

Contents

**Historical foreword on the centenary after Felix Hausdorff's classic
Set Theory — xiii**

Preface — xvii

1 Fundamentals of the theory of classes, sets, and numbers — 1

Introduction — 1

1.1 Classes and sets — 2

1.1.1 Symbols, symbol-strings, and texts of the theory of classes and sets — 2

1.1.2 Formulas and terms — 3

1.1.3 Axioms, deducibility, and theorems — 5

1.1.4 Logical axiom schemes of the theory of classes and sets — 9

1.1.5 First non-logical axioms and axiom schemes of the theory of classes and sets — 10

1.1.6 First axioms of existence of sets — 13

1.1.7 Correspondences — 16

1.1.8 Mappings — 17

1.1.9 Multivalued and simple collections — 22

1.1.10 The union and intersection of a multivalued collection — 25

1.1.11 The other axioms of existence of sets — 27

1.1.12 The product of a multivalued collection. The axiom of choice — 33

1.1.13 Formulas of the distributivity for union, intersection, and product of a multivalued collection — 40

1.1.14 Binary relations. Equivalence relations. Preorder and order relations — 43

1.1.15 Basic notions connected with preorder and order relations — 46

1.2 Ordinals and ordinal numbers — 55

1.2.1 The property of minimality. The principle of induction — 55

1.2.2 The relation of Neumann on the universal class. Ordinals — 56

1.2.3 Properties of ordinals — 58

1.2.4 Relations between well-ordered sets — 61

1.2.5 The correspondence between well-ordered sets and ordinal numbers — 62

1.2.6 Natural numbers. Multivalued and simple sequences — 65

1.2.7 The construction of mappings by natural induction — 69

1.2.8 The principle of transfinite induction. The constructions of mappings by transfinite induction — 73

1.2.9	The ordered disjoint union of well-ordered sets. The addition of ordinal numbers —	75
1.2.10	The connection between ordinal and natural numbers —	77
1.2.11	The other forms of the axiom of choice —	79
1.3	Cardinal numbers —	82
1.3.1	The definition of cardinal numbers. The cardinality of natural numbers. The first denumerable cardinal number —	82
1.3.2	The power of sets —	83
1.3.3	Properties of finite sets —	85
1.3.4	The first uncountable cardinal number. The enumeration of infinite cardinal numbers —	89
1.3.5	Derivative cardinal numbers —	90
1.3.6	Derivative natural numbers —	98
1.3.7	Ordered sets of natural numbers —	104
1.3.8	Properties of infinite cardinal numbers —	106
1.3.9	Properties of countable sets —	110
1.3.10	Properties of the class of all countable ordinal numbers —	111
1.4	Real numbers —	113
1.4.1	Integers —	113
1.4.2	Rational numbers —	126
1.4.3	Real and extended real numbers —	137
1.4.4	The Cantor completeness of the real line —	152
1.4.5	The Dedekind completeness and order properties of the extended real line —	157
1.4.6	Natural roots of positive real numbers. Raising to a rational degree —	164
1.4.7	Convergence of nets in the extended real line —	168
1.4.8	Netful and sequential series in the extended real line —	176
1.4.9	The order equivalence of intervals of the real line —	182

A	Characterization of all natural models of Neumann – Bernays – Gödel and Zermelo – Fraenkel set theories —	185
	Introduction —	185
A.1	First-order theories —	187
A.1.1	The language of first-order theories —	187
A.1.2	Deducibility in a first-order theory —	189
A.1.3	An interpretation of a first-order theory in a set theory —	191
A.2	Some elements of the Zermelo – Fraenkel set theory —	194
A.2.1	The proper axioms and axiom schemes of the ZF set theory —	194
A.2.2	Ordinals and cardinals in the ZF set theory —	198
A.3	Cumulative sets in the ZF set theory —	204
A.3.1	Construction of cumulative sets —	204

A.3.2	Properties of cumulative sets — 205
A.3.3	Properties of inaccessible cumulative sets — 210
A.4	Universal sets and their connection with inaccessible cumulative sets — 213
A.4.1	Universal sets and their properties — 213
A.4.2	Description of the class of all universal sets — 216
A.4.3	Enumeration of the class of all universal sets in the ZF+AU set theory and the structural form of the universality axiom — 218
A.4.4	Enumeration of the class of all inaccessible cardinals in the ZF+AI theory and the structural form of the inaccessibility axiom — 222
A.5	Weak forms of the universality and inaccessibility axioms — 224
A.5.1	The ω -universality and ω -inaccessibility axioms — 224
A.5.2	Comparison of various forms of the universality and inaccessibility axioms — 229
A.6	Characterization of all supertransitive standard models of the ZF and NBG set theories in the ZF set theory — 235
A.6.1	Supertransitive standard model sets with the strong substitution property for the ZF set theory — 235
A.6.2	Supertransitive standard model of the NBG set theory in the ZF set theory — 242
A.7	Characterization of all natural models of the NBG set theory — 251
A.7.1	Tarski sets and their properties — 251
A.7.2	Galactic sets and their connection with Tarski sets — 256
A.7.3	Characterization of Tarski sets. Characterization of all natural models of the NBG theory — 258
A.8	Characterization of all natural models of the ZF set theory in the ZF set theory — 260
A.8.1	Scheme-inaccessible cardinal numbers and scheme-inaccessible cumulative sets — 260
A.8.2	Scheme-universal sets and their connection with scheme-inaccessible cumulative sets — 265
A.8.3	Supertransitive standard models of the ZF set theory in the ZF set theory — 271
A.8.4	Tarski scheme sets. Characterization of all natural models of the ZF set theory — 277
B	Local theory of sets as a foundation for category theory and its connection with the Zermelo – Fraenkel set theory — 281
	Introduction — 281
B.1	The local theory of sets — 285
B.1.1	Proper axioms and axiom schemes of the local theory of sets — 285
B.1.2	Some constructions in the local theory of sets — 290

B.2	The MacLane problem on a set-theoretical foundation for the naive category theory. The solution of this problem within the framework of the local theory of sets —	291
B.2.1	The definition of a local category in the local theory of sets —	292
B.2.2	Functors and natural transformations and generated by them “the category of categories” and “the category of functors” in the local theory of sets —	293
B.3	Universal classes, ordinals, cardinals, and cumulative classes in the local theory of sets —	295
B.3.1	The relativization of formulas of the LTS to universal classes. The interpretation of the ZF set theory in universal classes —	295
B.3.2	The globalization of local constructions —	297
B.3.3	Ordinals and cardinals in the local theory of sets —	300
B.3.4	Cumulative classes in the LTS and their connection with universal classes —	305
B.3.5	The structure of the assemblies of all universal classes and all inaccessible cardinals in the local theory of sets —	308
B.4	Relative consistency between the LTS and the ZF set theory —	312
B.4.1	Additional axioms on inaccessible cardinals in the ZF set theory —	312
B.4.2	“Forks” of relative consistency —	317
B.5	The proof of relative consistency by the method of abstract interpretation —	330
B.5.1	Abstracts of a set theory —	331
B.5.2	The abstract interpretation of a first-order theory in a set theory —	331
B.6	Undedducibility of some axioms in the LTS —	333
B.6.1	The undedducibility of the axiom scheme of replacement —	333
B.6.2	The independence of axiom $AU(\omega)$ of the axioms of the LTS —	344
B.6.3	The locally minimal theory of sets —	345
B.7	The finite axiomatizability of the LTS and the NBG set theory —	346
B.7.1	Replacement of the full comprehension axiom scheme by finitely many axioms —	347
B.7.2	The deductive equivalence of the theories LTS and LTS^f —	349
B.7.3	The finite axiomatization of the NBG set theory by P. Bernays —	356
C	Compactness theorem for generalized second-order language —	361
	Introduction —	361
C.1	Types, formations, terminals, signatures, and formulas —	363
C.1.1	Types —	363
C.1.2	Formations and terminals —	364
C.1.3	Signatures and formulas —	364

C.2 Mathematical systems of the signature Σ^g with generalized equalities and belongings —	366
C.2.1 The definition of a mathematical system of the generalized signature Σ^g —	366
C.2.2 Concordance of mathematical systems of the generalized second-order signature —	367
C.2.3 Evaluations and models —	367
C.2.4 The generalized equality of values of evaluations and satisfiability —	369
C.2.5 An example of a good model for the second-order equality axioms —	372
C.3 Infraproducts, infrafiltration, and generalized compactness theorem —	373
C.3.1 Infraproducts of collections of evaluated mathematical systems of the generalized second-order signature Σ_2^g —	373
C.3.2 Infrafiltration of formulas of the second-order language $L(\Sigma_2^g)$ of the generalized second-order signature Σ_2^g —	375
C.3.3 Compactness theorem for formulas of the language $L(\Sigma_2^g)$ of the generalized second-order signature —	382
C.3.4 Uncountable models of the second-order generalized Peano – Landau arithmetic —	382
Index of terms —	387
Index of notations —	409
Bibliography —	417

