5 Spheroidal-type wavelets

5.1 Introduction

Wavelet analysis was introduced in the early 1980s in the context of signal analysis and exploration for petroleum to give a representation of signals and detect their characteristics. Several methods have been applied for the task; the most known is the Fourier transform. A major drawback of this method is its limitation to stationary and periodic signals. Furthermore, the description of signals is limited to the global behavior and cannot provide any detailed information. Also, in its numerical computer processing, Fourier analysis often yields nonfast algorithms.

Progress has been made by introducing the windowed Fourier transform (WFT) to address the problems of time-frequency localization. The WFT acts on signals by computing the classical Fourier transform of the signal multiplied by a time-localized function known as the window. However, the situation was not resolved, especially with the emergence of new problems, such as irregular signals or high-frequency variations.

The major drawback with the WFT is the fact that the shape of the window is fixed and may not be adapted to the fluctuations of nonstationary signals. Thus, the need for an analysis taking into account nonlinear algorithms, nonstationary signals, as well as nonperiodical and volatile ones has become a necessity for both theory and application. Wavelet analysis was introduced, developed, and has proved its power despite these obstacles. In this chapter, we review a special case of wavelet analysis adapted especially to spheroidal wavelets. We recall the strong relationship with orthogonal polynomials, homogenous polynomials, spherical harmonics, as well as special functions, and develop some details and examples.

5.2 Wavelets on the real line

Wavelet analysis is primarily based on an effective representation for standard functions on the real line and a robustness to the specification models. It also permits a reduction in time computation algorithms compared to other methods. This is essentially due to the simplicity of the analysis and the ease of generalization and efficiency according to the dimension. It permits one to analyze functions from different horizons starting from one horizon, which is not possible with Fourier analysis, for example. There, the number of coefficients to be computed is the standard point behind any approximation. Finally, wavelet analysis permits one to relate time localization to frequency.

Mathematically speaking, a wavelet or an analyzing wavelet on the real line is a function $\psi \in L^2(\mathbb{R})$, which satisfies some conditions, such as the admissibility condi-

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tion, which somehow describes Fourier–Plancherel identity and which says that

$$\int_{\mathbb{R}^+} |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} = C_\psi < \infty . \tag{5.1}$$

The function ψ has to also satisfy a number of vanishing moments, which is related in wavelet theory to its regularity order. It states that

$$p = 0, \dots, m-1, \quad \int_{\mathbb{R}} \psi(t) t^p \, dt = 0 . \tag{5.2}$$

Sometimes, we say that ψ is $\mathcal{C}^m$ on $\mathbb{R}$. The time-localization chart is a normalization form that is resumed in the identity

$$\int_{-\infty}^{+\infty} |\psi(u)|^2 du = 1 . \tag{5.3}$$

To analyze a signal by wavelets, one passes via the so-called wavelet transforms. A wavelet transform is a representation of the signal by means of an integral form similar to Fourier in which the Fourier sine and/or cosine is replaced by the analyzing wavelet ψ. In Fourier transform, the complex exponential source function yields the copies $e^{is\cdot}$ index by the indices $s \in \mathbb{R}$, which somehow represent frequencies. This transform is continuous in the sense that it is indexed by the whole line of indices $s \in \mathbb{R}$.

In wavelet theory, the situation is more unified. A continuous wavelet transform (CWT) is also well known. First, a frequency, scale, or a dilation/compression parameter $s > 0$ and a second one related to time or position $u \in \mathbb{R}$ have to be fixed. The source function ψ, known as the analyzing wavelet, is next transformed to yield some copies (replacing the $e^{is\cdot}$)

$$\psi_{s,u}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x-u}{s}\right) . \tag{5.4}$$

The CWT of a real valued function f defined on the real line at the position u and the scale s is defined by

$$d_{s,u}(f) = \int_{-\infty}^{\infty} f(t)\psi_{s,u}(t)dt, \quad \forall u, s . \tag{5.5}$$

By varying the parameters s and u, we can completely cover the time-frequency plane. This gives a full and redundant representation of the whole signal to be analyzed (see [99]). This transform is called continuous because of the nature of the parameters s and u that can operate at all levels and positions.

So, wavelets operate according to two parameters: the parameter u which permits one to translate the graph of the source wavelet mother ψ and the parameter s which permits one to compress or to dilate the graph of ψ. Computing or evaluating the coefficients $d_{u,s}$ means analyzing the function f with wavelets.

Properties 122. The wavelet transform $d_{s,u}(f)$ possesses some properties, such as
(1) the linearity, in the sense that

$$d_{s,u}(\alpha f + \beta g) = \alpha d_{s,u}(f) + \beta d_{s,u}(g), \quad \forall f, g \,,$$

(2) the translation-invariance, in the sense that

$$d_{s,u}(\tau_t f) = d_{s,u-t}(f), \quad \forall f; \text{ and } \forall u, s, t \,,$$

and where

$$(\tau_t f)(x) = f(x - t) \,,$$

(3) the dilation-invariance, in the sense that

$$d_{s,u}(f_a) = \frac{1}{\sqrt{a}} d_{as,au}(f), \quad \forall f; \text{ and } \forall u, s, a \,,$$

and where for $a > 0$,

$$(f_a)(x) = f(ax) \,.$$

The proof of these properties is easy and readers can refer to [8] for a review.

It holds in wavelet theory, as in Fourier analysis theory, that the original function f can be reproduced via its CWT by an L^2-identity.

Theorem 123. *For all $f \in L^2(\mathbb{R})$, we have the L^2-equality*

$$f(x) = \frac{1}{C_\psi} \int\int d_{s,u}(f) \psi\left(\frac{x-u}{s}\right) \frac{dsdu}{s^2} \,.$$

The proof of this result is based on the following lemma.

Lemma 124. *Under the hypothesis of Theorem 123, we have*

$$\int\int d_{s,u}(f) \overline{d_{s,u}(g)} \frac{dsdu}{s} = C_\psi \int f(x)\overline{g(x)} dx, \; \forall f, g \in L^2(\mathbb{R}) \,.$$

Proof. We have

$$d_{s,u}(f) = \frac{1}{s} f * \psi_s(u) = \frac{1}{s} \int f(x) \psi\left(\frac{x-u}{s}\right) dx = \frac{1}{2\pi} \mathcal{F}\left(\hat{f}(y) \overline{\hat{\psi}(sy)} e^{-iuy}\right) \,.$$

Consequently,

$$\int_u d_{s,u}(f)\overline{d_{s,u}(g)}du = \frac{1}{2\pi}\int_y \widehat{f}(y)\overline{\widehat{g}(y)}|\hat{\psi}(sy)|^2 dy\,.$$

By application of Fubini's rule, we get

$$\begin{aligned}\int_{s>0}\int_u d_{s,u}(f)\overline{d_{s,u}(g)}\frac{dsdu}{s} &= \frac{1}{2\pi}\int_{s>0}\int_y \hat{f}(y)\overline{\hat{g}(y)}|\hat{\psi}(sy)|^2\frac{dsdy}{s}\\ &= \frac{1}{2\pi}d_\psi\int_y \hat{f}(y)\overline{\hat{g}(y)}dy\\ &= C_\psi\int_y f(y)\overline{g(y)}dy\,.\end{aligned}$$

□

Proof of Theorem 123. By applying the Riesz rule, we get

$$\begin{aligned}&\left\|F(x) - \frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}\right\|_{L^2}\\ &= \sup_{\|G\|=1}\left(\int F(x) - \frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}\right)\overline{G(x)}dx\,.\end{aligned}$$

Next, using Fubini's rule, we observe that the last line is equal to

$$\begin{aligned}&= \sup_{\|G\|=1}\left(\int F(x)\overline{G(x)}dx - \frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\overline{C_{a,b}(G)}\frac{dadb}{a}\right)\\ &= \sup_{\|G\|=1}\frac{1}{C_\psi}\int_{(a,b)\notin[1/A,A]\times[-B,B]} C_{a,b}(F)\overline{C_{a,b}(G)}\frac{dadb}{a}\,,\end{aligned}$$

which by Cauchy–Schwartz inequality is bounded by

$$\le \frac{1}{C_\psi}\left[\int_{(a,b)\notin[1/A,A]\times[-B,B]} |C_{a,b}(F)|^2\frac{dadb}{a}\right]^{1/2}\left[\sup_{\|G\|=1}\int_{(a,b)\notin[1/A,A]\times[-B,B]} |C_{a,b}(G)|^2\frac{dadb}{a}\right]^{1/2}\,.$$

Now, Lemma 124 shows that the last quantity goes to 0 as R tends to $+\infty$. □

On the real line, the most well-known examples are Haar and Schauder wavelet, where explicit computations are always possible. The Haar example is the simplest example in the theory of wavelets. It is based on the wavelet mother expressed by

$$\psi(x) = \chi_{[0,1/2[}(x) - \chi_{[1/2,1[}(x)\,.$$

The Schauder wavelet is based on the explicit wavelet mother

$$\psi(x) = \frac{1}{2}(1 - |2x|)\chi_{[-1/2,1/2]}(x) - (1 - |2x-1|)\chi_{[0,1[}(x) + \frac{1}{2}(1 - |2x-2|)\chi_{[1/2,3/2]}(x)\,.$$

Readers can refer to [8, 75, 95, 99] for more details and examples of original wavelet analysis on the real line and Euclidian spaces in general.

5.3 Chebyshev wavelets

Chebyshev wavelets stem from one mother wavelet ψ^m depending on a parameter m, which represents the degree of Chebyshev polynomial of first kind associated with the wavelet. The source *Chebyshev wavelet* mother ψ^m is defined by

$$\psi^m(t) = \widetilde{T}_m(t),\ 0 \le t < 1 \quad \text{and} \quad 0,\ \text{else}$$

where

$$\widetilde{T}_m(t) = \sqrt{\frac{2}{\pi}}T_m(t), \qquad m = 0, 1, 2, \dots, M-1\,. \tag{5.6}$$

Here $T_m(t)$ are the Chebyshev polynomials of the first kind of degree m, given by

$$T_m(t) = \cos(m \arccos t)\,.$$

Next, we perform the usual translation–dilation actions using parameters $j \in \mathbb{N}$ for the level and a parameter $n = 1, 2, \dots, 2^{j-1}$ for the position. Thus, we obtain the dilation–translation copies of ψ^m explicitly expressed by

$$\psi^m_{j,n}(t) = \begin{cases} 2^{\frac{j}{2}}\widetilde{T}_m(2^j t - 2n + 1), & \frac{n-1}{2^{j-1}} \le t < \frac{n}{2^{j-1}} \\ 0, & \text{else}\,. \end{cases} \tag{5.7}$$

The Chebyshev wavelets are orthonormal with respect to the weight function

$$\omega_j(t) = \omega_{n,k}(t) = \omega(2^{j-1}t - n + 1), \quad n = 1, 2, \dots, 2^{k-1} \text{ and } \frac{n-1}{2^{j-1}} \le t < \frac{n}{2^{j-1}}\,.$$

Denote next

$$L^2_\omega([0,1]) = \left\{ f,\ \int_0^1 |f(x)|^2\ \omega(x)dx < \infty \right\}\,,$$

where $\omega(x) = \frac{1}{2\sqrt{x(1-x)}}$. A function $f \in L^2_\omega([0,1])$ can be approximated in a series form as

$$f = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\psi^m_{j,n}\,,$$

where

$$C^m_{j,n} = \langle f, \psi^m_{j,n} \rangle_{\omega_j} \,,$$

in which $\langle ., . \rangle_{\omega_j}$ is the inner product in $L^2_{\omega_j}([0,1])$.

5.4 Gegenbauer wavelets

Gegenbauer wavelets (GW) depend on four parameters: j, n, m, p. The parameter $j \in \mathbb{N}$ represents the level of resolution, $n \in \{1, 2, 3, \ldots, 2^{j-1}\}$, is related to the translation parameter, $m = 0, 1, 2, \ldots, M-1$, $M > 0$ is the degree of the Gegenbauer polynomial, and finally a real parameter $p > -\frac{1}{2}$. The mother Gegenbauer wavelet is defined on $[0, 1)$ by $\psi^{m,p}(x) = G^p_m(x)$, where G^p_m is the well-known Gegenbauer polynomial defined in Chapter 1. Next, the translation–dilation copies of $\psi^{m,p}$ are defined by

$$\psi^{m,p}_{j,n}(x) = \begin{cases} \frac{1}{\sqrt{L^p_m}} 2^{\frac{j}{2}} G^p_m(2^j x - 2n + 1), & \frac{2n-2}{2^j} \leq t < \frac{2n}{2^j} \,, \\ 0, & \text{elsewhere} \,. \end{cases}$$

Note here that the translation parameter takes only odd values.

Remark 125. For $p = \frac{1}{2}$, we get Legendre wavelets. For $p = 0$ and $p = 1$, we obtain the Chebyshev wavelet of first and second kind, respectively.

To obtain the mutual orthogonality of Gegenbauer wavelets $\psi^{m,p}_{j,n}$, the weight function associated with the Gegenbauer polynomials has to be dilated and translated as for the Gegenbauer wavelets. Thus, we obtain a translation–dilation copy of the weight ω as

$$\omega_{j,n}(x) = \omega(2^j x - 2n + 1) = (1 - (2^j x - 2n + 1)^2)^{p - \frac{1}{2}} \,.$$

At a fixed level of resolution, we get

$$\omega_{j,n}(x) = \begin{cases} \omega_{j,1}(x), & 0 \leq x < \frac{1}{2^{j-1}} \,, \\ \omega_{j,2}(x), & \frac{1}{2^{j-1}} \leq x < \frac{2}{2^{j-1}} \,, \\ \omega_{j,3}(x), & \frac{2}{2^{j-1}} \leq x < \frac{3}{2^{j-1}} \,, \\ \vdots & \\ \omega_{j,2^{j-1}}(x), & \frac{2^{j-1}-1}{2^{j-1}} \leq x < 1 \,. \end{cases}$$

According to such wavelets, a function $f \in L^2[0,1)$ can be expressed in terms of the GW as

$$f = \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} d_{j,n}^{m,p} \psi_{j,n}^{m,p} , \tag{5.8}$$

where the coefficient $d_{j,}^{m,p}$ are the so-called wavelet coefficients given by

$$d_{j,n}^{m,p} = \langle f, \psi_{j,n}^{m,p} \rangle = \int_0^1 \omega_{j,n}(x) \psi_{j,n}^{m,p}(x) f(x) dx .$$

For more details, refer to [124, 126, 135, 139].

5.5 Hermite wavelets

Hermite wavelets are based on the well-known Hermite polynomials. Recall that such polynomials consist of a sequence of orthogonal polynomials with respect to the special weight function $\omega(x) = e^{-x^2}$ and are explicitly given by

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} (e^{-x^2}) .$$

The Hermite mother wavelet is given by

$$\psi^m(t) = \begin{cases} \widetilde{H}_m(2t), & 0 \leq t < 1 , \\ 0, & \text{else} , \end{cases} \tag{5.9}$$

where

$$\widetilde{H}_m = \frac{1}{2^m l! \sqrt{\pi}} H_m .$$

The translation–dilation copies of ψ^m are next defined by

$$\psi_{k,n}^m(t) = \begin{cases} 2^{\frac{k}{2}} \widetilde{H}_m(2^{k+1}t - 2l + 1), & \frac{l-1}{2^k} \leq t < \frac{n}{2^k} \\ 0, & \text{else} . \end{cases} \tag{5.10}$$

Note that such wavelets depend essentially on the parameter m, which is the degree of the m-Hermite polynomial H_m. Hermite wavelets are orthonormal with respect to the weight function

$$\omega_{l,k}(t) = \omega(2^{k-1}t - l + 1), \quad l = 0, 1, \ldots, x, 2k, \ \frac{l-1}{2^k} \leq t < \frac{n}{2^k} .$$

Some propeties of Hermite wavelets are given in [1].

5.6 Laguerre wavelets

Laguerre wavelets are orthogonal wavelets defined in the interval (0, 1) and stem from one source mother function

$$\psi^m(t) = \widetilde{L}_m(t)\chi_{[0,1[}(t) = \frac{1}{m!}L_m(t)\chi_{[0,1[}(t)\,, \tag{5.11}$$

where L_m is the Laguerre polynomial of degree m. A wavelet basis is next expressed by

$$\psi^m_{k,n}(t) = \begin{cases} 2^{\frac{k}{2}}\widetilde{L}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}\,, \\ 0, & \text{else}\,. \end{cases} \tag{5.12}$$

For more details on these wavelets, see [79].

5.7 Bessel wavelets

There are several approaches to introduce Bessel wavelets [126, 127].

In the present section, we will present the most known approach. For $1 \le p < \infty$ and $\mu > 0$, denote

$$L^p_\sigma(\mathbb{R}_+) := \left\{ f \text{ such that } \|f\|_{p,\sigma} = \left(\int_0^\infty |f(x)|^p \, d\sigma(x) \right)^{\frac{1}{p}} < \infty \right\}\,,$$

where $d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}dx$. Denote also

$$j_\mu(x) = 2^{\mu-\frac{1}{2}}\Gamma\left(\mu + \frac{1}{2}\right)x^{\frac{1}{2}-\mu}J_{\mu-\frac{1}{2}}(x)\,,$$

where $J_{\mu-\frac{1}{2}}$ is the Bessel function of first kind and of order $\mu - \frac{1}{2}$. Denote next,

$$D(x,y,z) = \int_0^\infty j_\mu(xt)j_\mu(yt)j_\mu(zt)d\sigma(t)$$

and the translation

$$\tau_x f(y) = \widetilde{f}(x,y) = \int_0^\infty D(x,y,z)f(z)d\sigma(z), \quad \forall 0 < x, y < \infty\,.$$

Next, for a two-variable function f, we define the dilation operator

$$D_a f(x,y) = a^{-2\mu-1} f\left(\frac{x}{a}, \frac{y}{a}\right)\,.$$

Definition 126. Let $\Psi \in L^p_\sigma(\mathbb{R}_+)$. The Bessel wavelet copy $\Psi_{a,b}$ is defined by

$$\Psi_{a,b}(x) = D_a \tau_b \Psi(x) = a^{-2\mu-1} \int_0^\infty D\left(\frac{b}{a}, \frac{x}{a}, z\right) \Psi(z) d\sigma(z); \quad \forall a, b \geq 0 .$$

The Bessel wavelet transform (BWT) of a function $f \in L^q_\sigma(\mathbb{R}_+)$, at the scale a and the position b is defined by

$$(B_\Psi f)(a, b) = a^{-2\sigma-1} \int_0^\infty \int_0^\infty f(t)\overline{\Psi}(z) D\left(\frac{b}{a}, \frac{t}{a}, z\right) d\sigma(z) d\sigma(t) .$$

The following result shows one of the BWT of functions.

Theorem 127. *Let $f \in L^p_\sigma(\mathbb{R}_+)$, $\Psi \in L^q_\sigma(\mathbb{R}_+)$ with $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(B_\psi f)$ is continuous on $\mathbb{R}^2_+$.*

Proof. Let (a_0, b_0) be an arbitrary fixed point of $\mathbb{R}^2_+$. We have

$$\begin{aligned}
&|(B_\Psi f)(a, b) - (B_\Psi f)(a_0, b_0)| \\
&\leq a^{-2\mu-1} \left| \int_0^\infty \int_0^\infty f(t)\overline{\Psi(z)} \left[D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a}, z\right) \right] d\sigma(z) d\sigma(t) \right| \\
&\leq a^{-2\mu-1} \left[\int_0^\infty \int_0^\infty |f(t)|^p \left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right) \right|^{\frac{1}{p}} d\sigma(t) d\sigma(z) \right] \\
&\quad \times \left[\int_0^\infty \int_0^\infty |\overline{\psi}(z)|^q \left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right) \right|^{\frac{1}{q}} d\sigma(t) d\sigma(z) \right] .
\end{aligned}$$

Now, observe that

$$\left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right) \right| \leq 2 .$$

Moreover, using the dominated convergence theorem and the continuity of $D(\frac{b}{a}, \frac{t}{a}, z)$ with respect to (a, b), we get

$$\lim_{(a,b)\to(a_0,b_0)} |(B_\Psi f)(a, b) - (B_\Psi f)(a_0, b_0)| = 0 ,$$

which proves the continuity of the BWT on $\mathbb{R}^2_+$. □

Definition 128. Let $f, g \in L^p_\sigma(\mathbb{R}_+)$. We define the convolution product (usually known as the Hankel convolution) by

$$(f\sharp g)(x) = \int_0^\infty \tau_x f(y) g(y) d\sigma(y) \, .$$

The following result is a variant of Parseval/Plancherel rules for the case of BWT.

Theorem 129. *Let $\Psi \in L^2_\sigma(\mathbb{R}_+)$ and $f, g \in L^2_\sigma(\mathbb{R}_+)$. Then*

$$\int_0^\infty \int_0^\infty (B_\Psi f)(a,b)\overline{(B_\Psi g)}(a,b) \frac{d\sigma(a)}{a^{2\mu+1}} d\sigma(b) = C_\Psi \, \langle f, g \rangle \, ,$$

where

$$C_\Psi = \int_0^\infty t^{-2\mu-1} |\widehat{\Psi}(t)|^2 dt > 0 \, .$$

The proof follows similar techniques as for the case of real-line wavelets. Because of its importance, we reproduce it in detail.

Proof. Recall that

$$\begin{aligned}(B_\Psi f)(a,b) &= \int_{\mathbb{R}_+} f(t) \Psi_{a,b}(t) d\sigma(t) \\ &= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^2} f(t)\Psi(z) D\left(\frac{b}{a}, \frac{t}{a}, z\right) d\sigma(z) d\sigma(t) \, .\end{aligned}$$

Now observe that

$$D\left(\frac{b}{a}, \frac{t}{a}, z\right) = \int_{\mathbb{R}_+} j\left(\frac{b}{a}u\right) j\left(\frac{t}{a}u\right) j(zu) d\sigma(u) \, .$$

Hence,

$$
\begin{aligned}
(B_\Psi f)(a,b) &= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^3} f(t)\Psi(z) j\left(\frac{b}{a}u\right) j\left(\frac{t}{a}u\right) j(zu)\,d\sigma(u)\,d\sigma(z)\,d\sigma(t) \\
&= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^2} \widehat{f}\left(\frac{u}{a}\right) \Psi(z) j\left(\frac{b}{a}u\right) j(zu)\,d\sigma(u)\,d\sigma(z) \\
&= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+} \widehat{f}\left(\frac{u}{a}\right) \widehat{\Psi}(u) j\left(\frac{b}{a}u\right) d\sigma(u) \\
&= \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{\Psi}(a\eta) j(b\eta)\,d\sigma(\eta) \\
&= \left(\widehat{f}(\eta)\widehat{\Psi}(a\eta)\right)^{\wedge}(b)\,.
\end{aligned}
$$

As a result

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2} (B_\Psi f)(a,b)\overline{(B_\Psi g)}(a,b) \frac{d\sigma(a)}{a^{2\mu+1}}\,d\sigma(b) \\
&= \int_{\mathbb{R}_+^2} \widehat{f}(\eta)\widehat{\Psi}(a\eta)\widehat{g}(\eta)\widehat{\Psi}(a\eta)\,d\sigma(\eta) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{g}(\eta) \left(\int_{\mathbb{R}_+} |\widehat{\Psi}(a\eta)|^2 \frac{d\sigma(a)}{a^{2\mu+1}} \right) d\sigma(\eta) \\
&= C_\Psi \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{g}(\eta)\,d\sigma(\eta) \\
&= C_\Psi \,\langle \widehat{f}, \widehat{g} \rangle \\
&= C_\Psi \,\langle f, g \rangle\,.
\end{aligned}
$$

□

5.8 Cauchy wavelets

Cauchy wavelets are one step in the direction of introducing spherical wavelets as they aim to take into account the angular behavior of the analyzed signals. In the one-dimensional case, Cauchy wavelets are defined via their Fourier transform

$$
\widehat{\psi}_m(\omega) = \begin{cases} 0, & \text{for} \quad \omega < 0 \\ \omega^m e^{-\omega}, & \text{for} \quad \omega \geq 0\,, \end{cases}
$$

with $m > 0$. In 1D, the positive half-line is a convex cone. Thus a natural generalization to 2D will be a wavelet whose support in spatial frequency space is contained in a convex cone with an apex at the origin. Let $C \equiv C(\alpha, \beta)$ be the convex cone determined by the unit vectors e_α, e_β, where $\alpha < \beta$, $\beta - \alpha < \pi$ and for all θ, $e_\theta \equiv (\cos\theta, \sin\theta)$. The axis of the cone is $\xi_{\alpha\beta} = e_{\frac{\alpha+\beta}{2}}$. In other words,

$$\begin{aligned} C(\alpha, \beta) &= \{k \in \mathbb{R}^2, \quad \alpha \le \arg(k \le \beta)\} \\ &= \{k \in \mathbb{R}^2, \quad k.\xi_{\alpha\beta} \ge e_\alpha.\xi_{\alpha\beta} = e_\beta .> \mathbf{0}\} . \end{aligned}$$

The dual cone to $C(\alpha, \beta)$ is

$$\widetilde{C}(\alpha, \beta) = \{k \in \mathbb{R}^2, \quad k.k' > 0, \quad \forall k' \in C(\alpha, \beta)\} .$$

Note that $\widetilde{C}(\alpha, \beta)$ may also be seen as

$$\widetilde{C}(\alpha, \beta) = C(\widehat{\alpha}, \widehat{\beta}) ,$$

where $\widehat{\alpha} = \beta - \frac{\pi}{2}$, $\widehat{\beta} = \alpha + \frac{\pi}{2}$ and $e_\alpha.e_{\widehat{\alpha}} = e_\beta.e_{\widehat{\beta}} = 0$. Thus the axis of $\widetilde{C}$ is $\xi_{\alpha\beta}$.

The two-dimensional Cauchy wavelet is defined via its Fourier transform

$$\widehat{\psi}^{C,\eta}_{lm} = \begin{cases} (k.e_{\widetilde{\alpha}})^l (k.e_{\widetilde{\beta}})^m e^{-k.\eta}, & k \in C(\alpha, \beta) , \\ 0, & \text{otherwise} , \end{cases} \tag{5.13}$$

where $\eta \in \widetilde{C}$ and $l, m \in \mathbb{N}^*$. Note that such a wavelet is also supported by C. It satisfies the admissibility condition

$$c_{\psi^{C,\eta}_{lm}} \equiv (2\pi)^2 \int \frac{d^2k}{|k|^2} |\widehat{\psi}^{C,\eta}_{lm}(k)|^2 < \infty . \tag{5.14}$$

The following result obtained by Antoine et al. is proved in [12] and yields an explicit form for the two-dimensional Cauchy wavelet.

Proposition 130. *For even $\eta \in \widetilde{C}$ and $l, m \in \mathbb{N}^*$. The 2D Cauchy wavelet $\psi^{C,\eta}_{lm}(x)$ with support in C belongs to $L^2(\mathbb{R}^2, dx)$ and is given by*

$$\psi^{C,\eta}_{lm}(x) = \frac{i^{l+m+2}}{2\pi} l!m! \frac{[\sin(\beta - \alpha)]^{l+m+1}}{[(x + i\eta).e_\alpha]^{l+1}[(x + i\eta).e_\beta]^{m+l}} . \tag{5.15}$$

We can, with analogous techniques, define multidimensional Cauchy wavelets. See [12] and the references therein for more details.

5.9 Spherical wavelets

Spherical wavelets are adopted for understanding complicated functions defined or supported by the sphere. The classical spherical wavelets are essentially done by convolving the function against rotated and dilated versions of one fixed function ψ. To introduce a special wavelet analysis on the sphere related to zonals we first recall some useful topics. Let $F \in L^2[-1,1]$ and L_n be the Legendre polynomial of degree n. The coefficients

$$\widehat{F}(n) = 2\pi\langle F, L_n\rangle = 2\pi \int_{-1}^{1} F(x)L_n(x)dx, \quad n \in \mathbb{N}$$

are called *the Legendre coefficients* or *the Legendre transforms* of F. It is proved in harmonic Fourier analysis that F can be expressed in a series form

$$F = \sum_{n=0}^{\infty} \widehat{F}(n)\frac{2n+1}{4\pi}L_n \tag{5.16}$$

called *the Legendre series* of F.

Definition 131. A family $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ is called a *spherical scaling function system* if the following assertions hold.
(1) For all $n, j \in \mathbb{N}$, we have $\widehat{\phi_j}(n) \le \widehat{\phi_{j+1}}(n)$. In other words, for all $n \in \mathbb{N}$ the sequence $(\widehat{\phi_j}(n))_{j\in\mathbb{N}}$ is increasing
(2) $\lim_{j\longrightarrow\infty} \widehat{\phi_j}(n) = 1$ for all $n \in \mathbb{N}$
(3) $\widehat{\phi_j}(n) \ge 0$ for all $n, j \in \mathbb{N}$,
where $\widehat{\phi_j}(n)$ is the Legendre transform of $\widehat{\phi_j}$.

We will now investigate a way of constructing a scaling function [54].

Definition 132. A continuous function $\gamma\colon \mathbb{R}^+ \longmapsto \mathbb{R}$ is said to be admissible if it satisfies the *admissibility condition*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\left(\sup_{x\in[n,n+1]} |\gamma(x)|\right)^2 < +\infty\,. \tag{5.17}$$

In this case, γ is called *an admissible generator* of the function $\psi\colon [-1,1] \to \mathbb{R}$ given by

$$\psi = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\gamma(n)L_n\,. \tag{5.18}$$

We immediately obtain the following characteristics [162].

Proposition 133. *The following assertions are true:*
(1) *If γ is an admissible generator, then the generated function $\psi \in L^2[-1,1]$.*
(2) *For all $n \in \mathbb{N}$, $\widehat{\psi}(n) = \gamma(n)$.*

Proof. (1) Since the Legendre polynomials form an orthogonal basis for $L^2[-1,1]$ with $\langle L_n, L_n\rangle_{L^2[-1,1]} = \frac{4\pi}{2n+1}$, the admissibility condition imposed on γ yields that

$$\|\psi\|^2_{L^2[-1,1]} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}(\gamma_0(n))^2 \le \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\left(\sup_{x\in[n,n+1]} |\gamma_0(x)|\right)^2 < +\infty$$

(2) is an immediate result from (5.16). □

We now investigate the idea to construct a whole family of admissible functions starting from one source admissible function.

Definition 134. The *dilation operator* is defined for $\gamma\colon [0,\infty) \to \mathbb{R}$ and $a > 0$ by

$$D_a\gamma(x) = \gamma(ax) \quad \forall x \in [0,\infty)\,.$$

For $a = 2^{-j}$, $j \in \mathbb{Z}$ we denote $\gamma_j = D_j\gamma = D_{2^{-j}}\gamma$.

Definition 135. An admissible function $\varphi\colon [0,\infty) \to \mathbb{R}$ is said to be a *generator of a scaling function* if it is monotonously decreasing, continuous at 0 and satisfies $\varphi(0) = 1$.
The system $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$, defined by

$$\phi_j = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\varphi_j(n)L_n$$

is said to be *the corresponding spherical scaling function* associated with φ.

It holds sometimes that for all j, the sequence $(\widehat{\phi_j}(n))_n$ is stationary with zero stationary value. In this case, the system $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ is called *bandlimited*. It holds that for bandlimited scaling functions, each ϕ_j is a 1D polynomial, and for all $F \in L^2(S^2)$, $\phi_j * F$ is a polynomial on S^2. The following theorem affirms that scaling functions permit one to approximate L^2 functions with polynomial approximates (see [162]).

Now, we show that such scaling functions are suitable candidates to approximate functions in L^2 as it is needed in wavelet theory in general. Thus, they are suitable sources to define multiresolution analysis and/or a wavelet analysis on the sphere.

Theorem 136. *Let* $\{\phi_j\}_{j\in\mathbb{N}}$ *be a scaling function and* $F \in L^2(S^2)$. *Then*

$$\lim_{j\to\infty} \|F - \phi_j^{(k)} * F\|_{L^2(S^2)} = 0$$

for all levels of iterations $k \in \mathbb{N}$.

Here, for a function $\Phi \in L^2$, we designate by $\Phi^{(k)}$ the k-times self-convolution of Φ with itself. The last approximation is called *spherical approximate identity*. The next theorem shows the role of spherical scaling functions in the construction of multiresolution analysis on the sphere.

Proof. First observe that

$$\phi_j^{(k)} * F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_J(n)\widehat{F}(n,j)Y_{n,j} \,.$$

Thus,

$$F - \phi_j^{(k)} * F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (1 - \widehat{\Phi}_J(n))\widehat{F}(n,j)Y_{n,j} \,,$$

which by applying the Parseval identity yields that

$$\|F\phi_j^{(k)} * F\|_2^2 = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (1 - \widehat{\Phi}_J(n))^2(\widehat{F}(n,j))^2 \,.$$

Now, observing that the last series is J-uniformly convergent and the fact that

$$\lim_{J\to+\infty} (1 - \widehat{\Phi}_J(n)) = 0$$

for all n, it results that

$$\lim_{j\to\infty} \|F - \phi_j^{(k)} * F\|_{L^2(S^2)} = 0 \,. \qquad \square$$

Theorem 137. *Let for* $j \in \mathbb{Z}$,

$$V_j = \{\phi_j^{(2)} * F | F \in L^2(S^2)\} \,,$$

where $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ *is a scaling function. Then, the sequence* $(V_j)_j$ *defines a multiresolution analysis on the sphere. That is,*

(1) $V_j \subset V_{j+1} \subset L^2(S^2), \quad \forall j \in \mathbb{N}$.

(2) $\overline{\bigcup_{j=0}^{\infty} V_j} = L^2(S^2)$.

For $j \in \mathbb{Z}$, the spaces V_j represents the so-called *scale or approximation space* at the level j.

Proof. (1) As $\Phi \in L^2$ and also F, the convolution $\Phi * F$ is also L^2.
Consider next, for $J \in \mathbb{Z}$, the function

$$\gamma_J(n) = \left(\frac{\widehat{\Phi}_J(n)}{\widehat{\Phi}_{J+1}(n)}\right)^2 \widehat{F}(n,j) \quad \text{if} \quad \Phi_{J+1}(n) \neq 0$$

and 0 else, and define the function G by

$$G = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \gamma_j(n) Y_{n,j} \, .$$

It is straightforward that $G \in L^2$ and that $\widehat{G}(n,j) = \gamma_j(n)$. Furthermore,

$$\begin{aligned}\phi_{J+1}^{(2)} * G &= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_{J+1}(n) \widehat{G}(n,j) Y_{n,j} \\ &= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_J(n) \widehat{F}(n,j) Y_{n,j} \\ &= \phi_J^{(2)} * F \, .\end{aligned}$$

Hence, $\phi_J^{(2)} * F = \phi_{J+1}^{(2)} * G \in V_{J+1}$. Consequently, $V_J \subset V_{J+1}$.
(2) The density property is an immediate consequence of the spherical approximate identity proved in Theorem 136. □

Based on this multiresolution analysis of $L^2(S^2)$, we can introduce spherical wavelets.

Definition 138. Let $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset L^2[-1,1]$ be a scaling function and let $\Psi = \{\psi_j\}_{j \in \mathbb{N} \cup \{-1\}}$ and $\widetilde{\Psi} = \{\tilde{\psi}_j\}_{j \in \mathbb{N} \cup \{-1\}}$ be in $L^2[-1,1]$ satisfying the so-called *refinement equation*

$$\widehat{\psi_j}(n) \widehat{\tilde{\psi}_j}(n) = (\widehat{\phi_{j+1}}(n))^2 - (\widehat{\phi_j}(n))^2 \quad \forall n, j \in [0, +\infty) \, .$$

Then,
(a) Ψ and $\widetilde{\Psi}$ are called, respectively, (*spherical*) *primal wavelet* and (*spherical*) *dual wavelet* relative to Φ.
(b) The functions ψ_0 and $\tilde{\psi}_0$ are called the *primal mother wavelet* and the *dual mother wavelets*, respectively.
Here, we set $\psi_{-1} = \tilde{\psi}_{-1} = \phi_0$.

The following result obtained by Volker in [162] shows the existence of primal and dual wavelets.

Theorem 139. *Let φ_0 be a generator of a scaling function and $\psi_0, \tilde{\psi}_0$ be admissible function such that*

$$\psi_0\tilde{\psi}_0(x) = \left(\varphi_0\left(\frac{x}{2}\right)\right)^2 - (\varphi_0(x))^2 \quad \forall x \in \mathbb{R}^+ .$$

Then, ψ_0 and $\tilde{\psi}_0$ are generators of primal and dual mother wavelets, respectively.

Proof. We will prove precisely that the dilated copies $\{\psi_j\}_{j\in\mathbb{N}\cup\{-1\}}, \{\tilde{\psi}_j\}_{j\in\mathbb{N}\cup\{-1\}} \subset L^2[-1,1]$ defined via their Legendre coefficients by dilating ψ_0 and $\tilde{\psi}_0(x)$ as

$$\widehat{\psi_j}(n) = \psi_j(n) = \psi_0(2^{-j}n), \quad \widehat{\tilde{\psi}_j}(n) = \tilde{\psi}_j(n) = \tilde{\psi}_0(2^{-j}n); \quad \forall n, j \in \mathbb{N} .$$

and

$$\widehat{\psi_{-1}}(n) = \widehat{\tilde{\psi}_{-1}}(n) = \varphi_0(n); \quad \forall n \in \mathbb{N}$$

are a primal and dual wavelets, respectively. Indeed, considering these dilated copies we obtain for all $n, j \in \mathbb{N}$,

$$\begin{aligned}\widehat{\psi_j}(n)\widehat{\tilde{\psi}_j}(n) &= \psi_0(2^{-j}n)\tilde{\psi}_0(2^{-j}n)\\ &= (\varphi_0(2^{-j-1}n))^2 - (\varphi_0(2^{-j}n))^2\\ &= (\widehat{\phi_{j+1}}(n))^2 - (\widehat{\phi_j}(n))^2 .\end{aligned}$$

□

A fundamental property of spherical wavelets is the scale-step property proved below, which prepares us to introduce detail spaces.

Theorem 140. *Let $\Psi = \{\psi_j\}_{j\in\mathbb{N}\cup\{-1\}}$ and $\widetilde{\Psi} = \{\tilde{\psi}_j\}_{j\in\mathbb{N}\cup\{-1\}}$ be a primal and a dual wavelet corresponding to the scaling function $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$. The following assertions hold for all $F \in L^2(S^2)$.*
(i) $\phi_{J_2}^{(2)} * F = \phi_{J_1}^{(2)} * F + \sum_{j=J_1}^{J_2-1} \tilde{\psi}_j * \psi_j * F, \forall J_1 < J_2 \in \mathbb{N}$.
(ii) $F = \phi_J^{(2)} * F + \sum_{j=J}^{\infty} \tilde{\psi}_j * \psi_j * F, \forall J \in \mathbb{N}$.

Proof. (i) We will evaluate the last right-hand series term in the assertion. Using the definition of primal and dual wavelets, we obtain

$$\begin{aligned}\tilde{\psi}_j * \psi_j * F &= \sum_{n=0}^{+\infty}\sum_{s=1}^{2n+1} \widehat{\psi}_j(n)\widehat{\tilde{\psi}}_j(n)\widehat{F}(n,s)Y_{n,s}\\ &= \sum_{n=0}^{+\infty}\sum_{j=1}^{2n+1} \left[(\widehat{\phi}_{j+1}(n))^2 - (\widehat{\phi}_j(n))^2\right]\widehat{F}(n,s)Y_{n,s}\\ &= \phi_{j+1}^{(2)} * F - \phi_j^{(2)} * F .\end{aligned}$$

As a result,

$$\sum_{j=J_1}^{J_2-1} \tilde{\psi}_j * \psi_j * F = \phi_{J_2}^{(2)} * F - \phi_{J_1}^{(2)} * F \,.$$

(ii) is an immediate consequence of assertion (i). □

Theorem 141. *Denote for $j \in \mathbb{Z}$,*

$$W_j = \{\tilde{\psi}_j * \psi_j * F / F \in L^2(S^2)\} \,.$$

Then, for all $J \in \mathbb{Z}$,

$$V_{J+1} = V_J + W_J \,.$$

Proof. The inclusion $V_J \subset V_{J-1} + W_{J-1}$ is somehow easy and it is a consequence of Theorem 140. We will prove the opposite inclusion. So, let $F_1 \in V_J$ and $F_2 \in W_J$. We seek a function $F \in L^2$ for which we have

$$\Phi_{J+1}^{(2)} * F = F_1 + F_2 \,.$$

Since $F_1 \in V_J$ and $F_2 \in W_J$, there exist G_1 and G_2 in L^2 such that

$$F_1 = \Phi_J^{(2)} * G_1 \quad \text{and} \quad F_2 = \widetilde{\Psi}_J * \Psi_J * G_2 \,.$$

Now, consider the function γ defined by

$$\gamma(n,j) = \left(\frac{(\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) + ((\widehat{\Phi}_{J+1}(n))^2 - (\widehat{\Phi}_J(n))^2)\widehat{G}_2(n,j)}{\widehat{\Phi}_{J+1}(n)} \right)^2 ,$$

whenever $\Phi_{J+1}(n) \neq 0$ and 0 else, and define the function F by

$$F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \gamma(n,j) Y_{n,j} \,.$$

It is straightforward that $F \in L^2$ and that $\widehat{F}(n,j) = \gamma(n,j)$. Furthermore,

$$\begin{aligned}
\Phi_{J+1}^{(2)} * F &= \sum_{n=0}^{+\infty,*} \sum_{j=1}^{2n+1} (\widehat{\Phi}_{J+1}(n))^2 \widehat{F}(n,j) Y_{n,j} \\
&= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) Y_{n,j} \\
&\quad + \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} ((\widehat{\Phi}_{J+1}(n))^2 - (\widehat{\Phi}_J(n))^2) \widehat{G}_2(n,j) Y_{n,j}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) Y_{n,j} \\
&\quad + \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\widetilde{\Psi}}_J(n) \widehat{\Psi}_J(n) \widehat{G}_2(n,j) Y_{n,j} \\
&= \phi_J^{(2)} * G_1 + \widetilde{\Psi}_J * \Psi_J * G_2 \\
&= F_1 + F_2 \,.
\end{aligned}
$$

Consequently, $F_1 + F_2 \in V_{J+1}$. □

Definition 142. For $j \in \mathbb{Z}$, the space W_j is called the detail space at the level j and the mapping

$$
\begin{aligned}
(\mathrm{SWT})_j : L^2(S^2) &\to L^2(S^2) \\
F &\longmapsto \psi_j * F
\end{aligned}
$$

is called the *spherical wavelet transform* at the scale j.

Based on this definition and the results above, any function $F \in L^2(S^2)$ will be represented by means of an L^2-convergent series

$$
F = \sum_{j=-1}^{\infty} \tilde{\psi}_j * (SWT)_j(F) \,. \tag{5.19}
$$