S_1. For example,

$$\begin{aligned} 2U_1 - 5U_2 &= (2, 0, -2) + (-10, -15, -5) = (-8, -15, -7) \in S_1 \\ U_1 + U_2 &= (1, 0, -1) + (2, 3, 1) = (3, 3, 0) \in S_1 \\ U_1 + 0U_2 &= U_1 \in S_1. \end{aligned}$$

Since a scalar multiplication and then addition will create a linear combination the basic operations are scalar multiplication and addition. Then every element in S_1, elements are vectors, can be written as a linear combination of U_1 and U_2. In this case we say that S_1 is *spanned* or *generated* or *created* by U_1 and U_2. Then we say that the collection $\{U_1, U_2\}$ is a *spanning set* of S_1.

Definition 1.3.2 (Vector subspace)**.** Let S be a collection of vectors such that if $V_1 \in S$ then $cV_1 \in S$ where c is any scalar, including zero, and if $V_1 \in S$ and $V_2 \in S$ then $V_1 + V_2 \in S$. Then S is called a vector subspace.

Another way of defining S is that it is a collection which is closed under scalar multiplication and addition. When the elements of S are n-vectors (ordered set of n real numbers) then the operations “scalar multiplication” and “addition” are easily defined and many properties such as commutativity,

$$V_1 + V_2 = V_2 + V_1,$$

associativity

$$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$$

and so on are easily established. But if the elements of S are some general objects then the operations “scalar multiplication” and “addition” are to be redefined and then all types of extra properties are to be double-checked before constructing such a collection which is closed under “scalar multiplication” and “addition”. A more general definition of S will be introduced later. For the time being the elements in our S are all n-tuples of real numbers. The null vector is automatically an element of any such S. That is, $O \in S$. If $V \in S$ then $V + O = V$, $-V \in S$, $V - V = O$.

Definition 1.3.3 (A spanning set of a vector subspace)**.** A collection of vectors which span the whole of a given vector subspace is called a spanning set of that vector subspace.

Note that there can be a number of spanning sets for a given subspace S. In our illustrative example $C_1 = \{U_1, U_2\}$, where $U_1 = (1, 0, -1)$, $U_2 = (2, 3, 1)$, spans the subspace S_1. The same subspace could be spanned by $C_2 = \{U_1, U_2, U_1 + U_2\}$ or $C_3 = \{U_1, U_2 + 3U_1\}$ or $C_4 = \{U_2, U_1 - U_2, 2U_1 + 5U_2, U_1\}$ and so on. Thus, for a given subspace there can be infinitely many spanning sets. In all the spanning sets, $C_1, \dots, C_4$ above the smallest number of linearly independent vectors which can span S_1 or the maximum number of linearly independent vectors in all those spanning sets is 2.

Definition 1.3.4 (A basis for a vector subspace)**.** A set of all linearly independent vectors in a spanning set of a vector subspace is called a basis for that vector subspace. That is, a basis is a spanning set consisting of only linearly independent vectors.

As there can be many spanning sets for a given vector subspace there can be infinitely many bases for a given vector subspace. In our illustrative example $B_1 = \{U_1, U_2\}$ is a basis, $B_2 = \{U_1, U_2 + 3U_1\}$ is another basis, $B_3 = \{U_2, U_1 - U_2\}$ is a third basis, but $B_4 = \{U_2, U_1 - U_2, 2U_1 + U_2\}$ is not a basis because one vector, namely

$$2U_1 + U_2 = 2(U_1 - U_2) + 3U_2,$$

is a linear function of the other two. B_4 is a spanning set but not a basis. We are imposing two conditions for a basis of a vector subspace. (i) A basis is a spanning set for that vector subspace; (ii) A basis consists of only linearly independent vectors.

Example 1.3.7. Construct 3 bases for the vector subspace spanned by the following set of vectors:

$$U_1 = (1, 1, 1), \quad U_2 = (1, -1, 2), \quad U_3 = (2, 0, 3).$$

Solution 1.3.7. Our first step is to determine the number of linearly independent vectors in the given set so that one set of the maximum number of linearly independent vectors can be collected. Let us apply the sweep-out process, writing the vectors as rows.

$$\begin{array}{rrr} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{array} \quad -1(1)+(2); \quad -2(1)+(3) \Rightarrow \begin{array}{rrr} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{array}$$

$$-1(2)+(3) \Rightarrow \begin{array}{rrr} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{array}$$

Thus the whole vector subspace S, which is spanned by $\{U_1, U_2, U_3\}$, can also be spanned by $\{V_1, V_2\}$ where

$$V_1 = (1,1,1), \quad \text{and} \quad V_2 = (0,-2,1).$$

Hence one basis for S is $B_1 = \{V_1, V_2\}$. Any set of 2 linearly independent vectors that can be constructed by using V_1 and V_2 is also a basis for S. For example,

$$B_2 = \{2V_1, 3V_2\}, \quad B_3 = \{V_1, V_2 + V_1\}$$

are two more bases for S. Infinitely many such bases can be constructed for the same vector subspace S. This means that if we start with V_1 only then we can span only a part of S or a subset of S, say S_1. This S_1 consists of all scalar multiples of V_1. Similarly if we start with only V_2 we can only span a part of S or a subset of S, say S_2. This S_2 consists of scalar multiples of V_2. Note that the union of S_1 and S_2, $S_1 \cup S_2$, is not S. All linear functions of V_1 and V_2 are also in S. Hence $S_1 \cup S_2$ is only a subset of S.

Definition 1.3.5 (Dimension of a vector subspace). The maximum number of linearly independent vectors in a spanning set of S or the smallest number of linearly independent vectors which can span the whole of S or the number of vectors in a basis of S is called the dimension of the subspace S.

In our illustrative Example 1.3.7 the dimension of S is 2. In general, observe that for a given subspace S there cannot be two different bases B_1 and B_2 where in B_1 the number of linearly independent vectors is m_1 whereas in B_2 that number is m_2 with $m_1 \neq m_2$. If possible let $m_1 < m_2$. Then every vector in S is a linear function of these m_1 vectors and hence by definition there cannot be a vector in S which is linearly independent of these m_1 vectors. That means m_1 must be equal to m_2.

One more point is worth observing. Since every 3-vector can be written as a linear function of the basic unit vectors, the vectors $U_1 = (1,0,-1)$ and $U_2 = (2,3,1)$ in our illustrative example can be written as linear functions of the basic unit vectors

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1).$$

Note that

$$U_1 = e_1 - e_3 \quad \text{and} \quad U_2 = 2e_1 + 3e_2 + e_3.$$

In the set $B = \{e_1, e_2, e_3\}$ there are 3 linearly independent vectors. We have already seen that U_1 and U_2 can be written as linear functions of these unit vectors. Thus this set B could have spanned not only S of our Example 1.3.7, call it $\tilde{S}$, the vector subspace spanned by U_1 and U_2, but also a much larger space S where our $\tilde{S}$ is a subset or $\tilde{S}$ is contained in S or $\tilde{S} \subset S$ or $\tilde{S}$ is a subspace there. This is why we used the phrase "subspace" in our definitions. Incidently, since $S \subset S$ we can also call S itself a subspace.

Definition 1.3.6 (Orthogonal subspaces)**.** Consider two subspaces, S and S^* of n-vectors such that for every vector $U \in S$ and every vector $V \in S^*$, $U.V = 0$. That is, vectors in S are orthogonal to the vectors in S^* and vice versa. Then S and S^* are called subspaces orthogonal to each other.

Obviously, since the same vector cannot be orthogonal to itself (except the null vector) the same non-null vector cannot be present in S as well as in S^*. For example, if $U_1 = (1, 1, 1)$ is in S then $V_1 = (1, -2, 1)$ and $V_2 = (1, 0, -1)$ are two possible vectors in S^* since $U_1.V_1 = 0$ and $U_1.V_2 = 0$. But V_1 or V_2 or both need not be present in S^*.

Example 1.3.8. If $U = (1, 2, -1) \in S$ and if S is spanned by U itself then what is the maximum possible number of linearly independent vectors in a subspace S^* orthogonal to S? Construct a basis for such an S^*.

Solution 1.3.8. Let $X = (x_1, x_2, x_3)$ be in S^*. Then

$$U.X = 0 \;\Rightarrow\; x_1 + 2x_2 - x_3 = 0. \tag{1.3.7}$$

The maximum number of linearly independent 3-vectors possible is 3. Orthogonal vectors are linearly independent. Hence the maximum number of linearly independent X possible is $3 - 1 = 2$. In order to construct a basis we construct two linearly independent X from equation (1.3.7). For example, $X_1 = (-2, 1, 0)$ and $X_2 = (-1, 1, 1)$ are two linearly independent solutions of (1.3.7). Hence $\{X_1, X_2\}$ is a basis for the orthogonal space S^*. There can be many such bases for S^*, each basis will consist of two linearly independent solutions of (1.3.7). Note that the subspace spanned by $X_1 = (-2, 1, 0)$ alone will be orthogonal to S as well as the subspace spanned by $X_2 = (-1, 1, 1)$ alone will be orthogonal to S. But we were looking for that orthogonal subspace consisting of the maximum number of linearly independent solutions of (1.3.7).

Definition 1.3.7 (Orthogonal complement of a subspace)**.** Let S be a vector subspace and S^* a subspace orthogonal to S. If all the maximum possible number of linearly independent vectors, orthogonal to S, are in S^* then S^* is the orthogonal complement of S and it is usually written as $S^* = S^{\perp}$.

(viii) If the dimension of a vector subspace S of n-vectors is $m < n$ and if S^* is the orthogonal complement of S then the dimension of S^* is $n-m$. If the dimension of S is n then the dimension of S^* is zero which means S^* contains only the null vector.

1.3.2 Gram–Schmidt orthogonalization process

From a given set $U_1, \dots, U_k$ of k linearly independent n-vectors can we create another set $V_1, \dots, V_k$ of vectors which form an orthonormal system and each V_j is a linear function of the U_j's? That is, $V_i.V_j = 0,\ i \neq j$ and $\|V_j\| = 1,\ j = 1, \dots, k$. The answer to this question is in the affirmative and the process by which we obtain the set $V_1, \dots, V_k$ from the set $U_1, \dots, U_k$ is known as the Gram–Schmidt orthogonalization process. This process can be described as follows: Take the normalized U_1 as V_1. Construct a V_2 where

$$V_2 = \frac{W_2}{\|W_2\|}, \quad W_2 = U_2 + aV_1$$

where a is a scalar quantity. Since we require V_1 to be orthogonal to W_2 we have $W_2.V_1 = 0$ or $U_2.V_1 + aV_1.V_1 = U_2.V_1 + a = 0$ since $V_1.V_1 = 1$. Then $a = -U_2.V_1$. That is, $W_2 = U_2 - (U_2.V_1)V_1$ where $U_2.V_1$ is the dot product of U_2 and V_1. Note that

$$W_2.V_1 = U_2.V_1 - (U_2.V_1)V_1.V_1 = U_2.V_1 - U_2.V_1 = 0$$

since $V_1.V_1 = \|V_1\|^2 = 1$. Thus V_1 and V_2 are orthogonal to each other and each one is a normalized vector. Now, consider the general formula

$$\begin{aligned} W_j &= U_j - (U_j.V_1)V_1 - (U_j.V_2)V_2 \\ &\quad - \dots - (U_j.V_{j-1})V_{j-1} \quad \text{for } j = 2, \dots, k \text{ and} \\ V_j &= \frac{W_j}{\|W_j\|}. \end{aligned} \tag{1.3.8}$$

For example,

$$\begin{aligned} W_3 &= U_3 - (U_3.V_1)V_1 - (U_3.V_2)V_2, \\ V_3 &= \frac{W_3}{\|W_3\|}. \end{aligned}$$

Let us see whether W_3 is orthogonal to both V_1 and V_2. Take the dot product

$$W_3.V_1 = U_3.V_1 - (U_3.V_1)V_1.V_1 - (U_3.V_2)V_2.V_1.$$

It is already shown that $V_2.V_1 = 0$ and $V_1.V_1 = \|V_1\|^2 = 1$. Hence

$$W_3.V_1 = U_3.V_1 - (U_3.V_1) = 0.$$

Now take,

$$W_3.V_2 = U_3.V_2 - (U_3.V_1)V_1.V_2 - (U_3.V_2)V_2.V_2$$
$$= U_3.V_2 - 0 - (U_3.V_2) = 0$$

since $V_2.V_2 = \|V_2\|^2 = 1$ and $V_2.V_1 = 0$.

The formula (1.3.8) is constructed by writing W_j as a linear function of $U_j, V_1, \dots, V_{j-1}$ and then solving for the coefficients by using the conditions that the dot products of W_j with $V_1, \dots, V_{j-1}$ are all zeros. One interesting observation can be made on (1.3.8). V_j is a linear function of $V_1, \dots, V_{j-1}$ and U_j which implies that V_j is a linear function of $U_1, \dots, U_j$ only. That is, V_1 is a function of U_1 only, V_2 is a function of U_1 and U_2 only and so on, a triangular format.

Example 1.3.9. Given the vectors

$$U_1 = (1,1,-1), \quad U_2 = (1,2,1), \quad U_3 = (2,3,4)$$

construct an orthonormal system by using U_1, U_2 and U_3, if possible.

Solution 1.3.9. Let

$$V_1 = \frac{U_1}{\|U_1\|}, \quad \|U_1\| = \sqrt{(1)^2 + (1)^2 + (-1)^2} = \sqrt{3} \Rightarrow$$
$$V_1 = \frac{1}{\sqrt{3}}(1,1,-1).$$

Let

$$W_2 = U_2 - (U_2.V_1)V_1$$

where

$$V_1.U_2 = \frac{1}{\sqrt{3}}(1,1,-1).(1,2,1)$$
$$= \frac{1}{\sqrt{3}}[(1)(1) + (1)(2) + (-1)(1)] = \frac{2}{\sqrt{3}}.$$
$$W_2 = U_2 - (U_2.V_1)V_1$$
$$= (1,2,1) - \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,1,-1) = \left(\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right) = \frac{1}{3}(1,4,5),$$
$$\|W_2\| = \frac{1}{3}\sqrt{(1)^2 + (4)^2 + (5)^2} = \frac{\sqrt{42}}{3} \Rightarrow$$
$$V_2 = \frac{W_2}{\|W_2\|} = \frac{1}{\sqrt{42}}(1,4,5).$$

Note that for any vector U and for any nonzero scalar a, $\|aU\| = |a|\,\|U\|$ and hence keep the constants outside when computing the lengths. Consider

$$W_3 = U_3 - (U_3.V_1)V_1 - (U_3.V_2)V_2,$$

where

$$\begin{aligned} V_1.U_3 &= \frac{1}{\sqrt{3}}(1,1,-1).(2,3,4) = \frac{1}{\sqrt{3}}, \\ (V_1.U_3)V_1 &= \frac{1}{3}(1,1,-1), \\ V_2.U_3 &= \frac{1}{\sqrt{42}}(1,4,5).(2,3,4) = \frac{34}{\sqrt{42}}, \\ (V_2.U_3)V_2 &= \frac{34}{42}(1,4,5). \end{aligned}$$

Therefore

$$W_3 = (2,3,4) - \frac{1}{3}(1,1,-1) - \frac{34}{42}(1,4,5) = \frac{1}{7}(6,-4,2)$$

with

$$\|W_3\| = \frac{\sqrt{56}}{7} \Rightarrow V_3 = \frac{W_3}{\|W_3\|} = \frac{1}{\sqrt{56}}(6,-4,2).$$

Verification

$$\begin{aligned} V_1.V_2 &= \left[\frac{1}{\sqrt{3}}(1,1,-1)\right].\left[\frac{1}{\sqrt{42}}(1,4,5)\right] = 0; \\ V_1.V_3 &= \left[\frac{1}{\sqrt{3}}(1,1,-1)\right].\left[\frac{1}{\sqrt{56}}(6,-4,2)\right] = 0; \\ V_2.V_3 &= \left[\frac{1}{\sqrt{42}}(1,4,5)\right].\left[\frac{1}{\sqrt{56}}(6,-4,2)\right] = 0. \end{aligned}$$

Thus V_1, V_2, V_3 is the system of orthonormal vectors available from U_1, U_2, U_3.

Example 1.3.10. Given the vectors

$$U_1 = (1,1,-1), \quad U_2 = (1,2,1), \quad U_3 = (2,3,0)$$

construct an orthonormal system by using U_1, U_2, U_3, if possible.

Solution 1.3.10. Since U_1 and U_2 are the same as the ones in Example 1.3.9 we have

$$V_1 = \frac{1}{\sqrt{3}}(1,1,-1) \quad \text{and} \quad V_2 = \frac{1}{\sqrt{42}}(1,4,5).$$

Now, consider the equation

$$W_3 = U_3 - (U_3.V_1)V_1 - (U_3.V_2)V_2$$

where

$$V_1.U_3 = \left[\frac{1}{\sqrt{3}}(1,1,-1)\right].[2,3,0] = \frac{5}{\sqrt{3}},$$

$$(V_1.U_3)V_1 = \frac{5}{3}(1,1,-1)$$
$$V_2.U_3 = \left[\frac{1}{\sqrt{42}}(1,4,5)\right].[2,3,0] = \frac{14}{\sqrt{42}},$$
$$(V_2.U_3)V_2 = \frac{14}{42}(1,4,5).$$

Then

$$W_3 = (2,3,0) - \frac{5}{3}(1,1,-1) - \frac{14}{42}(1,4,5) = (0,0,0).$$

In this case the only orthogonal system possible is with a null vector and the non-null vectors V_1 and V_2. Here V_1 and V_2 are orthonormal but a null vector is orthogonal but not a normal vector. This situation arose because in the original set U_1, U_2, U_3, not all vectors are linearly independent. U_3 could have been written as a linear function of U_1 and U_2, in fact $U_3 = U_1 + U_2$, and that is why W_3 became null.

(ix) If there are m_1 dependent vectors and m_2 linearly independent vectors in a given system of $m_1 + m_2$ vectors of the same category then when the Gram–Schmidt orthogonalization process is applied on these $m_1 + m_2$ vectors we get only m_2 orthonormal vectors and the remaining m_1 will be null vectors.

When we start with a given set of vectors $U_1, \dots, U_k$ we do not know whether it is a linearly independent or dependent system. Hence, start with the orthogonalization process. If a W_j becomes null, ignore the corresponding U_j and proceed with the remaining to obtain a set of orthonormal vectors. This will be m_2 in number if in the original set $U_1, \dots, U_k$ only m_2 were linearly independent.

Note. For a more rigorous definition of a vector space we will wait until after the discussion of *matrices* so that these objects can also be included as elements in such a vector space.

Exercises 1.3

1.3.1. Check for linear dependence or independence in the following set of vectors:

(a) $$U_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 5 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

(b) $U_1 = (2,0,1,-1)$, $U_2 = (3,0,-1,2)$, $U_3 = (5,0,0,1)$;

(c) $U_1 = (3,1,-1,1,2)$, $U_2 = (5,1,2,-1,0)$, $U_3 = (7,-1,1,-1,0)$.

1.3.2. For each case in Exercise 1.3.1 find a basis for the vector subspace spanned by the vectors in the set.

1.3.3. For each of the subspaces spanned by the vectors in Exercise 1.3.1 construct a basis for the orthogonal complement and compute the dimensions of each of these orthogonal complements.

1.3.4. For each set of vectors in Exercise 1.3.1 construct a set of (i) mutually orthogonal vectors as linear functions of the given set of vectors, (ii) a set of orthonormal system of vectors as linear functions of the given set of vectors, if possible.

1.3.5. Let U_1 and U_2 be two linearly independent 2-vectors. Let V be an arbitrary 2-vector. Show that V can be written as a linear function of U_1 and U_2.

1.3.6. Illustrate the result in Exercise 1.3.5 geometrically.

1.3.7. Let U_1, U_2 and U_3 be three linearly independent 3-vectors and let V be an arbitrary 3-vector. Show that V can be written as a linear function of U_1, U_2 and U_3.

1.3.8. Treating vectors as arrowheads let $\vec{U}_1 = (1,1,-1) = \vec{i}+\vec{j}-\vec{k}$ and $\vec{U}_2 = (2,1,0) = 2\vec{i}+\vec{j}$ give a geometric interpretation of a basis for the subspace orthogonal to the subspace spanned by $\vec{U}_1$ and $\vec{U}_2$.

1.3.9. In the language of analytical geometry two lines in a plane are perpendicular to each other if the product of their slopes is -1. Express this statement in terms of the dot product of two vectors being zero.

1.3.10. Find all vectors which are orthogonal to both $U_1 = (1,1,1,-1)$ and $U_2 = (2,1,3,2)$.

1.3.11. If $U_1 = (1,1,1)$ and $U_2 = (1,1,-1)$, are the following true? Prove your assertions by using the definition of linear independence. (i) U_1 and $2U_1 - U_2$, (ii) $U_1 + U_2$ and $U_1 - U_2$, (iii) $U_1 - U_2$ and $2U_1 + 2U_2$, (iv) $U_1 + U_2$ and $2U_1 - 2U_2$, are all linearly independent.

1.3.12. Consider a subspace spanned by the vectors U_1 and U_2 in Exercise 1.3.11. Is it true that the sets in (i) to (iv) there, are bases for that subspace. Justify your answer.

1.3.13. Let S be the vector subspace spanned by U_1 and U_2 of Exercise 1.3.11. Construct 2 bases for the orthogonal complement S^* of S. What are the dimensions of S and S^*?

1.3.14. Consider a 3-space and two planes passing through the origin. Consider the normals to these planes. Construct 3 bases for the subspace spanned by these normals if (1) the planes are parallel, (2) the planes are perpendicular to each other, (3) the planes are neither parallel nor perpendicular to each other.

1.3.15. In Exercise 1.3.14 construct the orthogonal complements of the subspaces spanned in the three cases and find 2 bases each for these orthogonal complements.

1.3.16. Let $V_j \in S$, $j = 1,2,\dots$ be n-vectors where S is a vector space of dimension n. Show that any set of n linearly independent V_j's is a basis of S.

1.3.17. Let $V_j \in S$, $j = 1, \ldots, r$, $1 \le r \le n-1$ where the dimension of S is n and all V_j's are n-vectors. If $V_1, \ldots, V_r$ are linearly independent then show that there exist $n-r$ other elements $V_{r+1}, \ldots, V_n$ of S such that $V_1, \ldots, V_n$ is a basis of S.

1.3.18. Let S be the vector space of all 1×3 vectors. Let S_1 be spanned by $V_1 = (1,1,1)$, $V_2 = (1,0,-1)$, $V_3 = (2,1,0)$ and S_2 be spanned by $U_1 = (2,1,1)$, $U_2 = (3,1,-1)$. Show that (1) $S_1 \subset S$, $S_2 \subset S$, that is, S_1 and S_2 are subspaces in S. (2) $S_1 \cap S_2 \neq O$, that is, the intersection is not empty. (3) Determine the dimensions of S_1 and S_2. (4) Construct the subspace S_3 such that if $W \in S_3$ then $W = V + U$ where $V \in S_1$ and $U \in S_2$. [This $S_3 \subset S$ is called a simple sum of S_1 and S_2 and it is usually written as $S_3 = S_1 + S_2$.]

1.3.19. Consider the same S as in Exercise 1.3.18. Let

$$\mathrm{e}_1 = (1,0,0), \quad \mathrm{e}_2 = (0,1,0), \quad \mathrm{e}_3 = (0,0,1).$$

Let S_1 be spanned by e_1 and e_2 and S_2 be spanned by e_3. Show that (1) $S_1 \subset S$ and $S_2 \subset S$. (2) $S_1 \cap S_2 = O$. (3) Construct S_3 as in Exercise 1.3.18.

1.3.20. *Direct sum of subspaces*. Let S be a finite dimensional linear space (vector space) and let S_1 and S_2 be subspaces of S. Then the *simple sum* of S_1 and S_2, denoted by $S_1 + S_2$, is the set of all sums of the type $U + V$ where $U \in S_1$ and $V \in S_2$. Note that $S_1 + S_2$ is also a subspace of S. In addition, if $S_1 \cap S_2 = O$, that is, the intersection of S_1 and S_2 is null or empty then the simple sum is called a *direct sum*, and it will be denoted by $S_1 \hat{+} S_2$. Show that for the simple sums,

$$\dim(S_1 + S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2)$$

where $\dim(\cdot)$ denotes the dimension of $(\cdot)$ and $+$ the simple sum.

1.3.21. Let S_j, $j = 1, \ldots, k$ be subspaces of a finite dimensional space S. Show that, for the simple sums,

$$\dim(S_1 + \cdots + S_k) \le \sum_{i=1}^{k} \dim(S_i).$$

1.3.22. Let S_1 and S_2 be as in Exercise 1.3.20. Then show that every element $W \in (S_1 + S_2)$ can be written as $W = U + V$, $U \in S_1$, $V \in S_2$ and that this decomposition $W = U + V$ is unique if and only if $S_1 \cap S_2 = O$ where O means a null set.

1.3.23. Let $S_0, S_1, \ldots, S_k$ be subspaces of a finite dimensional linear space S. Show that the subspace S_0 can be written as a direct sum of the subspaces $S_1, \ldots, S_k$ if and only if the union of the bases for $S_1, \ldots, S_k$ forms a basis for S_0.

1.3.24. Let $S_j \in S$, $j = 0, 1, \ldots, k$ where S is a finite dimensional linear space. Show that

$$S_0 = S_1 \hat{+} \cdots \hat{+} S_k$$

if and only if

$$\dim(S_0) = \sum_{j=1}^{k} \dim(S_j).$$

1.3.25. Let S_j, $j = 0, 1, \dots, k$ be as in Exercise 1.3.24. Show that

$$S_0 = S_1 \hat{+} \cdots \hat{+} S_k$$

if and only if

$$S_i \cap (S_1 + \cdots + S_{i-1}) = O, \quad i = 1, \dots, k$$

where O is a null set.

1.3.26. By using vector methods prove that the segment joining the midpoints of two sides of any triangle is parallel to the third side and half as long.

1.3.27. By using vector methods prove that the medians of a triangle (the line segments joining the vertices to the midpoints of opposite sides) intersect in a point of trisection of each.

1.3.28. By using vector methods prove that the midpoints of the sides of any plane convex quadrilateral are the vertices of a parallelogram.

1.3.29. By using vector methods prove that the lines from any vertex of a parallelogram to the midpoints of the opposite sides trisect the diagonal they intersect.

1.3.30. If $U_1, \dots, U_k$ is a finite collection of vectors and if $\|U_j\|$ denotes the length of U_j then show that

$$\|U_1 + \cdots + U_k\| \le \|U_1\| + \|U_2\| + \cdots + \|U_k\|.$$

1.4 Some applications

We will explore a few applications of vector methods in multivariable calculus, statistical problems, model building and other related areas. The students who are not familiar with multivariable calculus may skip this section.

1.4.1 Partial differential operators

Consider a scalar function (as opposed to a vector function) of many real scalar (as opposed to vector) variables, $f(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are functionally independent (no variable can be written as a function of the other variables), or distinct, real variables. For example,

(i) $f = 2x_1^3 + x_2^2 - 3x_1x_2 + x_2 - 5x_1 + 8$
(ii) $f = 3x_1^2 + 2x_2^2 - x_1x_2 - x_2 - 2x_1 + 10$

are two such functions of two real scalar variables x_1 and x_2. Consider the vector of partial differential operators. Let us use the following notations:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \frac{\partial}{\partial X} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}, \quad \frac{\partial}{\partial X} f = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix},$$

$$\frac{\partial}{\partial X'} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right),$$

$$\frac{\partial}{\partial X'} f = \frac{\partial f}{\partial X'} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right). \tag{1.4.1}$$

For example, $\frac{\partial f}{\partial x_1}$ means to differentiate f with respect to x_1 partially which means assuming all other variables $x_2, \dots, x_n$ to be constants. In (ii) above $\frac{\partial}{\partial x_1}$ operating on f gives

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{\partial}{\partial x_1}(3x_1^2 + 2x_2^2 - x_1x_2 - x_2 - 2x_1 + 10) \\ &= \frac{\partial}{\partial x_1}(3x_1^2) + \frac{\partial}{\partial x_1}(2x_2^2) + \frac{\partial}{\partial x_1}(-x_1x_2) \\ &\quad + \frac{\partial}{\partial x_1}(-x_2) + \frac{\partial}{\partial x_1}(-2x_1) + \frac{\partial}{\partial x_1}(10) \\ &= 6x_1 + 0 - x_2 + 0 - 2 + 0 \\ &= 6x_1 - x_2 - 2. \end{aligned}$$

Similarly $\frac{\partial}{\partial x_2}$ operating on this f gives

$$\begin{aligned} \frac{\partial f}{\partial x_2} &= \frac{\partial}{\partial x_2}(3x_1^2 + 2x_2^2 - x_1x_2 - x_2 - 2x_1 + 10) \\ &= 0 + 4x_2 - x_1 - 1 - 0 + 0 \\ &= 4x_2 - x_1 - 1. \end{aligned}$$

Then $\frac{\partial}{\partial X}$ operating on f is a column vector, namely,

$$\frac{\partial}{\partial X} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6x_1 - x_2 - 2 \\ 4x_2 - x_1 - 1 \end{pmatrix}.$$

The transpose of this vector is denoted by $\frac{\partial f}{\partial X'}$ ($\frac{\partial}{\partial X'}$ operating on f). That is,

$$\frac{\partial f}{\partial X'} = (6x_1 - x_2 - 2,\ 4x_2 - x_1 - 1).$$

1.4.2 Maxima/minima of a scalar function of many real scalar variables

When looking for points where the function may have local maximum or local minimum we differentiate the function partially with respect to each variable, equate to zero and solve the system of equations to determine the critical points or turning points or points where the function may have local maximum or local minimum or saddle points. These steps, in vector notation, are equivalent to solving the equation

$$\frac{\partial f}{\partial X} = O \tag{1.4.2}$$

where O denotes the null vector. In our illustrative example

$$\frac{\partial f}{\partial X} = O \Rightarrow \begin{pmatrix} 6x_1 - x_2 - 2 \\ 4x_2 - x_1 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is,
(a) $6x_1 - x_2 - 2 = 0$,
(b) $4x_2 - x_1 - 1 = 0$.

When solving $\frac{\partial f}{\partial X} = O$ we need not write down the individual equations as in (a) and (b) above. One can use matrix methods, which will be discussed in the next chapter, and solve (1.4.2) directly. Solving (a) and (b) we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9/23 \\ 8/23 \end{pmatrix}.$$

In our illustrative example there is only one critical point

$$(x_1, x_2) = \left(\frac{9}{23}, \frac{8}{23}\right).$$

This critical point may correspond to a maximum or a minimum or it may be a saddle point. In order to check for maxima/minima we look for the whole configuration of the matrix of second order partial derivatives and look for definiteness of matrices. This aspect will be considered after introducing matrices in the next chapter.

1.4.3 Derivatives of linear and quadratic forms

Some obvious results when we use the operator $\frac{\partial}{\partial X}$ on linear and quadratic forms will be examined here. A linear form is available by taking a dot product of X with a constant vector. For example if

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

then

$$X.a = a.X = a_1x_1 + \cdots + a_nx_n \tag{1.4.3}$$

is a linear form. For example,

$$y_1 = 2x_1 - x_2 + 3x_3 + x_4$$
$$y_2 = x_1 + x_1 + x_3 + x_4 - 2x_5 + 7x_6$$

are two linear forms. In a linear form each term is of degree one and all terms are of degree one each or a linear form is homogeneous of degree 1 in the variables. For example, the degree of a term is determined as follows: $3x^5$ (degree $0+5=5$), $x_1^5 + 3x_2^5$ (each term is of degree 5), $2x_1^4x_2$ (degree $0+4+1=5$), $6x_1$ (degree $0+1=1$, linear), 5 (degree 0, constant).

What will be the result if a linear form is operated with the operator $\frac{\partial}{\partial X}$? Let $y = X.a$ then

$$\frac{\partial}{\partial X}y = \frac{\partial y}{\partial X} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a.$$

Hence we have the following important result:

(i) Consider the operator $\frac{\partial}{\partial X}$ and the linear form $X.a$ where a is a constant vector. Then

$$y = X.a = a.X \Rightarrow \frac{\partial}{\partial X}y = \frac{\partial y}{\partial X} = a$$

where a is the column vector of the coefficients in $X.a$.

Example 1.4.1. Evaluate $\frac{\partial y}{\partial X}$ if

$$y = x_1 - 5x_2 + x_3 - 2x_4.$$

Solution 1.4.1.

$$\frac{\partial y}{\partial x_1} = 1, \quad \frac{\partial y}{\partial x_2} = -5, \quad \frac{\partial y}{\partial x_3} = 1, \quad \frac{\partial y}{\partial x_4} = -2$$

and hence

$$\frac{\partial}{\partial X}y = \frac{\partial y}{\partial X} = \begin{bmatrix} 1 \\ -5 \\ 1 \\ -2 \end{bmatrix}.$$

Now, let us examine a simple quadratic form. Consider the sum of squares of a number of variables. Let

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{then} \quad X.X = x_1^2 + x_2^2 + \cdots + x_n^2.$$

This is a special case of a quadratic form. In a quadratic form, every term is of degree 2 each or it is a homogeneous function of many variables of degree 2. For the time being, we consider the above simple quadratic form. More general quadratic forms will be considered after introducing matrices in the next chapter. What will happen if a sum of squares is operated with the operator $\frac{\partial}{\partial X}$? Proceeding as in the linear case the result is the following:

(ii) Let $y = X.X = x_1^2 + \cdots + x_n^2$ then

$$y = X.X \Rightarrow \frac{\partial y}{\partial X} = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \end{pmatrix} = 2X.$$

1.4.4 Model building

Suppose that a gardener suspects that the growth of a particular species of plant (growth measured in terms of the height of the plant) is linearly related to the amount of a certain fertilizer used. Let the amount of the fertilizer used be denoted by x and the corresponding growth (height) be y. Then the gardener's suspicion is that

$$y = a + bx$$

where a and b are some constants, that is, y and x are linearly related. What exactly is this linear relationship? The gardener conducts an experiment to estimate the values of a and b. Suppose that the gardener applies the amounts $x_1, \dots, x_n$ of the fertilizer x on different plants of the same species, in a carefully planned experiment, and take the corresponding measurements $y_1, \dots, y_n$ on y. Thus the gardener has the following pairs of values (x_i, y_i), $i = 1, \dots, n$. For example, when one spoon of fertilizer (measured in spoon units) is applied the growth (measured after a fixed time) noted is 3 inches (growth measured in inches) then the corresponding pair is $(x_1, y_1) = (1, 3)$. If $y = a + bx$ is a mathematical relationship then every pair (x, y) should satisfy the equation $y = a + bx$. Then we need only two pairs of values on (x, y) to exactly evaluate a and b and then every other value on (x, y) must satisfy the relationship. But this is not the situation here. The gardener is thinking that there may be a relationship between x and y, that relationship may be a linear relationship and that she will be able to estimate y at a preassigned value of x. Then the error in estimating y by using such a relationship at a given value of x is $y - (a + bx)$. Denoting the error in the i-th pair by ϵ_i we have

$$\epsilon_i = y_i - a - bx_i.$$

One way of estimating the unknown parameters a and b is to minimize the sum of squares of the errors (error = observed value minus the modeled value, whatever be the model, linear or not). Such a method of estimating the parameters in a model by minimizing the error sum of squares is known as the *method of least squares*. The error vector and the error sum of squares in our linear model are given by

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix},$$

$$\epsilon.\epsilon = \epsilon_1^2 + \cdots + \epsilon_n^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2. \tag{1.4.4}$$

Equation (1.4.4) can be written in a more elegant way as a quadratic form after discussing matrices. Let the vector of unknowns be denoted by $\alpha = \binom{a}{b}$. Then the method of least squares implies that $\epsilon.\epsilon$ is minimized with respect to α. It is obvious that the maximum of $\epsilon.\epsilon$, being a non-negative arbitrary quantity, is at $+\infty$. Then the minimizing equations, often known as the *normal equations* in least square analysis, are the following:

$$\frac{\partial}{\partial \alpha}(\epsilon.\epsilon) = O \Rightarrow \begin{pmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \end{pmatrix} (\epsilon.\epsilon) = O \Rightarrow$$

$$\begin{pmatrix} -2\sum_{i=1}^{n}(y_i - a - bx_i) \\ -2\sum_{i=1}^{n} x_i(y_i - a - bx_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\sum_{i=1}^{n}(y_i - a - bx_i) = 0 \tag{a}$$

and

$$\sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0 \tag{b}$$

since $-2 \neq 0$. Opening up the sum we have, from (a) and (b),

$$\left(\sum_{i=1}^{n} y_i\right) - na - b\left(\sum_{i=1}^{n} x_i\right) = 0 \tag{c}$$

and

$$\left(\sum_{i=1}^{n} x_i y_i\right) - a\left(\sum_{i=1}^{n} x_i\right) - b\left(\sum_{i=1}^{n} x_i^2\right) = 0. \tag{d}$$

Denoting

$$\bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} \quad \text{and} \quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

and solving (c) and (d) we get the values of a and b. Let us denote these estimates by $\hat{a}$ and $\hat{b}$ respectively. Then we have

$$\hat{b} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{(\sum_{i=1}^{n} x_i y_i) - n(\bar{x}\bar{y})}{(\sum_{i=1}^{n} x_i^2) - n(\bar{x})^2} \tag{1.4.5}$$

and

$$\hat{a} = \bar{y} - \hat{b}\bar{x}. \tag{1.4.6}$$

From (1.4.5) and (1.4.6) we have the estimates for a and b, and the estimated linear model by using the method of least squares is then

$$y = \hat{a} + \hat{b}x. \tag{1.4.7}$$

Example 1.4.2. In a feeding experiment with beef cattle the farmer suspects that the increase in weight is linearly related to the quantity of a particular combination of feed. The farmer has obtained the following data. Construct the estimating function by the method of least squares and then estimate the weight if the quantity of feed is 2.2 kg.

Data:

$y =$ (gain in weight in kg)	0.5	0.8	1.5	2.0
$x =$ (quantity of feed in kg)	1.2	1.5	2.0	2.5

Solution 1.4.2.

$$\bar{x} = \frac{1.2 + 1.5 + 2.0 + 2.5}{4} = 1.8, \quad \bar{y} = \frac{0.5 + 0.8 + 1.5 + 2.0}{4} = 1.2.$$

For convenience of computations let us form the following table: [Use a calculator or computer to compute $\hat{a}$ and $\hat{b}$ directly.]

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(x - \bar{x})(y - \bar{y})$
1.2	0.5	−0.6	−.7	0.36	0.42
1.5	0.8	−0.3	−0.4	0.09	0.12
2.0	1.5	0.2	0.3	0.04	0.06
2.5	2.0	0.7	0.8	0.49	0.56
				----	----
				0.98	1.16

$$\hat{b} = \frac{1.16}{0.98} \approx 1.1837, \quad \hat{a} = 1.2 - \frac{(1.16)}{(0.98)}(1.8) \approx -0.9306.$$

The estimated model is

$$y = -0.9306 + 1.1837x.$$

Then the predicted value of y at $x = 2.2$ is

$$\hat{y} = -0.9306 + 1.1837(2.2) \approx 1.6735\,\text{kg}.$$

Exercises 1.4

1.4.1. Find the critical points for the following functions and then check to see whether these correspond to maxima or minima or something else:
(a) $f = 2x_1^2 + x_2^2 - 3x_2 + 5x_1x_2 - x_1 + 5$.
(b) $f = x_1^2 + x_2^2 - 2x_1x_2 - 5x_1 - 2x_2 + 8$.

1.4.2. Evaluate $\frac{\partial f}{\partial X}$ and write the results in vector notations:
(a) $f = 3x_1 - x_2 + 5x_3 - x_4 + 10$.
(b) $f = x_1^2 + 2x_1x_2 + x_1x_3 - x_2^2 + 3x_3^2$.
(c) $f = 2x_1^2 + x_2^2 + x_3^2 - 5x_1x_2 + x_2x_3$.

1.4.3. Write the operator $\frac{\partial}{\partial X'}$. Then on each element of this vector apply the operator $\frac{\partial}{\partial X}$. Explain what you have in this configurations of n rows and n columns.

1.4.4. Apply the operator $\frac{\partial}{\partial X}\frac{\partial}{\partial X'}$ on f in each of (a), (b), (c) in Exercise 1.4.2.

1.4.5. Fit linear models of the type $y = a + bx$ for the following data:
(a) $(x,y) = \{(0,2),(1,5),(2,6),(3,9)\}$.
(b) $(x,y) = \{(-1,1),(-2,-2),(0,3),(1,6)\}$.

1.4.6. Fit a model of the type $y = a + bx + cx^2$ to the following data:

$$(x,y) = \{(-1,2),(0,1),(1,5),(2,7),(3,21)\}.$$

1.4.7. In statistical distribution theory the moment generating function of a real vector $X' = (x_1,\dots,x_k)$ random variable is denoted by $M(T)$, $T' = (t_1,\dots,t_k)$ where T is a vector of parameters. When $M(T)$ is evaluated for the real multivariate Gaussian distribution we obtain

$$M(T) = \mathrm{e}^{\phi(T)}$$

where

$$\phi(T) = t_1\mu_1 + \cdots + t_k\mu_k + \frac{1}{2}\left[\sum_{i,j=1}^{k}\sigma_{ij}t_it_j\right]$$

where $\mu_1,\dots,\mu_k$ as well as σ_{ij}, $i = 1,\dots,k$, $j = 1,\dots,k$ are constants, free of T. When $M(T)$ is available and differentiable, then the expected value of X or the first moment of X, denoted by $\mu = E(X)$, is obtained as $\frac{\partial}{\partial T}M(T)|_{T=O}$, that is the first derivative evaluated at $T = O$, and the variance–covariance matrix is $\frac{\partial}{\partial T}\frac{\partial}{\partial T'}M(T)|_{T=O} - \mu\mu'$. Evaluate $E(X)$ and the variance–covariance matrix for the multivariate Gaussian distribution.

1.4.8. The exponential series is

$$\mathrm{e}^y = y^0 + \frac{y}{1!} + \frac{y^2}{2!} + \cdots, \quad y^0 = 1.$$

Consider the operator $D = \frac{d}{dx}$. Then

$$e^{xD} = (xD)^0 + \frac{xD}{1!} + \frac{x^2D^2}{2!} + \cdots .$$

where, for example, $D^r = DD \dots D$ stands for D operating repeatedly r times. Let $e^{xD}f_0$ denote e^{xD} operating on f and then $D^r f$ is evaluated at $x = 0$, $r = 0, 1, \dots$. Then

$$e^{xD}f_0 = f(0) + \frac{x}{1!}\left(\frac{d}{dx}f\right)_{x=0} + \frac{x^2}{2!}\left(\frac{d^2}{dx^2}f\right)_{x=0} + \cdots .$$

This is Taylor series in one variable. Now consider a two variable case. Let

$$\nabla = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad D_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2$$

and the increment vector at the point (a_1, a_2) is $\Delta' = (x_1 - a_1, x_2 - a_2)$. Then the dot product is given by

$$\nabla.\Delta = (x_1 - a_1)D_1 + (x_2 - a_2)D_2.$$

As before, let $e^{\nabla.\Delta}f_0$ denote $e^{\nabla.\Delta}$ operating on f where the various derivatives are evaluated at the point (a_1, a_2). Write down the Taylor series expansion for two variables (x_1, x_2) at the point (a_1, a_2) explicitly up to the terms involving all the second order derivatives.

1.4.9. By using the operator ∇ in Exercise 1.4.8 expand the following functions by using Taylor expansion, at the specified points:
(a) $x_1^2 + 2x_1x_2^2 + x_2^3 + 5x_1 - x_2 + 7$ at $(1, -1)$.
(b) $2x_1^2 + x_2^2 - 3x_1x_2 + 8$ at $(-2, -3)$.
(c) $x_1^4 + x_1^3x_2 + 3x_2^4 - x_1x_2 + 4$ at $(2, 0)$.

1.4.10. Extend the ideas in Exercise 1.4.8 to a scalar function $f(x_1, x_2, x_3)$ of 3 real variables x_1, x_2, x_3, at the point (a_1, a_2, a_3). In this case $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$. Evaluate the first few terms of the series explicitly up to the terms involving $(\nabla.\Delta)^3$.

1.4.11. Apply the result in Exercise 1.4.10 to expand the following function up to terms involving $(\nabla.\Delta)^3$, and at the point $(1, 0, -1)$:

$$x_1^2 e^{-x_1-x_2-x_3} + 5x_1^3x_2^2x_3 - e^{-2x_1+3x_2}.$$

1.4.12. For Exercise 1.4.5 (a) estimate y at (i) $x = 2.7$, (ii) $x = 3.1$. Is it reasonable to use the model to estimate y at $x = 10$?

1.4.13. For Exercise 1.4.6 estimate y at (i) $x = 0.8$, (ii) $x = 3.1$. Is it reasonable to predict y at (iii) $x = -4$, (iv) $x = 8$ by using the same model?

1.4.14. Use the method of least squares to fit the model

$$y = a_0 + a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2$$

to the following data:

$$(x_1, x_2, y) = (0,0,1), (0,1,0), (0,2,-2), (1,-1,-1), (2,1,8), (1,2,3).$$

1.4.15. Can the method of least squares, as minimizing the error sum of squares with error defined as "observed minus the modeled value", be used to fit the model $y = ab^x$ to the data

$$(x, y) = (x_1, y_1),\ (x_2, y_2), \dots, (x_n, y_n)$$

and if not what are the difficulties encountered?