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On a conjecture of Cheeger

4.1 Introduction

In [7] Cheeger, proved that in every doubling metric measure space (X, ρ, μ) satisfying a Poincaré inequality Lipschitz functions are differentiable μ-almost everywhere. More precisely, he showed the existence of a family $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ of Borel charts (that is, $U_i \subset X$ is a Borel set, $X = \bigcup_i U_i$ up to a μ-negligible set, and $\varphi_i \colon X \to \mathbb{R}^{d(i)}$ is Lipschitz) such that for every Lipschitz map $f \colon X \to \mathbb{R}$ at μ-almost every $x_0 \in U_i$ there exists a unique (co-)vector $df(x_0) \in \mathbb{R}^{d(i)}$ with

$$\limsup_{x\to x_0} \frac{\left|f(x) - f(x_0) - df(x_0)\cdot(\varphi(x) - \varphi(x_0))\right|}{\rho(x, x_0)} = 0.$$

This fact was later axiomatized by Keith [15], leading to the notion of *Lipschitz differentiability space*, see Section 4.2 below.

Cheeger also conjectured that the push-forward of the reference measure μ under every chart ϕ_i has to be absolutely continuous with respect to the Lebesgue measure, that is,

$$(\varphi_i)_\#(\mu \llcorner U_i) \ll \mathcal{L}^{d(i)}\,,$$

see [7, Conjecture 4.63]. Some consequences of this fact concerning existence of bi-Lipschitz embeddings of X into some $\mathbb{R}^N$ are detailed in [7, Section 14], also see [8, 9].

Let us assume that $(X, \rho, \mu) = (\mathbb{R}^d, \rho_\mathcal{E}, \nu)$ with $\rho_\mathcal{E}$ the Euclidean distance and ν a positive Radon measure, is a Lipschitz differentiability space when equipped with the (single) identity chart (note that it follows a-posteriori from the validity of Cheeger's conjecture that no mapping into a higher-dimensional space can be a chart in a Lipschitz differentiability structure of $\mathbb{R}^d$). In this case the validity of Cheeger's conjecture reduces to the validity of the (weak) converse of Rademacher's theorem, which states that a positive Radon measure ν on $\mathbb{R}^d$ with the property that all Lipschitz functions are differentiable ν-almost everywhere must be absolutely continuous with respect to $\mathcal{L}^d$. Actually, it is well known to experts that this converse of Rademacher's theorem implies Cheeger's conjecture in any metric space, see for instance [15, Section 2.4], [6, Remark 6.11], and [12].

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The (strong) converse of Rademacher's theorem has been known to be true in $\mathbb{R}$ since the work of Zahorski [21], where he characterized the sets $E \subset \mathbb{R}$ that are sets of non-differentiability points of some Lipschitz function. In particular, he proved that for every Lebesgue negligible set $E \subset \mathbb{R}$ there exists a Lipschitz function which is nowhere differentiable on E.

The same result for maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ has been proved by Alberti, Csörnyei & Preiss for $d = 2$ as a consequence of a deep structural result for negligible sets in the plane [1, 2]. In 2011, Csörnyei & Jones [14] announced the extension of the above result to every Euclidean space. For Lipschitz maps $f\colon \mathbb{R}^d \to \mathbb{R}^m$ with $m < d$ the situation is fundamentally different and there exists a null set such that every Lipschitz function is differentiable at at least one point from that set, see [17, 18]. We finally remark that the weak converse of Rademacher's theorem in $\mathbb{R}^2$ can also be obtained by combining the results of [4] and [5], see [5, Remark 6.2 (iv)].

Recently, a result concerning the singular structure of measures satisfying a differential constraint was proved in [10]. When combined with the main result of [5] this proves the weak converse of Rademacher's theorem in any dimension, see [10, Theorem 1.14].

In this note we detail how the results in [5, 10] in conjunction with Bate's result on the existence of a sufficient number of independent Alberti representations in a Lipschitz differentiability space [6] imply Cheeger's conjecture; see Section 4.2 for the relevant definitions.

Theorem 4.1.1. *Let (X, ρ, μ) be a Lipschitz differentiability space and let (U, ϕ) be a d-dimensional chart. Then, $\phi_\#(\mu \llcorner U) \ll \mathcal{L}^d$.*

Note that by the same arguments of this paper Cheeger's conjecture would also follow from the results announced in [1] and [14].

After we finished writing this note we learned that similar results have been proved by Kell and Mondino [16] and by Gigli and Pasqualetto [13].

4.2 Setup

4.2.1 Lipschitz differentiability spaces

Throughout this chapter, the triple (X, ρ, μ) will always denote a *metric measure space*, that is, (X, ρ) is a separable, complete metric space and $\mu \in \mathcal{M}_+(X)$ is a positive Radon measure on X.

We call a pair (U, φ) such that $U \subset X$ is a Borel set and $\varphi\colon X \to \mathbb{R}^d$ is Lipschitz, a *d-dimensional chart* or simply a *d-chart*. A function $f\colon X \to \mathbb{R}$ is said to be *differentiable with respect to a d-chart* (U, φ) at $x_0 \in U$ if there exists a unique (co-)vector $df(x_0) \in$

$\mathbb{R}^d$ such that

$$\limsup_{x\to x_0} \frac{\left|f(x) - f(x_0) - df(x_0)\cdot(\varphi(x) - \varphi(x_0))\right|}{\rho(x, x_0)} = 0.$$

We call a metric measure space (X, ρ, μ) a *Lipschitz differentiability space* (also called a metric measure space that admits a *measurable differentiable structure*) if there exists a countable family of $d(i)$-charts (U_i, φ_i) $(i \in \mathbb{N})$ such that $X = \bigcup_i U_i$ and any Lipschitz map $f\colon X \to \mathbb{R}$ is differentiable with respect to every (U_i, φ_i) at μ-almost every point $x_0 \in U_i$.

4.2.2 Alberti representations

We denote by $\Gamma(X)$ the set of *curves* in X, that is, the set of all Lipschitz maps $\gamma\colon \operatorname{Dom}\gamma \to X$, for which the domain $\operatorname{Dom}\gamma \subset \mathbb{R}$ is non-empty and compact. Note that we are not requiring $\operatorname{Dom}\gamma$ to be an interval and thus the set $\Gamma(X)$ is sometimes also called the set of *curve fragments* on X. We equip $\Gamma(X)$ with the Hausdorff metric $\operatorname{dist}_{\mathcal{H}}$ on graphs and we consider it as a subspace of the Polish space

$$\mathcal{K} = \left\{ K \subset \mathbb{R} \times X \,:\, K \text{ compact} \right\}, \tag{4.1}$$

endowed with the Hausdorff metric. Moreover, by arguing as in [19, Lemma 2.20], it is easy to see that $\Gamma(X)$ is an F_σ-subset of $\mathcal{K}$, i.e. a countable union of closed sets.

The decomposition of a measure into a family of 1-dimensional Hausdorff measures supported on curves leads to the notion of Alberti representation. First introduced in [4] for the study of the rank-one property of BV-derivatives, this decomposition has turned out to be a key tool in the study of differentiability properties of Lipschitz functions, see for instance [1, 2, 5, 6].

Definition 4.2.1. *Let (X, ρ, μ) be a metric measure space. An* Alberti representation *of μ on a μ-measurable set $A \subset X$ is a parameterized family $(\mu_\gamma)_{\gamma\in\Gamma(X)}$ of positive Borel measures $\mu_\gamma \in \mathcal{M}_+(X)$ with*

$$\mu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\gamma,$$

together with a Borel probability measure $\pi \in \mathcal{P}(\Gamma(X))$ such that

$$\mu(B) = \int \mu_\gamma(B)\, \mathrm{d}\pi(\gamma) \qquad \text{for all Borel sets } B \subset A. \tag{4.2}$$

Here, the measurability of the integrand is part of the requirement of being an Alberti representation.

Remark 4.2.2. *Note that this definition is slightly different from the one in [6, Definition 2.2] since there the set $\Gamma(X)$ consists of* bi-Lipschitz *curves. Clearly, the existence of a representation in the sense of [6] implies the existence of a representation in our* *sense and this will suffice for our purposes. Let us, however, point out that the converse holds true as well. Indeed, the part of γ that contributes to the integral in (4.2) can be decomposed into countably many bi-Lipschitz pieces, see [19, Remark 2.17].*

We will further need the notion of *independent* Alberti representations of a measure. Let $C \subset \mathbb{R}^d$ be a closed, convex, one-sided cone, i.e. a set of the form

$$C := \left\{ v \in \mathbb{R}^d \,:\, v \cdot w \geq (1-\theta)\|v\| \right\}$$

for some $w \in \mathbb{S}^{d-1}$ and $\theta \in (0,1)$. With a Lipschitz map $\phi \colon X \to \mathbb{R}^d$, we say that an Alberti representation $\int \nu_\gamma \, \mathrm{d}\pi(\gamma)$ has *φ-directions in C* if

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\} \qquad \text{for } \pi\text{-a.e. curve } \gamma \text{ and } \mathcal{H}^1\text{-a.e. } t \in \operatorname{Dom}\gamma.$$

A number of m Alberti representations of μ are *φ-independent* if there are linearly independent cones $C_1, \dots, C_m$ such that the i'th Alberti representation has φ-directions in C_i. Here, linear independence of the cones $C_1, \dots, C_m$ means that any collection of vectors $v_i \in C_i \setminus \{0\}$ is linearly independent. In the case $X = \mathbb{R}^d$ we will always consider $\phi = \mathrm{Id}$.

One of the main results of [6] asserts that a Lipschitz differentiability space necessarily admits many independent Alberti representations, also cf. [5, Theorem 1.1]. Recall that according to Remark 4.2.2 any representation in the sense of [6] is also a representation in the sense of Definition 4.2.1.

Theorem 4.2.3. *Let (X, ρ, μ) be a Lipschitz differentiability space with a d-chart (U, φ). Then, there exists a countable decomposition*

$$U = \bigcup_{k \in \mathbb{N}} U_k, \qquad U_k \subset U \text{ Borel sets,}$$

such that every $\mu \llcorner U_k$ has d φ-independent Alberti representations.

A proof of this theorem can be found in [6, Theorem 6.6].

4.2.3 One-dimensional currents

To use the results of [10] we need a link between Alberti representations and 1-dimensional currents. Recall that a 1-*dimensional current T* in $\mathbb{R}^d$ is a continuous linear functional on the space of smooth and compactly supported differential 1-forms on $\mathbb{R}^d$. The *boundary* of T, ∂T is the distribution (0-current) defined via $\langle \partial T, f\rangle := \langle T, df\rangle$ for every smooth and compactly supported function $f \colon \mathbb{R}^d \to \mathbb{R}$. The *mass* of T, denoted by $\mathbf{M}(T)$, is the supremum of $\langle T, \omega\rangle$ over all 1-forms ω such that $|\omega| \leq 1$ everywhere. A current T is called *normal* if both T and ∂T have finite mass; we denote the set of normal 1-currents by $\mathbf{N}_1(\mathbb{R}^d)$.

By the Radon–Nikodým theorem, a 1-dimensional current T with finite mass can be written in the form $T = \vec{T}\|T\|$ where $\|T\|$ is a finite positive measure and $\vec{T}$ is a vector field in $L^1(\mathbb{R}^d, \|T\|)$ with $|\vec{T}(x)| = 1$ for $\|T\|$-almost every $x \in \mathbb{R}^d$. In particular, the action of T on a smooth and compactly supported 1-form ω is given by

$$\langle T, \omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x), \vec{T}(x)\rangle \, \mathrm{d}\|T\|(x) \, .$$

An *integer-multiplicity rectifiable* 1*-current* (in the following called simply rectifiable 1-current) $T = [\![E, \tau, m]\!]$ is a 1-current which acts on 1-forms ω as

$$\langle T, \omega\rangle = \int_{E} \langle \omega(x), \tau(x)\rangle \, m(x) \, \mathrm{d}\mathcal{H}^1(x) \, ,$$

where E is a 1-rectifiable set, $\tau(x)$ is a unit vector spanning the approximate tangent space $\mathrm{Tan}(E, x)$ and m is an integer-valued function such that $\int_E m \, \mathrm{d}\mathcal{H}^1 < \infty$. More information on currents can be found in [11].

The relation between Alberti representations and normal 1-currents is partially encoded in the following decomposition theorem, due to Smirnov [20].

Theorem 4.2.4. *Let $T = \vec{T}\|T\| \in \mathbf{N}_1(\mathbb{R}^d)$ be a normal* 1*-current with $|\vec{T}(x)| = 1$ for $\|T\|$-almost every x. Then, there exists a family of rectifiable* 1*-currents*

$$T_\gamma = [\![E_\gamma, \tau_\gamma, 1]\!], \qquad \gamma \in \Gamma,$$

where Γ is a measure space endowed with a finite positive Borel measure $\pi \in \mathcal{M}_+(\Gamma)$, such that the following assertions hold:

(i) T can be decomposed as

$$T = \int_\Gamma T_\gamma \, \mathrm{d}\pi(\gamma)$$

and

$$\mathbf{M}(T) = \int_\Gamma \mathbf{M}(T_\gamma) \, \mathrm{d}\pi(\gamma) = \int_\Gamma \mathcal{H}^1(E_\gamma) \, \mathrm{d}\pi(\gamma) \, ;$$

(ii) $\tau_\gamma(x) = \vec{T}(x)$ for $\mathcal{H}^1$-almost every $x \in E_\gamma$ and for π-almost every $\gamma \in \Gamma$;

(iii) $\|T\|$ can be decomposed as

$$\|T\| = \int_\Gamma \mu_\gamma \, \mathrm{d}\pi(\gamma) \, ,$$

where each μ_γ is the restriction of $\mathcal{H}^1$ to the 1*-rectifiable set E_γ.*

An Alberti representation of an Euclidean measure splits it into measures concentrated on “fragments” of curves. In general, these fragments cannot be glued together to obtain a 1-dimensional normal current since the boundary may have infinite mass. Nevertheless, the "holes" of every curve appearing in an Alberti representation of a measure $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ can be "filled" in such a way as to produce a normal 1-current T with $\nu \ll \|T\|$. Moreover, if the representation has directions in a cone C then the constructed normal current T has orienting vector $\vec{T}$ in $C \setminus \{0\}$ almost everywhere (with respect to $\|T\|$). Indeed, we have the following lemma, which is essentially [5, Corollary 6.5]; it can be interpreted as a partial converse to Theorem 4.2.4:

Lemma 4.2.5. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a finite Radon measure. If there is an Alberti representation $\nu = \int \nu_\gamma \, d\pi(\gamma)$ with directions in a cone C, then there exists a normal 1-current $T \in \mathbf{N}_1(\mathbb{R}^d)$ such that $\vec{T}(x) \in C \setminus \{0\}$ for $\|T\|$-almost every $x \in \mathbb{R}^d$ and $\nu \ll \|T\|$.*

Proof. For the purpose of illustration we sketch the proof.

Step 1. Given ν as in the statement, we claim that there exists a normal 1-current $T = \vec{T}\|T\|$ with $\mathbf{M}(T) \le 1$ and $\mathbf{M}(\partial T) \le 2$ such that $\vec{T}(x) \in C$, for $\|T\|$-almost every x and that ν is not singular with respect to $\|T\|$.

The claim follows from the proof of [5, Lemma 6.12]. For the sake of completeness let us present the main line of reasoning. By arguing as in Step 1 of the proof of [5, Lemma 6.12], to every $\gamma \in \Gamma(\mathbb{R}^d)$ with $\gamma'(t) \in C$ and a Borel measure $\nu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\gamma$ we can associate a 1-Lipschitz map $\psi_{\nu_\gamma} : [0,1] \to \mathbb{R}^d$ satisfying

$$\nu_\gamma(\operatorname{Im}(\psi_{\nu_\gamma})) > 0 \qquad \text{and} \qquad \psi'_{\nu_\gamma}(t) \in C \setminus \{0\} \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [0,1].$$

This map can moreover be chosen such that $\gamma \mapsto \psi_{\nu_\gamma}$ coincides with a Borel measurable map π-almost everywhere once we endow the set of curves with the topology of uniform convergence, see Step 3 in the proof of [5, Lemma 6.12].

Let $T_{\nu_\gamma} := [\![\operatorname{Im}\psi_{\nu_\gamma}, \tau_{\psi_{\nu_\gamma}}, 1]\!]$ be the rectifiable 1-current associated to ψ_{ν_γ} and set

$$T := \int T_{\nu_\gamma} \, d\pi(\gamma) .$$

Since ψ_{ν_γ} is 1-Lipschitz, $\mathcal{H}^1(\operatorname{Im}\psi_{\nu_\gamma}) \le 1$ and thus $\mathbf{M}(T) \le 1$. Moreover, for all smooth compactly supported functions $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\langle \partial T, f \rangle = \langle T, df \rangle = \int f(\psi_{\nu_\gamma}(1)) - f(\psi_{\nu_\gamma}(0)) \, d\pi(\gamma) ,$$

so that $\mathbf{M}(\partial T) \le 2$.

By assumption, $\vec{T}(x) \in C \setminus \{0\}$ for $\|T\|$-almost every $x \in \mathbb{R}^d$. To show that $\|T\|$ and ν are not mutually singular, for π-almost every γ set

$$\nu'_\gamma := \nu_\gamma \llcorner \operatorname{Im}\psi_{\nu_\gamma} \qquad \text{and} \qquad \nu' := \int \nu'_\gamma \, d\pi(\gamma) ,$$

so that $\nu' \neq 0$ and $\nu' \le \nu$. We will now establish that $\nu' \ll \|T\|$, for which we will prove that ν and $\|T\|$ are not mutually singular. Let $E \subset \mathbb{R}^d$ be such that $\|T\|(E) = 0$. Using

$$T = \int [\![\operatorname{Im}\psi_{\nu_\gamma}, \tau_{\psi_{\nu_\gamma}}, 1]\!] \, d\pi(\gamma) \qquad \text{with} \qquad \tau_{\psi_{\nu_\gamma}} = \frac{\psi'_{\nu_\gamma}}{|\psi'_{\nu_\gamma}|} \in C ,$$

we get

$$\mathcal{H}^1(\operatorname{Im} \psi_{\nu_\gamma} \cap E) = 0 \qquad \text{for } \pi\text{-a.e. } \gamma.$$

Since by definition $\nu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im} \gamma$, we have that $\nu'_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im} \psi_{\nu_\gamma}$. Thus, $\nu'(E) = 0$.

Step 2. Let us define

$$\mathcal{T} := \left\{ T \in \mathbf{N}_1(\mathbb{R}^d) \, : \, \mathbf{M}(T) \le 1, \mathbf{M}(\partial T) \le 2 \text{ and } \vec{T} \in C \, \|T\|\text{-a.e.} \right\}$$

and

$$\mathcal{T}_\nu := \left\{ T \in \mathcal{T} \, : \, \nu \text{ and } T \text{ are not singular} \right\} .$$

Note that if $C = \{ v \in \mathbb{R}^d \, : \, v \cdot w \ge (1-\theta)\|v\| \}$ for some $w \in \mathbb{S}^{d-1}$, $\theta \in (0,1)$, then $\vec{T} \in C$ almost everywhere implies that

$$\|T\| \ge T \cdot w \ge (1-\theta)\|T\| \tag{4.3}$$

as measures (here we are identifying T with an $\mathbb{R}^d$-valued Radon measure and use the pointwise scalar product). Moreover, as a consequence of the Radon–Nikodým theorem, for every $T \in \mathcal{T}_\nu$ we may write

$$\nu = g_{\|T\|}\|T\| + \nu^s_{\|T\|} \qquad \text{with} \qquad \nu^s_{\|T\|} \perp \|T\| \, , \int g_{\|T\|} \, \mathrm{d}\|T\| > 0 \, .$$

Let us set $M := \sup_{T \in \mathcal{T}_\nu} \int g_{\|T\|} \, \mathrm{d}\|T\| > 0$ and let $T_k \in \mathcal{T}_\nu$ be a sequence with

$$\int g_{\|T_k\|} \, \mathrm{d}\|T_k\| \to M.$$

Define

$$T := \sum_k 2^{-k} T_k$$

and note that $T \in \mathcal{T}$. Moreover, by (4.3), $\|T_k\| \ll \|T\|$ for all $k \in \mathbb{N}$, so that there exist $h_k : \mathbb{R}^d \to \mathbb{R}$ with

$$\int_E h_k \, \mathrm{d}\|T\| = \int_E g_{\|T_k\|} \, \mathrm{d}\|T_k\| \le \nu(E) \qquad \text{for all Borel sets } E \subset \mathbb{R}^d.$$

In particular, $T \in \mathcal{T}_\nu$ and $h_k \le g_{\|T\|}$. Set $m_k = \max_{1 \le j \le k} h_j$. By the monotone convergence theorem, $m_k \to m_\infty \le g_{\|T\|}$ in $\mathrm{L}^1(\mathbb{R}^d, \|T\|)$ and

$$M \le \lim_{k\to\infty} \int m_k \, \mathrm{d}\|T\| = \int m_\infty \, \mathrm{d}\|T\| \le \int g_{\|T\|} \, \mathrm{d}\|T\| \le M.$$

Hence, M is actually a maximum and it is attained by T.

We now claim that $\nu \ll \|T\|$. Indeed, assume by contradiction that $\nu = g_{\|T\|} \, \mathrm{d}\|T\| + \nu^s_{\|T\|}$ with $\nu^s_{\|T\|} \neq 0$. Since the Alberti representation of ν induces an Alberti representation of $\nu^s_{\|T\|}$, we can apply Step 1 to find a normal 1-current

$$S \in \mathcal{T}_{\nu^s_{\|T\|}} \subset \mathcal{T}_\nu$$

such that $\nu^s_{\|T\|}$ and $\|S\|$ are not mutually singular. In particular, if $\nu = g_{\|S\|}\,\mathrm{d}\|S\| + \nu^s_{\|S\|}$, then there exists a Borel set $F \subset \mathbb{R}^d$ such that

$$\|T\|(F) = 0 \qquad \text{and} \qquad \int_F g_{\|S\|}\,\mathrm{d}\|S\| > 0. \tag{4.4}$$

Let us define $W := (T + S)/2$ and note that by (4.3) it holds that $\|T\|, \|S\| \ll \|W\|$ so that $W \in \mathcal{T}_\nu$. Moreover, there are functions $h_T, h_S \le g_{\|W\|}$ such that

$$\int_E h_T\,\mathrm{d}\|W\| = \int_E g_{\|T\|}\,\mathrm{d}\|T\|\,, \qquad \int_E h_S\,\mathrm{d}\|W\| = \int_E g_{\|S\|}\,\mathrm{d}\|S\|$$

for all Borel sets E. However, for F as in (4.4) we obtain

$$M \ge \int_{\mathbb{R}^d} g_{\|W\|}\,\mathrm{d}\|W\| \ge \int_{\mathbb{R}^d} g_{\|T\|}\,\mathrm{d}\|T\| + \int_F g_{\|S\|}\,\mathrm{d}\|S\| > M,$$

a contradiction. □

4.3 Proof of Cheeger's conjecture

The key tool to prove Cheeger's conjecture is the following result from [10, Corollary 1.12]:

Theorem 4.3.1. *Let $T_1 = \vec{T}_1\|T_1\|, \dots, T_d = \vec{T}_d\|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$ be 1-dimensional normal currents. Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that*
(i) $\nu \ll \|T_i\|$ for $i = 1, \dots, d$, and
(ii) $\operatorname{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$ for ν-almost every x.
Then, $\nu \ll \mathcal{L}^d$.

Combining the above result with Lemma 4.2.5 we immediately get the following:

Lemma 4.3.2. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ have d independent Alberti representations. Then, $\nu \ll \mathcal{L}^d$.*

Proof. Denote by $C_1, \dots, C_d$ independent cones such that there are d Alberti representations having directions in these cones. By Lemma 4.2.5 there are d normal 1-dimensional currents $T_1 = \vec{T}_1\|T_1\|, \dots, T_d = \vec{T}_d\|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$ such that

$$\nu \ll \|T_i\| \qquad \text{for } i = 1, \dots, d,$$

and $\vec{T}_i(x) \in C_i$ for ν-almost every $x \in \mathbb{R}^d$. By the independence of the cones,

$$\operatorname{span}\left\{\vec{T}_1(x), \dots, \vec{T}_d(x)\right\} = \mathbb{R}^d \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R}^d.$$

This implies $\nu \ll \mathcal{L}^d$ via Theorem 4.3.1. □

In order to use the above result to prove Theorem 8.1.1 one also needs the following "push-forward lemma".

Lemma 4.3.3. *Let (X, ρ, μ) be a Lipschitz differentiability space with a d-chart (U, φ). If $\mu \llcorner U$ has d φ-independent Alberti representations, then also the push-forward $\varphi_\#(\mu \llcorner U) \in \mathcal{M}_+(\mathbb{R}^d)$ has d independent Alberti representations.*

Proof. It is enough to show that if there exists a representation of the form $\mu \llcorner U = \int \mu_\gamma \, d\pi(\gamma)$ with φ-directions in a cone C (i.e. such that $(\phi \circ \gamma)'(t) \in C \setminus \{0\}$ for almost all $t \in \operatorname{Dom} \gamma$ and for π-almost every γ), then we can build an Alberti representation

$$\varphi_\#(\mu \llcorner U) = \int \nu_{\bar{\gamma}} \, d\bar{\pi}(\bar{\gamma}) \qquad \text{with} \qquad \bar{\pi} \in \mathcal{P}(\Gamma(\mathbb{R}^d)),$$

with $\bar{\gamma}'(t) \in C \setminus \{0\}$ for $\bar{\pi}$-almost every $\bar{\gamma}$ and almost every $t \in \operatorname{Dom} \bar{\gamma}$. To this end consider the map $\Phi \colon \Gamma(X) \to \Gamma(\mathbb{R}^d)$ given by $\Phi(\gamma) := \varphi \circ \gamma$ and let $\bar{\pi} := \Phi_\# \pi \in \mathcal{M}_+(\Gamma(\mathbb{R}^d))$. Note that, by the very definition of the push-forward measure, for $\bar{\pi}$-almost every $\bar{\gamma}$ it holds that $\bar{\gamma} = \phi \circ \gamma$ for some $\gamma \in \Gamma(X)$.

By considering π as a probability measure defined on the Polish space $\mathcal{K}$ defined in (4.1), and noting that π is concentrated on $\Gamma(X)$, we can apply the disintegration theorem for measures [3, Theorem 5.3.1] to show that for $\bar{\pi}$-almost every $\bar{\gamma}$ there exists a Borel probability measure $\eta_{\bar{\gamma}}$ concentrated on $\Phi^{-1}(\bar{\gamma})$ and such that

$$\pi(A) = \int \eta_{\bar{\gamma}}(A) \, d\bar{\pi}(\bar{\gamma}) \qquad \text{for all Borel sets } A \subset \Gamma(X).$$

Note also that, by the disintegration theorem, the map $\bar{\gamma} \mapsto \eta_{\bar{\gamma}}$ is Borel measurable. Let us now set

$$\nu_{\bar{\gamma}} := \int_{\Phi^{-1}(\bar{\gamma})} \varphi_\#(\mu_\gamma) \, d\eta_{\bar{\gamma}}(\gamma).$$

Clearly, we have the representation

$$\varphi_\#(\mu \llcorner U) = \int \nu_{\bar{\gamma}} \, d\bar{\pi}(\bar{\gamma})$$

and $\bar{\gamma}'(t) = (\phi \circ \gamma)'(t) \in C \setminus \{0\}$ for $\bar{\pi}$-almost every $\bar{\gamma}$ and almost every $t \in \operatorname{Dom} \bar{\gamma}$. Hence, to conclude the proof we only have to show that

$$\nu_{\bar{\gamma}} \ll \mathcal{H}^1 \llcorner \operatorname{Im} \bar{\gamma} \qquad \text{for } \bar{\pi}\text{-a.e. } \bar{\gamma}.$$

Let E be a set with $\mathcal{H}^1(E \cap \operatorname{Im} \bar{\gamma}) = 0$. Since $\bar{\gamma}'(t) \neq 0$ for almost every $t \in \operatorname{Dom} \gamma$, the area formula implies that $\mathcal{L}^1(\bar{\gamma}^{-1}(E)) = 0$. If $\gamma \in \Phi^{-1}(\bar{\gamma})$, say $\bar{\gamma} = \phi \circ \gamma$, then

$$\mathcal{H}^1(\phi^{-1}(E) \cap \operatorname{Im} \gamma) \le \mathcal{H}^1(\gamma(\bar{\gamma}^{-1}(E))) = 0 \qquad \text{for all } \gamma \in \Phi^{-1}(\bar{\gamma}).$$

Hence, $\mu_\gamma(\phi^{-1}(E)) = 0$ for all $\gamma \in \Phi^{-1}(\bar{\gamma})$ which immediately gives

$$\nu_{\bar{\gamma}}(E) = \int_{\Phi^{-1}(\bar{\gamma})} \mu_\gamma(\phi^{-1}(E)) \, d\eta_{\bar{\gamma}}(\gamma) = 0\,.$$

This concludes the proof. □

Proof of Theorem 8.1.1. Let (U, φ) be a d-chart. By Theorem 4.2.3 there are d φ-independent Alberti representations of $\mu \llcorner U_k$, where $U = \bigcup_{k\in\mathbb{N}} U_k$ is the decomposition from Bate's theorem. Then, via Lemma 4.3.3, the push-forward $\varphi_\#(\mu \llcorner U_k)$ also has d independent Alberti representations. Finally, Lemma 4.3.2 yields $\varphi_\#(\mu \llcorner U_k) \ll \mathcal{L}^d$ and this concludes the proof. □

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Bibliography

[1] G. Alberti, M. Csörnyei, and D. Preiss: Structure of null sets in the plane and applications. In Proceedings of the Fourth European Congress of Mathematics (Stockholm, 2004), pages 3–22. European Mathematical Society, 2005.

[2] G. Alberti, M. Csörnyei, and D. Preiss: Differentiability of lipschitz functions, structure of null sets, and other problems. In Proceedings of the International Congress of Mathematicians 2010 (Hyderabad 2010), pages 1379–1394. European Mathematical Society, 2010.

[3] L. Ambrosio, N. Gigli, and G. Savaré: Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zürich. Birkhäuser, 2005.

[4] G. Alberti: Rank one property for derivatives of functions with bounded variation. Proc. Roy. Soc. Edinburgh Sect. A, 123:239–274, 1993.

[5] G. Alberti and A. Marchese: On the differentiability of lipschitz functions with respect to measures in the Euclidean space. Geom. Funct. Anal., 26:1–66, 2016.

[6] D. Bate: Structure of measures in Lipschitz differentiability spaces. J. Amer. Math. Soc., 28:421–482, 2015.

[7] J. Cheeger: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal., 9:428–517, 1999.

[8] J. Cheeger and B. Kleiner: On the differentiability of Lipschitz maps from metric measure spaces to Banach spaces. In Inspired by S. S. Chern, volume 11 of Nankai Tracts Math., pages 129–152. World Scientific, 2006.

[9] J. Cheeger and B. Kleiner: Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon-Nikodým property. Geom. Funct. Anal., 19:1017–1028, 2009.

[10] G. De Philippis and F. Rindler: On the structure of $\mathcal{A}$-free measures and applications. Ann. of Math., 2016. to appear, arXiv:1601.06543.

[11] H. Federer: Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[12] J. Gong: Rigidity of derivations in the plane and in metric measure spaces. Illinois J. Math., 56:1109–1147, 2012.

[13] N. Gigli and E. Pasqualetto: Behaviour of the reference measure on RCD spaces under charts. arXiv:1607.05188, 2016.

[14] P. Jones: Product formulas for measures and applications to analysis and geometry, 2011. Talk given at the conference “Geometric and algebraic structures in mathematics”, Stony Brook University, May 2011, http://www.math.sunysb.edu/Videos/dennisfest/.

[15] S. Keith: A differentiable structure for metric measure spaces. Adv. Math., 183:271–315, 2004.

[16] M. Kell and A. Mondino: On the volume measure of non-smooth spaces with Ricci curvature bounded below. arXiv:1607.02036, 2016.

[17] D. Preiss: Differentiability of lipschitz functions on banach spaces. J. Funct. Anal., 91:312–345, 1990.

[18] D. Preiss and G. Speight: Differentiability of Lipschitz functions in Lebesgue null sets. Invent. Math., 199:517–559, 2015.

[19] A. Schioppa: Derivations and Alberti representations. Adv. Math., 293:436–528, 2016.

[20] S. K. Smirnov: Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. Algebra i Analiz, 5:206–238, 1993. translation in St. Petersburg Math. J. 5 (1994), 841–867.

[21] Z. Zahorski: Sur l’ensemble des points de non-dérivabilité d’une fonction continue. Bull. Soc. Math. France, 74:147–178, 1946.