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Brief survey on ∞-Poincaré inequality and existence of ∞-harmonic functions

7.1 Introduction

Recent work on analysis in metric measure spaces saw the development of Sobolev-type function theory and associated potential theory in the non-smooth setting of metric measure spaces. A significant part of this development is based on the theory of upper gradients first proposed by Heinonen and Koskela in [21], see [5] for potential theory based on this notion, and [22] for the corresponding study of metric space-valued Sobolev-type function theory in non-smooth settings.

In much of the above-mentioned analysis the basic requirements on the metric measure space under study are that the measure should be doubling and support a p-Poincaré inequality for some fixed p with $1 \le p < \infty$. The weakest of all the p-Poincaré inequalities is the ∞-Poincaré inequality. The paper [9] proved an analog of Rademacher's theorem for metric measure spaces whose measure is doubling and supports a p-Poincaré inequality for some $1 \le p < \infty$ and showed that for such metric measure spaces, the minimal p-weak upper gradient of a Lipschitz function f on the metric space *is* its pointwise Lipschitz constant function $\operatorname{Lip} f$. The conclusions of [9] did not depend on the precise value of p. Furthermore, it is known that spaces X whose measure is doubling and supports a p-Poincaré inequality are *quasiconvex*, that is, there is a constant $C \ge 1$ such that whenever $x, y \in X$ there is a rectifiable curve in X with end points x, y with length $\ell(\gamma) \le C\, d(x, y)$. Again, this consequence does not depend on the precise value of p.

It was therefore natural to ask whether one could obtain the results of [9] for metric measure spaces whose measure is doubling and supports an ∞-Poincaré inequality. The series of papers [12–16, 18] studied metric measure spaces equipped with a doubling measure supporting an ∞-Poincaré inequality, with this goal in mind. The purpose of this present article is to give an overview of the results obtained in these papers.

Throughout this article we will assume that the metric space X is complete and that the measure μ on X is a Radon measure and is doubling, that is, there is a constant $C_d \ge 1$ such that whenever $x \in X$ and $r > 0$, we have

$$0 < \mu(B(x, 2r)) \le C_d\, \mu(B(x, r)) < \infty.$$

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The structure of this paper is as follows. In Section 2 we describe the basic notions needed to study first order calculus in the metric setting. In Section 3 we describe properties related to spaces supporting an ∞-Poincaré inequality, and in Section 4 we describe results from [15] demonstrating that, unlike in [26], one needs only ∞-Poincaré inequality in order to know existence and uniqueness of ∞-harmonic functions with prescribed Lipschitz boundary data. The final section, Section 5, will give examples demonstrating sharpness of the results in the previous section.

7.2 Background

In the non-smooth metric setting (including weighted Euclidean spaces and sub-Riemannian manifolds) the derivative of a function is not available. However, there are some possible notions that take on the role of magnitude of the derivative. Of these possible notions, the notion of (weak) upper gradients, developed in [21], has the so-called strong locality property.

Definition 7.2.1. *Given a function $f : X \to \mathbb{R}$, we say that a non-negative Borel-measurable function $g : X \to [0, \infty]$ is an upper gradient of f if whenever γ is a non-constant compact rectifiable curve in X,*

$$|f(y) - f(x)| \le \int_\gamma g\, ds, \tag{7.1}$$

where x and y denote the two end points of γ.

See [2] or [22] for more on path integrals, and [5, 19–22] for more on the notion of upper gradients.

Definition 7.2.2. *Let $1 \le p \le \infty$. A non-negative Borel-measurable function $g : X \to [0, \infty]$ is a p-weak upper gradient of f if the collection Γ of all non-constant compact rectifiable curves γ for which* (7.1) *fails has p-modulus zero, that is, there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that for each $\gamma \in \Gamma$ we have $\int_\gamma \rho\, ds = \infty$.*

It was shown in [12] that a family Γ of non-constant compact rectifiable curves in X has ∞-modulus zero if and only if there is a non-negative Borel measurable function ρ on X with $\rho = 0$ μ-a.e. in X such that $\int_\gamma \rho\, ds = \infty$ whenever $\gamma \in \Gamma$. For $1 \le p < \infty$ such a strong result does not hold, but in this case Γ has p-modulus zero if and only if there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma$, see [29].

For $1 \le p \le \infty$ we set $D_p(f)$ to be the collection of all p-weak upper gradients of f that also belong to $L^p(X)$. For $1 < p < \infty$, the uniform convexity of $L^p(X)$ together with the lattice properties and convexity property of $D_p(f)$ imply the existence of a “minimal" function $g_f \in D_p(f)$ such that whenever $g \in D_p(f)$, we have $g_f \leq g$ μ-a.e. in X.

Remark 7.2.3. It turns out that the existence of such $g_f \in D_p(f)$ holds for all $1 \leq p \leq \infty$, see [31, Theorem 4.6]. We can say even more for the case $p = \infty$. From the work of [12] mentioned above, we can modify every $g \in D_\infty(f)$ on a set of μ-measure zero such that the modified function is an *upper gradient* of f. Thus, without loss of generality, we may assume that every function in $D_\infty(f)$ is an upper gradient of f.

It was shown in [34] and [5] that if f is a measurable function with a p-weak upper gradient $g \in D_p(f)$, and if f is constant on an open set $U \subset X$, then $g\chi_{X\setminus U} \in D_p(f)$. This property, called the *strong locality property* of p-weak upper gradients, is highly useful in the development of potential theory in the metric setting, see [5].

Definition 7.2.4. *For $1 \leq p < \infty$ we say that X supports a p-Poincaré inequality if there are constants $C > 0$ and $\lambda \geq 1$ such that whenever $f \in L^p(X)$ and $g \in D_p(X)$, for all balls $B = B(x, r) \subset X$ we have*

$$\fint_B |f - f_B|\, d\mu \leq C\, r \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}.$$

Here $f_B := \mu(B)^{-1} \int_B f\, d\mu =: \fint_B f\, d\mu$ is the average of f on the ball B, and $\lambda B := B(x, \lambda r)$. We say that X supports an ∞-Poincaré inequality if there are constants $C > 0$ and $\lambda \geq 1$ such that whenever $f \in L^\infty(X)$ and $g \in D_\infty(X)$, for all balls $B = B(x, r) \subset X$ we have

$$\fint_B |f - f_B|\, d\mu \leq C\, r\, \|g\|_{L^\infty(\lambda B)}.$$

The Sobolev-type function spaces under consideration here are the Newton-Sobolev spaces, first developed in [34].

Definition 7.2.5. *The set $\widetilde{N^{1,p}}(X)$ is the collection of all functions $f : X \to \mathbb{R}$ such that $f \in L^p(X)$ and $D_p(f)$ is non-empty. For functions $f \in \widetilde{N^{1,p}}(X)$ we set*

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{g \in D_p(f)} \|g\|_{L^p(X)}.$$

From the above discussion it is clear that when $f \in \widetilde{N^{1,p}}(X)$,

$$\|f\|_{N^{1,p}(X)} = \|f\|_{L^p(X)} + \|g_f\|_{L^p(X)}.$$

We say that $f_1 \sim f_2$ if $f_1, f_2 \in \widetilde{N^{1,p}}(X)$ and $\|f_1 - f_2\|_{N^{1,p}(X)} = 0$. It was shown in [34] that if $f_1, f_2 \in \widetilde{N^{1,p}}(X)$, then $f_1 \sim f_2$ if and only if $f_1 = f_2$ μ-a.e. in X and that $\sim$ is an equivalence relation on $\widetilde{N^{1,p}}(X)$.

Definition 7.2.6. *The Newton-Sobolev space $N^{1,p}(X)$ is the collection of all equivalence classes of functions from $\widetilde{N^{1,p}}(X)$ from the equivalence relation $\sim$.*

It was shown in [34] and [12] that $N^{1,p}(X)$ is a Banach space when equipped with the norm $\|\cdot\|_{N^{1,p}(X)}$.

A function $f : X \to \mathbb{R}$ is said to be L-Lipschitz on X if for all $x, y \in X$ we have $|f(x) - f(y)| \leq L\, d(x, y)$. For such functions f and $x \in X$, we set

$$\operatorname{Lip} f(x) = \lim_{r\to 0} \sup_{x \neq y \in B(x,r)} \frac{|f(x) - f(y)|}{d(x, y)}$$

and call this the pointwise Lipschitz constant function of f. For $A \subset X$ we set

$$\operatorname{LIP}(f, A) = \sup_{x,y\in A, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Note that $\operatorname{Lip} f$ is an upper gradient of the Lipschitz function f, see for example [20]. The Sobolev-type spaces studied in [9] are also based on the notion of upper gradients, combined with the idea of relaxation. It was shown in [34] that when $1 < p < \infty$ the Sobolev-type space of [9] agrees with the above $N^{1,p}(X)$. Combining the results of [9] with [34] it is seen that if X supports a p-Poincaré inequality for some $1 \leq p < \infty$ and f is Lipschitz continuous on X, then $g_f = \operatorname{Lip} f$ μ-a.e. in X. Thus g_f in this case becomes independent of the choice of p. More specifically, $\operatorname{Lip} f$ is the minimal q-weak upper gradient of f in the class $D_q(f)$ for all $q \geq p$.

If X supports a p-Poincaré inequality for some $1 \leq p < \infty$, then by Hölder's inequality it follows that X supports a q-Poincaré inequality for all $p \leq q \leq \infty$. It is a highly non-trivial result of Keith and Zhong [27] that if X is complete (recall that we also assume the measure to be doubling in our paper), and X supports a p-Poincaré inequality for some $1 < p < \infty$, then there is some $q \in [1, p)$ such that X supports a q-Poincaré inequality. The exposition of the proof of this result, given in [22], shows that q depends only on p and the doubling and p-Poincaré constants of μ and X. Such Gehring-type self-improvement is highly useful in regularity theory for p-harmonic functions in the metric setting, see for example [28], [5], and [22].

Definition 7.2.7. *Let $\Omega \subset X$ be a bounded domain such that $X \setminus \Omega$ has positive measure, and let $f : X \to \mathbb{R}$ be in $N^{1,p}(X)$. For $1 < p < \infty$ we say that a function $u \in N^{1,p}(X)$ with $u = f$ on $X \setminus \Omega$ is p-harmonic in Ω with boundary data f if whenever $\phi \in N^{1,p}(X)$ with $\phi = f$ on $X \setminus \Omega$, we have*

$$\|g_u\|_{L^p(\Omega)} \leq \|g_\phi\|_{L^p(\Omega)}. \tag{7.2}$$

Because of the strongly local nature of p-weak upper gradients (that is, if two functions in $N^{1,p}(X)$ agree on a Borel set, then their minimal p-weak upper gradients agree almost everywhere on that set, see [34] or [5, Lemma 2.19], *and* the local nature of integrals, we see that

$$\int_X g_\phi^p \, d\mu = \int_{\mathrm{supt}(\phi-u)} g_\phi^p \, d\mu + \int_{X\setminus\mathrm{supt}(\phi-u)} g_u^p \, d\mu.$$

Therefore, u is a p-harmonic function on Ω with boundary data f if and only if $u = f$ on $X \setminus \Omega$ and whenever $V \subset \Omega$ is an open set and $\phi \in N^{1,p}(X)$ such that $\phi = u$ on $X \setminus V$, we have

$$\|g_u\|_{L^p(V)} \le \|g_\phi\|_{L^p(V)}.$$

While the notion of p-harmonicity from (7.2) will not yield a local notion of ∞-harmonicity when $p \to \infty$, the above notion does. Thus we have the following definition of ∞-harmonicity.

Definition 7.2.8. *We say that $u \in N^{1,\infty}(X)$ is ∞-harmonic in Ω with boundary data $f \in N^{1,\infty}(X)$ if $u = f$ on $X \setminus \Omega$ and whenever $V \subset \Omega$ is an open set and $\phi \in N^{1,\infty}(X)$ such that $\phi = u$ on $X \setminus V$, we have*

$$\|g_u\|_{L^\infty(V)} \le \|g_\phi\|_{L^\infty(V)}.$$

With the above definition, a function u that is ∞-harmonic in Ω is also ∞-harmonic in every subdomain U of Ω, that is, ∞-harmonicity is a local property.

Note that a function f is in $N^{1,\infty}(X)$ if and only if $\|f\|_{L^\infty(X)} = \operatorname{esssup}_{x\in X}|f(x)|$ is finite and if it has an ∞-weak upper gradient g such that $\|g\|_{L^\infty(X)}$ is also finite. We do not claim that in general such functions are Lipschitz continuous on X; the paper [12] has examples of functions in $N^{1,\infty}(X)$ that fail to be Lipschitz continuous on X. As we will see in the next section, if X in addition supports an ∞-Poincaré inequality, then indeed such functions must be Lipschitz continuous on X.

Unlike ∞-harmonicity, the notion of minimal Lipschitz extension is not a local property. A Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ is said to be a minimal Lipschitz extension of $f = u|_{\partial\Omega}$ if

$$\mathrm{LIP}(u, \Omega) \le \mathrm{LIP}(u, \partial\Omega) = \mathrm{LIP}(f, \partial\Omega). \tag{7.3}$$

For every Lipschitz function $w : \overline{\Omega} \to \mathbb{R}$, we always have that

$$\mathrm{LIP}(w, \Omega) \ge \mathrm{LIP}(w, \partial\Omega),$$

and hence the minimality of u in the above definition. Every Lipschitz function $f : \partial\Omega \to \mathbb{R}$ has a minimal Lipschitz extension to Ω, as demonstrated by McShane [32]. In fact, the proof given in [32] also shows the non-uniqueness of such extension, for both the following two extensions are minimal Lipschitz extensions to Ω:

$$u^+(x) = \inf\{f(y) + L\, d(x, y) \,:\, y \in \partial\Omega\},$$
$$u^-(x) = \sup\{f(y) - L\, d(x, y) \,:\, y \in \partial\Omega\}.$$

Here $L = \mathrm{LIP}(f, \partial\Omega)$. In the seminal paper [3] Aronsson sought the optimal extensions that are minimal Lipschitz extensions locally as well; that is, (7.3) is satisfied not only for Ω but also for every non-empty open subset U of Ω (by replacing Ω with U in (7.3)). Functions satisfying this condition are called *absolute minimal Lipschitz extensions*, or AMLEs for short.

It was shown in [3] that AMLEs F in Euclidean domains satisfy $\Delta_\infty F = 0$, that is, they are ∞-harmonic. Here,

$$\Delta_\infty F = \sum_{i,j=1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_i \partial x_j}$$

is the ∞-Laplacian of F. Indeed, a function on an Euclidean domain is an AMLE if and only if it is ∞-harmonic, see for example [11] or [4, Theorem 4.13].

There are at least two ways of constructing AMLEs in the Euclidean setting, see [3] or [4] for a general overview of the topic. The first of the two methods employed in [3] uses a Perron method (which, in [4] is also called *comparison with cones*) and requires only the knowledge of the metric, see [25] for the extension of this method to metric spaces that are length spaces. The second method employed in [3] was to construct an ∞-harmonic extension, and since ∞-harmonicity and the AMLE property concide in the Euclidean setting, this construction will also yield an AMLE. This latter method used the non-linear potential theory to construct p-harmonic extensions u_p of the Lipschitz boundary data $f : \partial\Omega \to \mathbb{R}$ for $1 < p < \infty$, and showed that there exists a sequence $p_k \to \infty$ for which u_{p_k} converges uniformly to a limiting Lipschitz function u_∞, which was then shown to be ∞-harmonic. This method was extended to the setting of doubling metric measure spaces supporting a p-Poincaré inequality for some finite p in [26] to construct ∞-harmonic extensions of the boundary data f. It was shown in [26] that if the underlying metric measure space satisfies a p-weak Fubini property (see Section 4), then ∞-harmonic functions are AMLEs. In Section 4 we will explain how to construct ∞-harmonic functions and describe some connections between AMLEs and ∞-harmonic functions in the setting of metric measure spaces that support an ∞-Poincaré inequality but might not support any p-Poincaré inequality for any finite $p \geq 1$. In Section 5 we will describe examples of such metric measure spaces (such as the Sierpinski Carpet).

7.3 Characterizations of ∞-Poincaré inequality

The notion of p-modulus zero family of curves, as described in Definition 7.2.2, is extended to the notion of p-modulus as an outer measure on the collection of all rectifiable curves in X as follows.

Definition 7.3.1. *Let Γ be a family of rectifiable curves in X, and let $\mathcal{A}(\Gamma)$ denote the collection of all non-negative Borel measurable functions ρ on X such that $\int_\gamma \rho\, ds \geq 1$*

for each $\gamma \in \Gamma$. *For* $1 \le p < \infty$, *it is traditional to set*

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p \, d\mu,$$

see [20] for example. For $p = \infty$ *we extend the above notion by considering the p-th root of* Mod_p *and letting* $p \to \infty$*:*

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \|\rho\|_{L^\infty(X)}.$$

Recall that X is a complete metric space equipped with a doubling measure μ. The following characterizations of ∞-Poincaré inequality were established in [13] and [14]. In what follows, we will automatically assume that every rectifiable path (and these are the only paths we will consider in this article) is arc-length parametrized.

Theorem 7.3.2. ([14, Theorem 3.1], [13, Theorem 4.7]) *Let X be complete, connected, and μ be doubling. Then the following are equivalent:*

1. *X supports an ∞-Poincaré inequality.*
2. *There exist constants* $C, \lambda \ge 1$ *such that if* $f \in L^\infty(X)$ *with an upper gradient* $g \in L^\infty(X)$, *then f is* $C\|g\|_{L^\infty(X)}$*-Lipschitz continuous on X and is* $C\|g\|_{L^\infty(\lambda B)}$*-Lipschitz continuous on each ball* $B \subset X$.
3. *There is a constant* $C \ge 1$ *such that for all* $N \subset X$ *with* $\mu(N) = 0$ *and* $x, y \in X$ *with* $x \ne y$, *there is a rectifiable curve* γ *with end points* x, y *such that* $\ell(\gamma) \le Cd(x, y)$ *and* $\mathcal{H}^1(\gamma^{-1}(N)) = 0$.
4. *There is a constant* $C \ge 1$ *such that whenever* $x, y \in X$ *are two distinct points, setting* $\Gamma(x, y, C)$ *to be the collection of all rectifiable curves* γ *in X with end points* x, y *such that* $\ell(\gamma) \le C\, d(x, y)$, *we have*
$$\mathrm{Mod}_\infty(\Gamma(x, y, C)) > 0.$$
5. *There is a constant* $C \ge 1$ *such that whenever* $x, y \in X$ *are two distinct points, with* $\Gamma(x, y, C)$ *as above we have*
$$\frac{1}{C\, d(x, y)} \le \mathrm{Mod}_\infty(\Gamma(x, y, C)) \le \frac{C}{d(x, y)}.$$

Note that while Property (4) gives only a qualitative control of the ∞-modulus of the family $\Gamma(x, y, C)$, Property (5) gives quantitative control. Property (3) on the other hand is a purely geometric measure-theoretic property; thus it is clear that ∞-Poincaré inequality is a geometric measure-theoretic notion.

Remark 7.3.3. As a consequence of the above theorem, more specifically Property (2) of the theorem, we know that if $f : X \to \mathbb{R}$ has an ∞-weak upper gradient g in X and that $g \le L$ μ-a.e. in X, then f is CL-Lipschitz continuous on X. This is of particular use to us in Section 4.

Observe also that if X does *not* support any ∞-Poincaré inequality, then for each positive integer n we can find two distinct points $x_n, y_n \in X$ and a set $N_n \subset X$ with $\mu(N_n) = 0$ such that N_n separates x_n from y_n, that is, every rectifiable curve γ in X with end points x_n, y_n and with $\ell(\gamma) \le n\, d(x_n, y_n)$ must see N_n as a large set ($\mathcal{H}^1(\gamma^{-1}(N_n)) > 0$). One can always choose N_n to be independent of n by replacing N_n with $N := \bigcup_{k\in\mathbb{N}} N_k$. It would be interesting to know whether one can choose x_n, y_n to be also independent of n. In Section 5 we will give an example (Sierpiński Carpet) of a complete doubling metric measure space X for which there is a set $N \subset X$ with $\mu(N) = 0$ such that *every* pair of distinct points $x, y \in X$ plays the role of x_n, y_n in the above discussion.

We now compare the above result regarding ∞-Poincaré inequality to analogous results concerning p-Poincaré inequalities for finite $p \ge 1$. For the sake of brevity, we focus on metric measure spaces whose measure μ is Ahlfors Q-regular for some $Q \ge 1$, that is, there is a constant $C \ge 1$ such that whenever $x \in X$ and $0 < r < 2\mathrm{diam}\,(\mathrm{X})$,

$$\frac{r^Q}{C} \le \mu(B(x, r)) \le C\, r^Q.$$

A metric measure space supports a 1-Poincaré inequality if and only if it supports a *relative isoperimetric inequality*, that is, with $P(E, A)$ denoting the perimeter measure of the set $E \subset X$ inside an open set $A \subset X$ (see for example [1]),

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \le C\ \mathrm{rad}(B)\, P(E, \lambda B)$$

for all balls $B \subset X$. The perimeter measure $P(E, B)$ is comparable to the co-dimension 1 Hausdorff measure of the part of the measure-theoretic boundary of E that is contained in B when E is of finite perimeter in the sense of [1], that is,

$$P(E, B) \approx \lim_{r\to 0^+} \inf \left\{ \sum_{i\in I} \frac{\mu(B_i)}{\mathrm{rad}(B_i)} \ :\ I \subset \mathbb{N},\ \text{each } B_i \text{ is a ball, } E\cap B \subset \bigcup_{i\in I} B_i, \mathrm{rad}(B_i) \le r \right\}.$$

An Ahlfors Q-regular metric measure space supports a Q-Poincaré inequality if and only if it is a *Q-Loewner space* in the sense of Heinonen and Koskela [21]. A space is Q-Loewner if there is a decreasing homeomorphism $\phi : (0, \infty) \to (0, \infty)$ such that whenever $F, K \subset X$ are two compact connected sets with at least two points each such that $F \cap K$ is empty, then the p-modulus of the family of all rectifiable curves in X with one end point in F and the other in K is at least $\phi(\mathrm{d}(F, K)/\mathrm{diam}\,(\mathrm{F}) \wedge \mathrm{diam}\,(\mathrm{K}))$. Compared to the above characterization of 1-Poincaré inequality, the characterizaton of Q-Poincaré inequality is more intimately connected with the number of rectifiable curves in X, and is more similar to Property (5) of Theorem 7.3.2 above. A refinement of the argument found in [21] would allow us to refine the notion of Q-Loewner property by letting us to restrict attention to C-quasiconvex curves (as in the sense of Properties (4) and (5) above) connecting F to K. In the Q-Loewner property one needs the quantitative lower bound for the p-modulus of the relevant family of curves; surprisingly, from

Property (4) of Theorem 7.3.2 we know that a characterization of ∞-Poincaré inequality only requires that the ∞-modulus of the relevant family of curves be positive.

This connection between the amount of rectifiable curves in X and the support of p-Poincaré inequality becomes stronger when $p > Q$.

Proposition 7.3.4 ([14, Theorem 5.1(3)]). *Suppose that X is Ahlfors Q-regular and that $p > Q$. Then X supports a p-Poincaré inequality if and only if there is a constant $C \geq 1$ such that whenever $x, y \in X$ are two distinct points, then*

$$\operatorname{Mod}_p(\Gamma(x, y, C)) \geq \frac{1}{C\, d(x, y)^{p-Q}}.$$

Note that the lower bound above for $\operatorname{Mod}_p(\Gamma(x, y, C))^{1/p}$ tends to the lower bound found in Property (4) of Theorem 7.3.2. This gives us hope that other geometric properties related to p-Poincaré inequalities persist also for ∞-Poincaré inequality. Unfortunately this is not the case. Properties such as persistence of p-Poincaré inequalities under pointed measured Gromov-Hausdorff limits, self-improvement of p-Poincaré inequality to q-Poincaré inequality for some $q < p$ when $p > 1$, Rademacher-type differentiability of Lipschitz functions in the sense of Cheeger [9] all fail for spaces that support ∞-Poincaré inequality but no p-Poincaré inequality for any finite $p \geq 1$. We will describe some examples regarding this in Section 5. These examples are from [18] and [14].

Now we revert back to our standing assumptions that X is complete and μ is doubling. If X supports a p-Poincaré inequality and $\Omega \subset X$ is a *uniform domain*, then Ω supports a p-Poincaré inequality, see [6]. A domain Ω is a uniform domain if there is a constant $C \geq 1$ such that whenever $x, y \in \Omega$ there is a curve γ, called a *uniform curve*, with end points x, y such that $\ell(\gamma) \leq C\, d(x, y)$ and whenever z is a point in γ, and $\gamma_{x,z}$, $\gamma_{y,z}$ are two subcurves of γ with end points x, z and y, z respectively, we have

$$\min\{\ell(\gamma_{x,z}), \ell(\gamma_{y,z})\} \leq C \operatorname{dist}(z, X \setminus \Omega).$$

No geometric characterization is known for domains in X that would inherit the property of supporting a p-Poincaré inequality for $1 \leq p < \infty$. However, we have the following geometric characterization for inheritance of ∞-Poincaré inequality.

Lemma 7.3.5. *Suppose that X supports an ∞-Poincaré inequality. Let $\Omega \subset X$ be a domain such that the restriction of μ to Ω is doubling. Then Ω, equipped with the restriction of the measure μ and the metric d to Ω, supports an ∞-Poincaré inequality if and only if it is quasiconvex.*

Recall that a set $A \subset X$ is quasiconvex if there is a constant $C \geq 1$ such that whenever $x, y \in A$ there is a rectifiable curve γ in A with end points x, y such that $\ell(\gamma) \leq C\, d(x, y)$. Since this characterization has not appeared in any other current literature, we provide its proof here. The proof relies heavily on the characterization (3) of Theorem 7.3.2.

As stated, this theorem requires X to be complete. However, it does remain valid when X is locally complete as well, as demonstrated in [14]. If X is complete and Ω is a domain in X, then Ω is necessarily locally complete.

Proof. Let $N \subset \Omega$ such that $\mu(N) = 0$, and $x, y \in \Omega$. By assumption, there is a quasiconvex curve γ in Ω connecting x to y. For each point $z \in \gamma$, there exists $r_z > 0$ such that $B(z, 2Cr_z) \subset \Omega$, where C is the constant for X from Theorem 7.3.2(3) (which exists because X supports an ∞-Poincaré inequality). The collection of balls $B(z, r_z)$ forms a cover of the compact set γ, and so there is a finite subcover, say $B_i = B(z_i, r_i)$, $i = 1, \cdots, k$. Let a_i be the location at which γ first enters B_i, and b_i be the last time γ exits B_i. Note that

$$\sum_{i=1}^{k} d(a_i, b_i) \le 2\ell(\gamma) \le 2Cd(x, y).$$

Now we use the fact that X supports an ∞-Poincaré inequality (see Theorem 7.3.2(3)) to find a quasiconvex curve β_i (with $\ell(\beta_i) \le Cd(a_i, b_i)$) connecting a_i to b_i such that $\beta_i \subset 2CB_i \subset \Omega$ and $\mathcal{H}^1(\beta_i^{-1}(N)) = 0$. Let γ_0 be the concatenation of the curves β_i, $i = 1, \cdots, k$. Then $\ell(\gamma_0) \le 2C^2 d(x, y)$ and γ_0 lies in Ω and connects x to y, with $\mathcal{H}^1(\gamma_0^{-1}(N)) = 0$. By Theorem 7.3.2(3), the support of an ∞-Poincaré inequality follows.

Conversely, if Ω supports an ∞-Poincaré inequality, then by Theorem 7.3.2 we know that Ω has to be quasiconvex. This completes the proof. □

7.4 Existence of ∞-harmonic extensions of Lipschitz functions

Throughout this section we will assume in addition to the doubling property of μ that μ also supports an ∞-Poincaré inequality.

We fix a bounded domain $\Omega \subset X$ such that $\mu(X \setminus \Omega) > 0$, and an L-Lipschitz function $f : X \to \mathbb{R}$. In this section we seek to find a function $u \in N^{1,\infty}(X)$ that is ∞-harmonic in Ω and $u = f$ on $X \setminus \Omega$.

The definition of ∞-harmonic functions can be found in Definition 7.2.8 above. From Remark 7.2.3 we know that the minimal ∞-weak upper gradient g_f of a function f with $g_f \in L^\infty(X)$ can be modified on a set of measure zero such that the modified function is an upper gradient of f. Thus from now on g_f will denote such a minimal *upper gradient* of f.

Definition 7.4.1. *The space $N_0^{1,\infty}(\Omega)$ consists of all the functions $v \in N^{1,\infty}(X)$ that satisfy $v = 0$ on $X \setminus \Omega$.*

It follows from Definition 7.2.8 that a function $u \in N^{1,\infty}(X)$ is ∞-harmonic in Ω if and only if for each $v \in N_0^{1,\infty}(X)$, we have

$$\|g_u\|_{L^\infty(\mathrm{supt}(v)\cap\Omega)} \le \|g_{u+v}\|_{L^\infty(\mathrm{supt}(v)\cap\Omega)}. \tag{7.4}$$

Note that Lipschitz functions belong to $N^{1,p}_{loc}(X)$ for each $1 \le p \le \infty$, and hence if X supports a p-Poincaré inequality, for $\infty > q \ge p$ one can find a Hölder continuous q-harmonic function on Ω that agrees with f in $X \setminus \Omega$, with Hölder continuity constant dependent solely on q, the doubling constant and the constants related to the Poincaré inequality, and the bound on f and the Lipschitz constant of f. The uniform limit of a subsequence of the sequence of q-harmonic functions, as $q \to \infty$, will yield an ∞-harmonic function that solves the above-stated problem, see for example [26]. There are many complete doubling metric measure spaces that support an ∞-Poincaré inequality but support no p-Poincaré inequality for any finite $p \ge 1$. For such spaces this approach might fail to give an ∞-harmonic function. In [15] one modifies the above approach by considering the following rather than q-harmonic functions.

From now on, L will denote the essential supremum $\|g_f\|_{L^\infty(X)}$ of the minimal ∞-weak upper gradient of f.

Definition 7.4.2. *We fix $L > 0$ as above, and set $N^{1,\infty}_L(X)$ to be the collection of all functions u on X that have an upper gradient g with $\|g\|_{L^\infty(X)} \le L$. For $u \in N^{1,\infty}_L(X)$ the set $D_L(u)$ is the collection of all upper gradients g of u such that $\|g\|_{L^\infty(X)} \le L$.*

Functions in $N^{1,\infty}_L(X)$ might not be L-Lipschitz, but given that X supports an ∞-Poincaré inequality, they are CL-Lipschitz where C is the constant given by the ∞-Poincaré inequality on X.

Definition 7.4.3. *Fix $1 < p < \infty$. For $u \in N^{1,\infty}(X)$ we set $I^p_L(u) := \inf_{g \in D_L(u)} \int_\Omega g^p \, d\mu$, and let*

$$J^p_f = \inf_{u \in N^{1,\infty}(X)\,:\, u=f \text{ on } X\setminus\Omega} I^p_L(u).$$

In the above, if $u \in N^{1,\infty}(X)$ but $D_L(u)$ is empty, then $I^p_L(u) = \infty$.

Note that $J^p_f \le I_L(f)^p \le L^p \mu(\Omega) < \infty$, and so we can find a sequence $u_k \in N^{1,\infty}_L(X)$ with $u_k = f$ on $X\setminus\Omega$ such that $\lim_k I^p_L(u_k) = J^p_f$. Since each u_k is CL-Lipschitz, the family $\{u_k\}_k$ is equicontinuous on X, and since $u_k = f$ on $X\setminus\Omega$ with Ω bounded, it follows that the family is also equibounded on X. Thus an application of the Arzelà-Ascoli theorem allows us to, by passing to a subsequence if necessary, find a CL-Lipschitz function u_p on X such that $u_k \to u_p$ uniformly on X.

It was shown in [15] that $J^p_f = I^p_L(u_p)$ and that $u_p \in N^{1,\infty}_L(X)$ with $u_p = f$ on $X \setminus \Omega$. It was also shown there that such u_p is unique given f. This uniqueness result was used to show that solutions u_p satisfy a weak comparison principle: if $F \in N^{1,\infty}_L(X)$ such that $f \le F$ on $X \setminus \Omega$, then the solution U_p associated with the boundary data F satisfies $u_p \le U_p$ on Ω. In the proof of this comparison principle the local nature of the L^p-integral was a key tool.

The next step was to fix a monotone increasing sequence $\{p_k\}_k$ with $1 < p_k < \infty$ and for each $k \in \mathbb{N}$ consider u_{p_k} as above. By passing to a subsequence if necessary, it was possible to obtain a uniform limit ϕ of the equicontinuous equibounded sequence

$\{u_{p_k}\}_k$ of CL-Lipschitz functions, and show that ϕ is ∞-harmonic in Ω. Thus we have the following theorem (recall the standing assumption for this section that X supports an ∞-Poincaré inequality).

Theorem 7.4.4 ([15]). *The function ϕ is ∞-harmonic in Ω with $\phi \in N^{1,\infty}(X)$ and $\phi = f$ on $X \setminus \Omega$.*

The next natural question to ask is whether ∞-harmonic functions are necessarily AMLEs. It was shown in [15] that this is not the case, see the next section for a description of this example. In [26] it was shown that if the metric measure space satisfies a p-weak Fubini property, then ∞-harmonic functions are AMLEs and that AMLEs are also ∞-harmonic.

Definition 7.4.5. *Let $1 \le p \le \infty$. A metric measure space is said to satisfy a p-weak Fubini property if there are positive constants C and τ_0 such that whenever $0 < \tau < \tau_0$ and $B_1, B_2 \subset X$ are measurable sets with positive measure such that $\mathrm{d}(B_1, B_2) > \tau \max\{\mathrm{diam}\,(B_1), \mathrm{diam}\,(B_2)\}$, then $\mathrm{Mod}_p(\Gamma(B_1, B_2, \tau) > 0$ where $\Gamma(B_1, B_2, \tau)$ is the collection of all rectifiable curves γ in X with one end point in B_1, the other in B_2, and $\ell(\gamma) < \mathrm{d}(B_1, B_2) + C\tau$.*

A space satisfying a p-weak Fubini property will necessarily satisfy a q-weak Fubini property for each $p \le q \le \infty$. The following simple geometric characterization of ∞-weak Fubini property, akin to that of Theorem 7.3.2(3), holds. No such characterization is known to hold for p-weak Fubini property for finite $p \ge 1$. Note also by Theorem 7.3.2 that if X satisfies an ∞-weak Fubini property, then X supports an ∞-Poincaré inequality.

Lemma 7.4.6 ([15]). *X satisfies an ∞-weak Fubini property if and only if for every set $N \subset X$ with $\mu(N) = 0$, every $\varepsilon > 0$, and every pair of distinct points $x, y \in X$, there is a rectifiable curve γ with end points x, y such that $\ell(\gamma) \le d(x, y) + \varepsilon$ and $\mathcal{H}^1(\gamma^{-1}(N)) = 0$.*

Theorem 7.4.7 ([15]). *If X satisfies an ∞-weak Fubini property and $f : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X)$, then every ∞-harmonic extension of f to Ω is also an AMLE. Furthermore, every AMLE of f to Ω is necessarily ∞-harmonic in Ω.*

The ∞-weak Fubini property is not an unreasonable property to consider, as the following proposition shows. Given a set $N \subset X$ with $\mu(N) = 0$ we set the function $d_N : X \times X \to [0, \infty]$ as follows:

$$d_N(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is rectifiable with end points } x, y \text{ and } \mathcal{H}^1(\gamma^{-1}(N)) = 0\}.$$

If X supports an ∞-Poincaré inequality, then by Property (3) of Theorem 7.3.2 we see that d_N is a metric on X and that $d(x, y) \le d_N(x, y) \le C\, d(x, y)$. We set

$$d_\mu(x, y) := \sup\{d_N(x, y) : N \subset X \text{ with } \mu(N) = 0\}.$$

By above, if X supports an ∞-Poincaré inequality, then d_μ is biLipschitz equivalent to the original metric d.

Proposition 7.4.8 ([15]). *If X supports an ∞-Poincaré inequality, then the metric measure space (X, d_μ, μ) satisfies an ∞-weak Fubini property.*

Thus if X supports an ∞-Poincaré inequality, then the class of AMLEs and the class of ∞-harmonic functions with respect to the metric d_μ are the same class and this class is non-empty. It was shown in [24] and [33] that AMLEs of Lipschitz boundary functions in a geodesic space are unique. It follows then that if X supports an ∞-weak Fubini property, then $d_\mu = d$ and so ∞-harmonic extensions of Lipschitz functions are unique.

Going one step further, it was shown in [15] that the class of ∞-harmonic functions with respect to the metric d and the class of ∞-harmonic functions with respect to the metric d_μ coincide if X supports an ∞-Poincaré inequality. Hence even if X does not satisfy an ∞-weak Fubini property, uniqueness of ∞-harmonic functions in (X, d, μ) follows under the uniqueness of ∞-harmonic functions in (X, d_μ, μ). We will demonstrate in the next section that if the ∞-weak Fubini property fails, then ∞-harmonic extensions need not be unique even if the metric space is a geodesic space. We do not know whether if X is a *geodesic space* supporting an ∞-Poincaré inequality then $d_\mu = d$.

7.5 Examples

In this section we give some examples that show the optimality of the results presented in this paper. We begin with an example of a complete metric measure space whose measure is doubling and is quasiconvex, but does not support any ∞-Poincaré inequality.

Example 7.5.1. Let X be the Sierpiński carpet, equipped with the Euclidean metric and the natural Hausdorff measure $\mu = \mathcal{H}^{\log(8)/\log(3)}$. Then μ is doubling, and indeed is Ahlfors $\log(8)/\log(3)$-regular. Furthermore, X is $\sqrt{2}$-quasiconvex. However, by the results of [7] X cannot support an ∞-Poincaré inequality. Indeed, the projection of the measure μ to $[0, 1]$ via the first coordinate projection map yields a measure μ_0 that is singular with respect to the 1-dimensional Lebesgue measure $\mathcal{L}^1$ on $[0, 1]$ (see [7]). Thus there is a set $N_1 \subset [0, 1]$ with $\mu_0(N_1) = 0$ but $\mathcal{L}^1(N_1) = 1$, and since both μ_0 and $\mathcal{L}^1$ are Radon measures, we can even choose N to be a Borel set. Denoting by $\Pi_1 : X \to [0, 1]$ the first coordinate projection map, we define a function $f : X \to \mathbb{R}$ by

$$f(x) := \int_0^{\Pi_1(x)} \chi_{N_1}(t)\,dt.$$

It is easy to see that f is $\sqrt{2}$-Lipschitz continuous on X and that $g = \chi_{N_1} \circ \Pi_1$ is an upper gradient of f, see for example [13, Lemma 4.13]. Therefore $f \in N^{1,\infty}(X)$, and we note that f is non-constant on X (since N_1 has full measure in $[0, 1]$ with respect to $\mathcal{L}^1$, the measure with respect to which the above integral was taken). On the other hand, $X = B((0, 0), 2) \cap X =: B$, and we have that $⨍_B |f - f_B|\, d\mu > 0$ but $\|g\|_{L^\infty(\lambda B)} = 0$; hence X cannot support any ∞-Poincaré inequality.

The above example shows that one needs the full strength of Property (3) of Theorem 7.3.2 in order to characterize ∞-Poincaré inequality.

The next two examples explore the Sierpinski carpet further in the context of ∞-harmonic functions and AMLEs, see [15] for more details.

Example 7.5.2. If X satisfies the stronger requirement of ∞-weak Fubini property, it is directly seen that $d_\mu = d$ on X. As explained above, the Sierpiński carpet does not support any ∞-Poincaré inequality and hence cannot satisfy any ∞-weak Fubini property. Since the length metric on this carpet is biLipschitz equivalent to the Euclidean metric, it follows that the above statement holds also when the carpet is equipped with the length metric. Recall the set N_1 from Example 7.5.1 above, and let $N = (\Pi_1^{-1}(N_1) \cup \Pi_2^{-1}(N_1)) \cap X$ where X is the carpet. Here Π_1 and Π_2 are the first coordinate and the second coordinate projection maps from the carpet to $[0, 1]$. Note that $\mu(N) = 0$, but given any curve γ in the carpet with end points x, y such that $(x_1, x_2) = x \neq y = (y_1, y_2)$, we must have

$$\mathcal{H}^1(\gamma^{-1}(N)) \geq \max\{\mathcal{H}^1(\Pi_1 \circ \gamma(\gamma^{-1}(N))), \mathcal{H}^1(\Pi_2 \circ \gamma(\gamma^{-1}(N)))\} \geq \max\{|x_1 - y_1|, |x_2 - y_2|\} > 0.$$

It follows that $d_N(x, y) = \infty$, and so $d_\mu \neq d$ in the carpet.

Example 7.5.3. The Sierpiński carpet also gives a situation where an ∞-harmonic function is not necessarily an AMLE. To construct such a function, set $g := \chi_N$, where N is the set given in Example 7.5.2. Let $E := \{0\} \times [0, 1]$, and for points $x = (x_1, x_2)$ in the carpet we define

$$f(x) := \inf_\gamma \int_\gamma g\, ds,$$

where the infimum is over all rectifiable curves γ in the carpet with one end point at x and the other at E. Note that f is zero on E, but for $x \notin E$ we have $f(x) \geq x_1 > 0$. Hence f is non-constant. From [15] we know that f is $\sqrt{2}$-Lipschitz on the carpet with respect to the Euclidean metric, and is 1-Lipschitz with respect to the length metric on X. Its minimal ∞-weak upper gradient g_f satisfies $g_f \leq g$ μ-almost everywhere, and so by the fact that $\mu(N) = 0$, we have $\|g_f\|_{L^\infty(X)} \leq \|g\|_{L^\infty(X)} = 0$, and therefore f is automatically ∞-harmonic in the carpet. However, $\mathrm{LIP}(f, X) > 0$ because f is non-constant. Let Ω be the domain $X \setminus E$; then f is not AMLE in Ω since the only AMLE extension of the zero function on $E = \partial\Omega$ is the zero extension.

We next turn our attention to examples related to spaces supporting an ∞-Poincaré inequality.

Example 7.5.4. In this example we describe a metric measure space that is complete, doubling, supports an ∞-Poincaré inequality, but does *not* support any p-Poincaré inequality for any finite $p \geq 1$. The details regarding this example can be found in [18, Example 3.7].

Let $Q_1 = [0,1]^2$, and let Q_2 be the set obtained from Q_1 in the first step of the construction of the Sierpiński carpet, that is, Q_2 is obtained from Q_1 by first dividing Q_1 into 9 equal squares, each of side length $1/3$, and then removing the middle *open* square $(1/3, 2/3)^2$. Note that Q_2 is the union of 8 squares, each of side length $1/3$. Q_3 is obtained from Q_2 by repeating the above process for each of the 8 squares that make up Q_2 to obtain 8^2 squares, each of side length $1/3^2$. Proceeding inductively, for each positive integer n we have a union of 8^n *closed* squares, each of side length $1/3^n$, making up the set Q_n. Note that the Sierpinski carpet is the set $\bigcap_{n\in\mathbb{N}} Q_n$. In this example we are not interested in this carpet, as we saw in the previous examples that the carpet does not support any ∞-Poincaré inequality. Instead we consider the complete set $X \subset [0,\infty) \times [0,1] \subset \mathbb{R}^2$ given by

$$X = \bigcup_{n\in\mathbb{N}} Q_n + (n-1, 0).$$

Thus X is obtained from the strip $[0,\infty) \times [0,1]$ by removing the first middle-third square from the first unit square $[0,1]^2$ in the strip, removing the first and second steps of the construction of the carpet from the second unit square $[1,2] \times [0,1]$ and so on. The metric on X is the Euclidean metric, but the measure μ is not the Lebesgue measure $\mathcal{L}^2$ restricted to X (since this would fail to be doubling on X at large scales), but the following measure: for each $n \in \mathbb{N}$ we set

$$\mu_n = \left(\frac{9}{8}\right)^{n-1} \mathcal{L}^2|_{Q_n+(n-1,0)},$$

and set

$$\mu = \sum_{n\in\mathbb{N}} \mu_n.$$

Then X, equipped with the Euclidean metric and the measure μ, is doubling and is complete. It is directly verifiable by the use of Fubini theorem that X satisfies Property (3) of Theorem 7.3.2 (with constant $C = \sqrt{2}$), and so X supports an ∞-Poincaré inequality.

Suppose that X supports a p-Poincaré inequality for some finite $p \geq 1$. The domains $Q_n + (n-1, 0)$ are uniform domains in X with uniformity constant 2; hence, it support a p-Poincaré inequality with constants that do not depend on n, see [6]. Thus the spaces $X_n := Q_n$, equipped with the Euclidean metric and the measure

$\nu_n := (\frac{9}{8})^{n-1}\mathcal{L}^2|_{Q_n}$ will be doubling and support a p-Poincaré inequality, with the relevant constants independent of n. Note that the Sierpiński carpet is the pointed (at the point $(0,0) \in X$) measured Gromov-Hausdorff limit of the sequence $(X_n, d_{Euc}, \nu_n, p_n)$ with $p_n = (0,0)$, and so by a result of Cheeger [9] (see also [22, Chapter 11]) the Sierpinski carpet should support a p-Poincaré inequality as well and hence should support an ∞-Poincaré inequality. Since this is not possible (see Example 7.5.1 above), it follows that X does not support any p-Poincaré inequality for any finite $p \geq 1$ either.

The above example serves two purposes. It shows that the self-improvement property of Keith and Zhong [27] does not hold for $p = \infty$, and it also shows that the support of an ∞-Poincaré inequality does *not* persist under pointed measured Gromov-Hausdorff limits. Indeed, the sequence of spaces (X_n, ν_n) each support an ∞-Poincaré inequality with associated constants independent of n, but this sequence converges under Gromov-Hausdorff limit to the Sierpinski carpet equipped with its natural Hausdorff measure, which in turn does *not* support an ∞-Poincaré inequality.

The technique of sphericalization, studied in [8, 16, 17, 23, 30], applied to the above-constructed X yields a *compact* doubling metric measure space supporting an ∞-Poincaré inequality (see [16]) but *no* p-Poincaré inequality for any finite $p \geq 1$. The failure of supporting a finite p-Poincaré inequality happens locally and asymptotically at one point in this compact space, and so the results of [26] do not apply in this space when the domain of interest contains this bad point in it.

The final example of this section deals with the lack of Rademacher-type differentiability for spaces that support ∞-Poincaré inequality but no better.

Example 7.5.5. Since the Euclidean space $\mathbb{R}$, equipped with the Lebesgue measure $\mathcal{L}^1$ and the Euclidean metric, supports a 1-Poincaré inequality, it clearly also supports an ∞-Poincaré inequality. Let ν be a singular doubling measure on $\mathbb{R}$ (for example, the Riesz product measure, see [36], [18, Section 4], or [35, page 40, Section 8.8(a)]). Then the measure $\mu = \mathcal{L}^1 + \nu$ is a doubling measure which, by Theorem 7.3.2(3) and by the fact that null sets for μ are necessarily null sets for $\mathcal{L}^1$, also supports an ∞-Poincaré inequality.

Since ν is singular to $\mathcal{L}^1$, there is a set $N \subset \mathbb{R}$ with $\mathcal{L}^1(N) = 0$ such that $\nu(N) > 0$. By a result of Choquet [10], there is a Lipschitz function f on $\mathbb{R}$ that is not Euclidean differentiable anywhere in N. Suppose that $X = \mathbb{R}$, equipped with the Euclidean metric and the measure μ, supports a Cheeger type differentiable structure as in [9]. Then, the Cheeger differential Df exists μ-almost everywhere in $\mathbb{R}$, and hence as shown in [14, Example 4.7], (note that N must have infinitely many points as ν cannot have finite support) f has to be Euclidean differentiable at ν-almost every point in N, which violates the choice of f.

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