4 Review of special functions

4.1 Introduction

The main motivation behind this chapter about special functions is that these functions are applied in the quasi-field of mathematical physics and that there is no literature dedicated on them and their basic properties with efficient proofs and original references.

Special functions are, as their name indicates, special in their definitions, applications, proofs as well as their interactions with other fields. It is thus important to understand their basic properties.

They appear in the treatment of differential equations, such as heat and Schrödinger equations, quantum mechanics, approximation theory, communication systems, wave propagation, probability theory, and number theory.

Special functions are also related to orthogonal polynomials, as both of them are generated by second-order ordinary differential equations. We cite mainly Legendre, Gegenbauer, and Jacobi polynomials. They are also associated with infinite series, improper integrals, and Fourier transforms, yielding special transforms, such as Bessel, Jakobi, Hankel, and Dunkl transforms of functions.

Historically, special functions differ from elementary ones, such as powers, roots, trigonometric, and their inverses, mainly with the limitations that these latter classes have known. Many fundamental problems such as orbital motion, simultaneous oscillatory chains, and spherical body gravitational potential were not best described using elementary functions. This makes it necessary to extend elementary function classes to more general ones that may describe unresolved problems.

In the present chapter, we aim to recall special functions most frequently applied in scientific fields, such as Bessel functions, Mathieu functions, the Gamma function, the Beta function, and Jacobi functions.

4.2 Classical special functions

4.2.1 Euler's Γ function

Euler's Gamma function was introduced by Bernoulli and Christian Goldbach in the 17th century by extending the factorial to nonintegers. But the problem remained unsolved until the work of Leonhard Euler, who was the first to point out a rigorous formulation based on infinite products. Next, Euler's Gamma function has been applied in numerous contexts in both mathematics and physics, such as integration theory, number theory, probability, group theory, and partial differential equations (PDEs), and has also been extended to the meromorphic function on the whole complex plane.

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Definition 57. Euler's Γ function is defined by the following integral expression known sometimes as the second-kind Euler integral, defined for $x \in \mathbb{R}_+^*$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \,.$$

Proposition 58.
(1) *Euler's Γ integral converges for all $x > 0$.*
(2) *The function Γ is C^∞ on $]0, +\infty[$ and we have*

$$\Gamma^{(k)}(x) = \int_0^{+\infty} e^{-t} (\ln t)^k t^{x-1} dt, \qquad \forall x > 0, \forall k \in \mathbb{N} \,.$$

(3) *Euler's Γ function can be extended on the half-plane* $\mathrm{Re}(z) > 0$.

Proof. (1) For $x > 0$, denote $f(t, x) = t^{x-1} e^{-t}$. First note that $f(t, x) > 0$ for all $t \in (0, +\infty)$. When $t \to 0$, $f(t, x) \sim t^{x-1}$ and

$$\int_0^1 t^{x-1} dt = \frac{1}{x}$$

is convergent. So

$$\int_0^1 f(t, x) dt$$

is also convergent. Now, note that there exist $A, M > 0$ constants such that $t^2 f(t, x) < M$ whenever $t > A$ and thus

$$\int_A^{+\infty} f(t, x) dt \leq \int_A^{+\infty} \frac{1}{t^2} dt \,.$$

The last integral is convergent. So

$$\int_A^{+\infty} f(t, x) dt$$

is also convergent. Finally, the integral $\Gamma(x)$ is convergent for all $x > 0$.

(2) Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $\phi\colon [a,b]\times]0,+\infty[\longrightarrow \mathbb{R}$ such that $\phi(x,t) = t^{x-1}e^{-t}$. It consists of a C^∞ function that satisfies

$$\frac{\partial^{(k)}\phi}{\partial x^k}(x,t) = (\ln t)^k t^{x-1}e^{-t}\,,$$

which is also continuous on $[a,b]\times]0,+\infty[$. In addition, for $k \in \mathbb{N}^*$, we have

- $\forall x \in [a,b]$, the function $t \mapsto \frac{\partial^k\phi}{\partial x^k}(x,t)$ is continuous on $]0,+\infty[$.
- $\forall t \in]0,+\infty[$, the function $x \mapsto \frac{\partial^k\phi}{\partial x^k}(x,t)$ is continuous on $[a,b]$.
- $\forall (x,t) \in [a,b]\times]0,+\infty[$, we have

$$\left|\frac{\partial^k\phi}{\partial x^k}(x,t)\right| \le (\ln t)^k \max(t^{a-1}, t^{b-1})e^{-t}\,.$$

Hence, the function Γ is $\mathcal{C}^\infty$ on $]0,+\infty[$ and $\forall k \in \mathbb{N}^*$, $\forall x > 0$,

$$\Gamma^{(k)}(x) = \int_0^{+\infty} (\ln t)^k t^{x-1}e^{-t}dt\,.$$

Hence (3.3).

(3) We will prove by recurrence the proposal

$$P_m\colon \Gamma \text{ can be extended on } -m+1 > \operatorname{Re}(z) > -m, \forall m \in \mathbb{N}.$$

Indeed, P_0 holds because Γ is analytic on $\{\operatorname{Re}(z) > 0\}$. Therefore, it is analytic on $\{1 > \operatorname{Re}(z) > 0\}$. Next, for $0 > \operatorname{Re}(z) > -1$, we have $1 > \operatorname{Re}(z+1) > 0$. Hence, $\Gamma(z+1)$ is analytic. In addition, $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. Thus, Γ is holomorphic on $\{0 > \operatorname{Re}(z) > -1\}$ with 0 being a simple pole corresponding to the residues 1. So, Γ can be extended to a meromorphic function $\{\operatorname{Re}(z) > -1\}$ with a simple pole at 0. Hence, the property P_1. Next, applying the recurrence rule, we obtain

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{k=0}^{n-1}(z+k)}\,.$$

□

Properties 59. The following assertions are satisfied.

(1) $\Gamma(x+1) = x\Gamma(x); \quad \forall x > 0$.

(2) $\Gamma(n+1) = n!, \forall n \in \mathbb{N}$.

(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(4) $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}, \forall n \in \mathbb{N}$.

Proof. (1) An integration by parts gives

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}dt = x\int_0^{+\infty} t^{x-1}e^{-t}dt = x\Gamma(x)\,.$$

(2) Putting $x = n \in \mathbb{N}^*$ in assertion (1), we get

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n!\Gamma(1) = n!\,.$$

Hence the appointment of *generalized factorial function* for Γ.
(3) We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt\,.$$

Putting $x = \sqrt{t}$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx\,.$$

Hence,

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy\,.$$

Now, using the polar coordinates system, $x = r\cos\theta$ and $y = r\sin\theta$, with $r \in (0,\infty)$ and $\theta \in (0, \frac{\pi}{2})$ we get

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta = \pi\,.$$

Therefore, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(4) By recurrence on n. For $n = 0$, we have $\Gamma(0 + \frac{1}{2}) = \Gamma(\frac{1}{2})$ on the left and $\sqrt{\pi}$ on the right. So, the assertion is true for $n = 0$. Assume next that it is true for n. We shall then check it for $n + 1$.

$$\begin{aligned}\Gamma\left(n+1+\frac{1}{2}\right) &= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\\ &= \left(n+\frac{1}{2}\right)\frac{(2n)!\sqrt{\pi}}{2^{2n}n!}\\ &= \frac{(2n+2)!\sqrt{\pi}}{2^{2n+2}(n+1)!}\\ &= \frac{(2(n+1))!\sqrt{\pi}}{2^{2(n+1)}(n+1)!}\,.\end{aligned}$$

□

The next result shows some asymptotic behaviors of Euler's Γ function.

Theorem 60. *Euler's Γ function satisfies the so-called Stirling formula,*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \quad as \quad x \longrightarrow +\infty\,.$$

Proof. Recall that

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt\,.$$

By setting $t = x + \sqrt{x}u$, we obtain

$$\begin{aligned}\Gamma(x+1) &= \int_{-\sqrt{x}}^{+\infty} e^{-x-\sqrt{x}u} e^{x\ln(x+\sqrt{x}u)} \sqrt{x} du \\ &= \left(\frac{x}{e}\right)^x \sqrt{x} \int_{-\sqrt{x}}^{+\infty} e^{-\sqrt{x}u + x\ln(1+\frac{u}{\sqrt{x}})} du\,.\end{aligned}$$

Now, it suffices to prove that the last integral tends to $\sqrt{2\pi}$ as $x \to +\infty$. Denote $\Gamma_1(x)$ as this integral and let

$$f(x,u) = \begin{cases} e^{-\sqrt{x}u + x\ln(1+\frac{u}{\sqrt{x}})} & \text{if } u \geq -\sqrt{x} \\ 0 & \text{if not}\,. \end{cases}$$

We get

$$\Gamma_1(x) = \int_{-\infty}^{+\infty} f(x,u) du\,.$$

For fixed $u \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{x\longrightarrow+\infty} f(x,u) &= \lim_{x\longrightarrow+\infty} \exp\left(-\sqrt{x}u + x\ln\left(1+\frac{u}{\sqrt{x}}\right)\right) \\ &= \lim_{x\longrightarrow+\infty} \exp\left(-\sqrt{x}u + x\left(\frac{u}{\sqrt{x}} - \frac{1}{2}\frac{u^2}{x} + \theta\left(\frac{1}{x}\right)\right)\right) \\ &= \exp\left(\frac{-u^2}{2}\right)\,.\end{aligned}$$

On the other hand, if $u \in]-\sqrt{x}, 0]$, as $\frac{|u|}{\sqrt{x}} < 1$, we obtain

$$f(x,u) \leq \exp\left(-\frac{u^2}{2}\right)\,.$$

Finally, for $u \in]0,+\infty[$, $f(x,u)$ is a decreasing function of x on $]0,+\infty[$. We deduce for $u > 0$ and $x \in [1,+\infty[$ that

$$f(x,u) \leq f(1,u) = (1+u)e^{-u}\,.$$

So, for all $u \in \mathbb{R}$ and all $x \in [1,+\infty[$, we have $0 \leq f(x,u) \leq g(u)$, where g is the integrable function defined by

$$g(u) = \begin{cases} e^{-\frac{u^2}{2}}, & \text{if } u \leq 0 \\ (1+u)e^{-u}, & \text{if not } u \geq 0\,. \end{cases}$$

By the dominated convergence theorem, we obtain

$$\Gamma_1(x) \to \int_{-\infty}^{+\infty} e^{\frac{-u^2}{2}}\, du = \sqrt{2\pi}, \quad \text{as } x \to +\infty\,.$$

Now, by setting $t = \frac{\sqrt{x}}{u}$, we get $f(x,u) = e^{h(t,u)}$, where

$$h(t,u) = u^2 t^2 \left(-\frac{1}{t} + \ln\left(1 + \frac{1}{t}\right)\right),$$

which is decreasing in t. ☐

Proposition 61. *Euler's Γ function satisfies the so-called Gauss formula for all $x > 0$,*

$$\frac{1}{\Gamma(x)} = \lim_{n\longrightarrow+\infty} \frac{x(x+1)\cdots(x+n)}{n!n^x}\,.$$

Proof. Applying the recurrence relation n times, we obtain

$$\Gamma(x)x(x+1)\cdots(x+n) = \Gamma(x+n+1)\,.$$

Therefore,

$$\begin{aligned}
\frac{x(x+1)\cdots(x+n)}{n!n^x} &= \frac{\Gamma(x+n+1)}{\Gamma(x)n!n^x} \\
&\sim \frac{\sqrt{2\pi}(x+n+1)^{x+n+\frac{1}{2}} e^{-(x+n+1)}}{\Gamma(x)\sqrt{2\pi}(n+1)^{n+\frac{1}{2}} e^{-n-1} n^x} \\
&\sim \frac{1}{\Gamma(x)} \left(\frac{x+n+1}{n}\right)^x \left(\frac{x+n+1}{n+1}\right)^{n+1} e^{-x} \left(1 + \frac{x}{n}\right)^n \\
&= \frac{1}{\Gamma(x)}\,.
\end{aligned}$$

☐

Proposition 62.
(1) *The infinite product $\prod_{k=1}^{+\infty}(1+\frac{z}{k})e^{\frac{-z}{k}}$ is normally convergent on every compact of $\mathbb{C}$ and therefore defines an analytic function of z.*
(2) *Euler's Γ function satisfies the so-called Gauss–Weierstrass formula for $z \notin -\mathbb{N}$,*

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to+\infty} \frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} \\
&= z e^{\gamma z} \lim_{n\longrightarrow+\infty} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{\frac{-z}{k}}\,,
\end{aligned}$$

where γ is the Euler–Mascheroni constant given by

$$\gamma = \lim_{n\longrightarrow+\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right).$$

Proof. (1) Put $u_k(z) = (1+\frac{z}{k})e^{\frac{-z}{k}} - 1$. A simple Taylor development gives

$$|u_k(z)| \le \frac{|z|^2}{n^2}$$

whenever $\frac{|z|}{n}$ is bounded independent of n and z. Hence, the infinite product converges uniformly on every compact set in $\mathbb{C}$ to an analytic function.
(2) Observe that

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt = \lim_{n\to+\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt .$$

This is a consequence of the application of the dominated convergence theorem. So, next, integrating by parts, we obtain

$$\Gamma(z) = \lim_{n\to+\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}$$

whenever $z \notin -\mathbb{N}$. So the first part is proved. Next, observe that

$$\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = zn^{-z}\prod_{k=1}^{n}\frac{z+k}{k} = ze^{z(\gamma_n-\log n)}\prod_{k=1}^{n}e^{-\frac{z}{k}}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)$$

where $\gamma_n = \sum_{k=1}^{n-1}\frac{1}{k}$. Next, we obtain

$$\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = ze^{z(\gamma_n-\log n)}\prod_{k=1}^{n}e^{-\frac{z}{k}}\left(1+\frac{z}{k}\right),$$

which implies by the limit on n that

$$\frac{1}{\Gamma(z)} = \lim_{n\to+\infty}\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = ze^{\gamma z}\lim_{n\longrightarrow+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{\frac{-z}{k}} .$$ □

Proposition 63. *For $z \in \mathbb{C}\setminus\mathbb{Z}$,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} .$$

The proof is based on the following lemma, which canbe obtained by simple application of Fourier series theory on the function $f(t) = \cos(st)$, where $s \in \mathbb{C}$. The result may also be established by direct methods based on the simple relation $e^z = \lim_{k\to+\infty}(1+\frac{z}{k})^k$. Thus, the proof of this lemma is left to the reader.

Lemma 64. *$\forall z \in \mathbb{C}$, we have*

$$\sin(\pi z) = \pi z \prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right) . \tag{4.1}$$

Proof of Proposition 63. Let

$$a_n(z) = \frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} .$$

Proceeding as in the proof of Proposition 62, we obtain

$$a_n(z)a_n(1-z) = z\left(1+\frac{1-z}{n}\right)\prod_{k=1}^{n}\left(1-\frac{z^2}{k^2}\right) .$$

The limit on n gives

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \lim_{n\to+\infty} a_n(z)a_n(1-z) = z\prod_{k\geq 1}\left(1-\frac{z^2}{k^2}\right) = \frac{\sin(\pi z)}{\pi z} .$$

Consequently,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} .$$

□

Remark 65. The meromorphic function Γ has no roots on $\mathbb{C}$.

Proposition 66.
- *Convexity of Γ: Γ is strictly convex on* $]0,+\infty[$.
- *Asymptotic behavior of Γ at ∞:* $\lim_{x\longrightarrow+\infty}\Gamma(x) = +\infty$.
- *Asymptotic behavior of Γ at ∞:* $\lim_{x\longrightarrow+\infty}\frac{\Gamma(x)}{x} = +\infty$.
- *Asymptotic behavior of Γ at 0^+:* $\lim_{x\longrightarrow 0^+}\Gamma(x) = +\infty$.

Proof. (1) Recall that the function Γ is twice differentiable on $]0,+\infty[$ and $\forall x > 0$, so we have

$$\Gamma''(x) = \int_0^{+\infty}(\ln t)^2 t^{x-1}e^{-t}dt > 0 .$$

Hence, it is convex.

(2) Since the function Γ is increasing on $)0, +\infty[$, for x big enough, we have

$$\Gamma(x) = (x-1)\Gamma(x-1) \geq (x-1)\Gamma(1) = x - 1 .$$

We deduce that $\lim_{x \longrightarrow +\infty} \Gamma(x) = +\infty$.
(3) For $x > 1$, we have

$$\frac{\Gamma(x)}{x} = \frac{(x-1)}{x}\Gamma(x-1) \longrightarrow +\infty \quad \text{as } x \longrightarrow +\infty .$$

We deduce that the graph of the function Γ has at $+\infty$ a vertical asymptotic direction.
(4) For $x > 0$, $\Gamma(x) = \frac{\Gamma(x+1)}{x} \to \frac{\Gamma(1)}{0^+} = +\infty$ when $x \longrightarrow 0^+$.
So $\lim_{x \longrightarrow 0^+} \Gamma(x) = +\infty$. In addition, we have precisely, $\Gamma(x) \sim \frac{1}{x}$ as $x \to 0^+$. □

4.2.2 Euler's beta function

The origin of Euler's beta function goes back to differential calculus and integrals. It was introduced in the Arithmetica Infinitorum published by Wallis. Newton next discovered the binomial formula and introduced Euler's beta function, which was then developed for other versions, such as the incomplete and the corrected versions. The beta function is given by Euler in the following form:

$$\beta(p, q) = \int_0^1 t^p (1-t)^q \, dt$$

and is known as the first-kind Euler integral. But since Legendre's work, it appears in a slightly modified form

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, \quad p > 0, q > 0 .$$

It is apparent that such a function is symmetrical in (p, q), i.e.,

$$\beta(p, q) = \beta(q, p) .$$

The beta function also has another integral representation. Indeed, by setting $t = \frac{y}{a}$, $a > 0$, it becomes

$$\beta(p, q) = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1}(a-y)^{q-1} \, dy .$$

Again, setting $t = \sin^2 \theta$, we get a trigonometric form

$$\beta(p, q) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} \, d\theta .$$

Finally, with the variable change $t = \frac{y}{(1+y)}$, we get

$$\beta(p,q) = \int_0^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}} dy .$$

In the following, we will apply one of these representations without mentioning it each time. The form applied will be understood from the development.

Proposition 67.
(1) *The beta integral converges whenever* $x, y > 0$.
(2) *The beta integral is continuous on* $]0,+\infty[\times]0,+\infty[$.
(3) *The beta integral remains valid on the quarter complex plane* $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

Proof. (1) Whenever $p, q > 0$ we have

$$t^p(1-t)^q \sim t^p,\ t \to 0^+ \quad \text{and} \quad t^p(1-t)^q \sim (1-t)^q,\ t \to 1^- .$$

Hence, the integral is convergent.
(2) On $]0,+\infty[\times]0,+\infty[$, the function $(p,q) \mapsto f_t(p,q) = t^p(1-t)^q$ is continuous for all $t \in (0,1)$. Furthermore, $t^p(1-t)^q \le 1$, $\forall t, p, q$. Thus, the integral is uniformly convergent to a continuous function on $]0,+\infty[\times]0,+\infty[$. Next, by recurrence on $k \in \mathbb{N}$, we can prove that beta is k-times differentiable according to p and q. We can also prove that

$$\frac{\partial^k \beta}{\partial p^k}(p,q) = \int_0^1 (\log t)^k t^p (1-t)^q dt ,$$

$$\frac{\partial^k \beta}{\partial q^k}(p,q) = \int_0^1 (\log(1-t))^k t^p (1-t)^q dt$$

and for $n + m = k$,

$$\frac{\partial^k \beta}{\partial p^n q^m}(p,q) = \int_0^1 (\log t)^n (\log(1-t))^m t^p (1-t)^q dt .$$

(3) For $p, q \in \mathbb{C}$, we have

$$|t^p(1-t)^q| = t^{\mathrm{Re}(p)}(1-t)^{\mathrm{Re}(q)}, \quad \forall t \in (0,1) .$$

□

Properties 68.

(1) $p\beta(p, q+1) = q\beta(p+1, q)$, $\forall p, q \geq 0$.

(2) $\beta(p, 1) = \frac{1}{p}$.

(3) $\beta(\frac{1}{2}, \frac{1}{2}) = \pi$.

(4) $\forall n \in \mathbb{N}$ and $\forall p > 0$, $\beta(p, n) = \frac{n-1}{p}\beta(p+1, n-1)$.

(5) $\forall n \in \mathbb{N}$ and $\forall p > 0$, $\beta(p, n) = \frac{(n-1)(n-2)\cdots 2\cdot 1}{p(p+1)\cdots(p+n-1)}$.

(6) $\forall m, n \in \mathbb{N}$, $\beta(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$.

(7) $\forall p, q > 0$, $\beta(p, q) = \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}\,dy$.

(8) $\forall p$, $0 < p < 1$, $\beta(p, 1-p) = \int_0^1 \frac{y^{p-1}+y^{-p}}{(1+y)}\,dy$.

(9) $\forall p$, $0 < p < 1$, $\beta(p, 1-p) = \frac{\pi}{\sin \pi p}$.

(10) $\forall p, q > 0$, $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Proof. (1) Integrating by parts, we get

$$\beta(p, q+1) = \int_0^1 \frac{q}{p} x^p (1-x)^{q-1}\,dx = \frac{q}{p}\beta(p+1, q)\,.$$

(2) We have

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx\,.$$

So,

$$\beta(p, 1) = \int_0^1 x^{p-1}\,dx = \frac{1}{p}\,.$$

(3) Taking $q + 1 = n \in \mathbb{N}$, we get

$$\beta(p, n) = \frac{n-1}{p}\beta(p+1, n-1)\,.$$

(4) Observing that $\beta(p, 1) = \frac{1}{p}$, we get by iteration

$$\beta(p, n) = \frac{1\cdot 2\cdots(n-1)}{p(p+1)\cdots(p+n-1)}\,.$$

(5) If we take $p = m \in \mathbb{N}$ in the previous equation, we obtain

$$\beta(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}\,.$$

(6)

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx \quad (x = u^2)$$
$$= 2\int_0^1 \frac{du}{\sqrt{1-u^2}}$$
$$= \pi\,.$$

(7) We have

$$\beta(p,q) = \int_0^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}}dy$$
$$= \int_0^1 \frac{y^{p-1}}{(1+y)^{p+q}}dy + \underbrace{\int_1^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}}dy}_{I} \quad \left(y \text{ with } \frac{1}{y} \text{ in I}\right)$$
$$= \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}dy\,.$$

(8) For $0 < p < 1$, we get

$$\beta(p,1-p) = \int_0^{\infty} \frac{y^{p-1}}{(1+y)}dy$$
$$= \int_0^1 \frac{y^{p-1}}{(1+y)}dy + \int_1^{\infty} \frac{y^{p-1}}{(1+y)}dy$$
$$= \int_0^1 \frac{y^{p-1}+y^{-p}}{(1+y)}\,.$$

(9) Recall that $\frac{1}{1+y} = \sum_{n=0}^{\infty}(-1)^n y^n$ whenever $0 < y < 1$. Hence,

$$\int_0^1 \frac{y^{p-1}}{1+y}dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+n}\,.$$

Similarly, we have

$$\int_0^1 \frac{y^{-p}}{1+y}dy = \sum_{n=1}^{\infty} \frac{(-1)^n}{p-n}\,.$$

Therefore,

$$\beta(p,1-p) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{p-n} = \frac{\pi}{\sin \pi p}\,.$$

(10) By setting $t = y^2$ in the Γ integral, we obtain

$$\Gamma(p) = 2 \int_0^{+\infty} y^{2p-1} e^{-y^2} dy \,.$$

Thus,

$$\Gamma(p)\Gamma(q) = 4 \int_0^{+\infty} \int_0^{+\infty} x^{2q-1} y^{2p-1} e^{-(x^2+y^2)} dxdy \,.$$

Next, applying polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, this yields that

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4 \int_0^{+\infty} \int_0^{\frac{\pi}{2}} (r\cos\theta)^{2q-1} (r\sin\theta)^{2p-1} e^{-r^2} drd\theta \\ &= 4 \int_0^{+\infty} r^{2(p+q-1)} e^{-r^2} dr \int_0^{+\frac{\pi}{2}} (\cos\theta)^{2q-1} (\sin\theta)^{2p-1} d\theta \\ &= 4 \frac{1}{2} \Gamma(p+q) \frac{1}{2} \beta(p,q) \\ &= \Gamma(p+q)\beta(p,q) \,.\end{aligned}$$

□

The following result relates the differentiability of beta to Euler's Γ function. The proof is an immediate consequence of the last property above.

Proposition 69. *The function beta is differentiable and we have*

$$\frac{\partial}{\partial p}\beta(p,q) = \beta(p,q)\left(\frac{\Gamma'(p)}{\Gamma(p)} - \frac{\Gamma'(p+q)}{\Gamma(p+q)}\right) = \beta(p,q)\left(\psi(p) - \psi(p+q)\right) \,,$$

where ψ is the so-called di-Gamma function defined by $\psi(p) = \frac{\Gamma'(p)}{\Gamma(p)}$.

In the following, we introduce the complete and incomplete beta functions.

Definition 70. The *complete Beta function* is defined for $a, b > 0$ by

$$\beta(p; a, b) = \int_0^p t^{a-1}(1-t)^{b-1} dt \,. \tag{4.2}$$

The incomplete (regularized) beta function is

$$I_p(a,b) = \frac{\beta(p; a, b)}{\beta(a,b)}; \quad a, b > 0 \,. \tag{4.3}$$

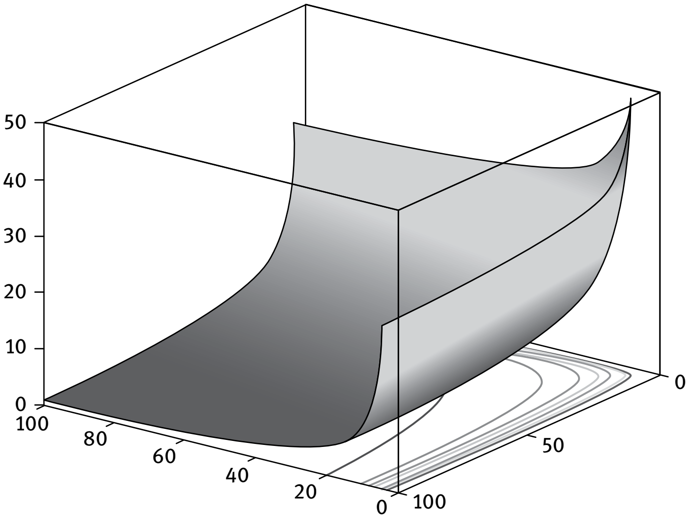

Fig. 4.1: Representations of the beta function.

Figure 4.1 illustrates the graph of the beta function.

4.2.3 Theta function

The theta function appears in many areas, such as manifolds, quadratic forms, soliton theory, and quantum theory.

Definition 71. The function θ is defined for $(z, \tau) \in \mathbb{C}^2$ such that $\operatorname{Im}(\tau) > 0$, by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2in\pi z} . \tag{4.4}$$

Proposition 72. *We have*
(1) $\forall \tau$ *such that* $\operatorname{Im}(\tau) > 0$, $\theta(., \tau)$ *is a holomorphic function on* $\mathbb{C}$.
(2) $\theta(z + 1, \tau) = \theta(z, \tau)$, $\forall \tau$ *such that* $\operatorname{Im}(\tau) > 0$.
(3) $\theta(z + \tau, \tau) = e^{-i\pi\tau} e^{-2i\pi z} \theta(z, r)$.

Proof. (1) For all τ, the function $z \mapsto e^{i\pi n^2 \tau} e^{2i\pi n z}$ is holomorphic on $\mathbb{C}$. Moreover, for all compact $K \subset \mathbb{C}$, we have

$$\sup_{z \in K} |e^{i\pi n^2 \tau} e^{2i\pi n z}| \le e^{-\pi n^2 \operatorname{Im}(\tau)} e^{2\pi R n} ,$$

with R such that $K \subset D(0,R)$. Hence, the series $\sum_n e^{-\pi n^2 \operatorname{Im}(\tau)} e^{Rn}$ is convergent, which yields that $\sum_n e^{i\pi n^2\tau} e^{2i\pi nz}$ is holomorphic.
(2) $\forall z, \tau$, we have

$$\begin{aligned}\theta(z+1,\tau) &= \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi n(z+1)} \\ &= \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi nz} (e^{2i\pi})^n \\ &= \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi nz} \\ &= \theta(z,\tau)\ .\end{aligned}$$

(3) $\forall z, \tau$, we have

$$\begin{aligned}\theta(z+\tau,\tau) &= \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi n(z+\tau)} \\ &= \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi nz} (e^{2i\pi})^n \\ &= \sum_{n\in\mathbb{Z}} e^{i\pi (n^2+2n)\tau} e^{2i\pi nz} \\ &= \sum_{n\in\mathbb{Z}} e^{i\pi ((n+1)^2-1)\tau} e^{2i\pi nz} \\ &= e^{-i\pi\tau} \sum_{n\in\mathbb{Z}} e^{i\pi ((n+1)^2)\tau} e^{2i\pi nz} \\ &= e^{-i\pi\tau} \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2i\pi (n-1)z} \\ &= e^{-i\pi\tau} e^{-2i\pi z} \theta(z,r)\ .\end{aligned}$$

□

Proposition 73.
(1) *For all τ such that* $\operatorname{Im}(\tau) > 0$, *we have*

$$\sqrt{\frac{\tau}{i}}\theta(z,\tau) = e^{\frac{-i\pi z^2}{\tau}} \theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right)\ . \tag{4.5}$$

(2) *For $t > 0$, let $\Theta(t) = \theta(0, it)$. Then,*

$$\sqrt{t}\Theta(t) = \Theta\left(\frac{1}{t}\right)\ .$$

Proof. Denote for $x \in \mathbb{R}$, $f(x) = e^{i\pi x^2\tau} e^{2i\pi xz}$. From the well-known Poisson summation formula, we obtain

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\ .$$

On the other hand, note that

$$f(x) = G_\alpha\left(x + \frac{z}{\tau}\right)e^{\frac{-i\pi z^2}{\tau}} \text{ with } \alpha = -2i\pi\tau \quad \text{and} \quad G_\alpha(t) = e^{-\frac{\alpha t^2}{2}} .$$

Therefore,

$$\begin{aligned}
\widehat{f}(\omega) &= \widehat{G_\alpha}\left(. + \frac{z}{\tau}\right)(\omega)e^{\frac{-i\pi z^2}{\tau}} \\
&= e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{-2\pi^2\omega^2}{\alpha}}\sqrt{\frac{2\pi}{\alpha}}e^{\frac{-i\pi z^2}{\tau}} \\
&= e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{\pi\omega^2}{i\tau}}\sqrt{\frac{1}{-i\tau}}e^{\frac{-i\pi z^2}{\tau}} \\
&= e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{-i\pi\omega^2}{\tau}}\sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}} .
\end{aligned}$$

Hence,

$$\sum_{n\in\mathbb{Z}}\widehat{f}(n) = \sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\sum_{n\in\mathbb{Z}}e^{2i\pi\frac{z}{\tau}n}e^{\frac{-i\pi n^2}{\tau}} = \sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) .$$

Consequently,

$$\theta(z,\tau) = \sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) ,$$

or equivalently,

$$\sqrt{\frac{\tau}{i}}\theta(z,\tau) = e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) . \qquad \square$$

4.2.4 Riemann zeta function

The Riemann zeta function is often known in number theory and in particular in the study of the distribution of prime numbers.

Definition 74. The Riemann zeta function is defined for $x > 1$ by

$$\zeta(x) = \sum_{n=1}^{+\infty}\frac{1}{n^x} . \tag{4.6}$$

Remark 75. The definition may be extended to complex numbers $x = a + ib$ with $a > 1$.

Proposition 76. *The ζ Riemann's function satisfies the so-called Euler's multiplication*

$$\zeta(x) = \prod_{p \in \mathcal{P}} \frac{1}{(1-p^{-x})}, \quad \forall x > 1 ,$$

where $\mathcal{P}$ is the set of prime numbers.

Proof. For $x > 1$, we have

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \cdots$$

Thus,

$$\frac{1}{2^x}\zeta(x) = \frac{1}{2^x} + \frac{1}{4^x} + \frac{1}{6^x} + \frac{1}{8^x} + \frac{1}{10^x} + \cdots$$

Or equivalently,

$$\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \frac{1}{3^x} + \frac{1}{5^x} + \frac{1}{7^x} + \frac{1}{9^x} + \cdots$$

Multiplying again by $\frac{1}{3^x}$, we get

$$\frac{1}{3^x}\left(1 - \frac{1}{2^x}\right)\zeta(x) = \frac{1}{3^x} + \frac{1}{9^x} + \frac{1}{15^x} + \frac{1}{21^x} + \frac{1}{27^x} + \cdots$$

Hence,

$$\left(1 - \frac{1}{3^x}\right)\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \frac{1}{5^x} + \frac{1}{7^x} + \cdots$$

Next, by following the same process we get for $p \in \mathcal{P}$,

$$\left(1 - \frac{1}{p^x}\right)\cdots\left(1 - \frac{1}{11^x}\right)\left(1 - \frac{1}{7^x}\right)\left(1 - \frac{1}{5^x}\right)\left(1 - \frac{1}{3^x}\right)\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \sum_{n>p} \frac{1}{n^x} .$$

Next, note that the last summation goes to 0 as $p \to \infty$. Therefore,

$$\zeta(x) \prod_{p \in \mathcal{P}} (1 - p^x) = 1 .$$

Hence,

$$\zeta(x) = \prod_{p \in \mathcal{P}} \frac{1}{(1-p^{-x})} .$$

□

Proposition 77.
(1) *ζ is continuous, nonincreasing, and convex on* $]1, +\infty[$.
(2) *ζ is* $\mathcal{C}^\infty$ *on* $]1, +\infty[$ *and*

$$\zeta^{(k)}(x) = (-1)^k \sum_{n=2}^{\infty} \frac{(\ln n)^k}{n^x}; \quad \forall k \in \mathbb{N} \text{ and } x > 1 .$$

Proof. (1) Let $a > 1$. For $n \in \mathbb{N}^*$, the function $x \mapsto \frac{1}{n^x}$ is continuous on $[a, +\infty[$. Moreover, $\forall x \in [a, +\infty[$,

$$|\frac{1}{n^x}| = \frac{1}{n^x} \le \frac{1}{n^a} .$$

Thus, the series $\sum_n \frac{1}{n^a}$ is normally convergent. Hence, the sum ζ is continuous on $[a, +\infty[$. This being true for all real $a \in]1, +\infty[$. Henceforth, ζ is continuous on $]1, +\infty[$.

Next, the *monotony of* ζ follows from the fact that for all $n \in \mathbb{N}$, the functions $x \mapsto \frac{1}{n^x}$ is nonincreasing on $]1, +\infty[$.

Finally, to prove the *convexity of the function* ζ, recall that for all $n \in \mathbb{N}$, the functions $x \mapsto \frac{1}{n^x}$ is convex on $]1, +\infty[$. So, ζ is convex on $]1, +\infty[$ as a sum of convex functions $]1, +\infty[$.

(2) Let $a > 1$. For all $n \in \mathbb{N}$, the function $f_n : x \mapsto \frac{1}{n^x}$, is $\mathcal{C}^\infty$ on $[a, +\infty[$ and for $x \ge a$ and $k \ge 1$, we have

$$\left|f_n^k(x)\right| = \left|(-1)^k \frac{(\ln n)^k}{n^a}\right| \le \frac{(\ln n)^k}{n^a} .$$

Note that $\sum_n \frac{(\ln n)^k}{n^a}$ converges by the Bertrand rule of numerical series. So, we deduce that for $k \ge 1$, the series $\sum_n f_n^{(k)}$ is normally convergent on $[a, +\infty[$. As a result, ζ is $\mathcal{C}^k$ on $[a, +\infty[$ for all k. Hence, it is $\mathcal{C}^\infty$ on $[a, +\infty[$ for all $a > 1$. So, it is $\mathcal{C}^\infty$ on $]1, +\infty[$ and the derivatives are obtained as stated above. □

Proposition 78. *The ζ function satisfies*
- $\lim_{x \longrightarrow +\infty} \zeta(x) = 1$.
- $\lim_{x \longrightarrow 1^+} \zeta(x) = +\infty$.

Proof. (1) Note first that the series $\sum_{n \ge 1} \frac{1}{n^2}$ converges. Henceforth, the series $\zeta(x)$ is uniformly convergent on the interval $[2, +\infty[$. Furthermore,

$$\lim_{x \to +\infty} \frac{1}{n^x} = \begin{cases} 1, & \text{for } n = 1 , \\ 0, & \text{for } n > 1 . \end{cases}$$

So, by applying the limit on $\zeta(x)$ at infinity we get

$$\lim_{x \longrightarrow +\infty} \zeta(x) = 1 + \sum_{n \ge 2} 0 = 1 .$$

(2) holds from the fact that ζ is nonincreasing on $]1, +\infty[$ and that $\sum_{n \ge 1} \frac{1}{n} = +\infty$. □

Proposition 79. *The ζ function can be extended on the band $\Omega = \{z \in \mathbb{C}; \operatorname{Re}(s) > 1\}$ in a holomorphic function. With higher derivative $\zeta^{(k)}$, $k \in \mathbb{N}$ is given by*

$$\zeta^{(k)}(z) = \sum_{n=1}^{+\infty} \frac{(-1)^k \ln^k n}{n^z} . \tag{4.7}$$

Proof. (1) The function $f_n(z) = \frac{1}{n^z}$, $n \geq 1$ is holomorphic, and the series $\sum_n f_n$ is uniformly convergent on all sets of the form $\Omega_a = \{z \in \mathbb{C}; \operatorname{Re}(z) > a\}$ for all $a > 1$. So the sum ζ is holomorphic on $\operatorname{Re}(z) > 1$.
(2) For $k \in \mathbb{N}$, we have $f_n^{(k)}(z) = \frac{(-1)^k \ln^k n}{n^z}$. On any set Ω_a, the series $\sum_n f_n^{(k)}$ is uniformly convergent. Hence, ζ is $\mathcal{C}^k$ and its derivative of order k is given by (4.7). ☐

Proposition 80. *The function ζ has a meromorphic extension on $\mathbb{C}$, with a single pole in 1 which is simple.*

To prove this result, we need to recall that the well-known Bernoulli numbers, denoted by B_n, form a sequence of rational numbers. These numbers were first studied by Jacques Bernoulli in the context of computing summations of the form $S_m(n) = \sum_{k=0}^{n-1} k^m$ for different integer values m. It holds that these quantities are polynomials of the variable n with degree $m + 1$. Hence, we can write them in the form

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} C_{m+1}^k B_k n^{m+1-k} . \tag{4.8}$$

The numbers B_k are called the Bernoulli numbers. These numbers may also be defined by means of a generator function as

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k . \tag{4.9}$$

Generally, these numbers may be extended to polynomials. The well-known Bernoulli polynomials are obtained from the following relation:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} x^k . \tag{4.10}$$

It yields a sequence of polynomials of degree k in x. For more details, refer to [67]. These are applied in numerous fields. We recall here one application that will be used later. It consists of the well-known Euler–Maclaurin summation rule for functions.

Proposition 81. *Let f be* $\mathcal{C}^{2k}$ *function on* $[p, q]$, $p, q \in \mathbb{Z}$ *and* $k \in \mathbb{N}$. *It holds that*

$$\frac{f(p) + f(q)}{2} + \sum_{i=p+1}^{q-1} f(i) = \sum_{j=1}^{k} \frac{B_{2j}}{(2j!)} \left(f^{(2j-1)}(q) - f^{(2j-1)}(p)\right) + \int_p^q f(x)dx + R_{p,q}^k ,$$

where $R_{p,q}^k$ *is the rest*

$$R_{p,q}^k = -\frac{1}{(2k)!} \int_p^q f^{(2k)}(x) B_{2k}(x - [x]) dx$$

where $B_{2k}(.)$ *is the Bernoulli polynomial of degree* $2k$.

Proof of Proposition 80. By applying Euler–Maclaurin summation to the function $f(x) = \frac{1}{(1+x)^z}$ on the interval $[0, n]$, we get

$$\frac{1 + (1+n)^{-z}}{2} + \sum_{i=1}^{n-1} f(i) = \int_0^n f(x)dx + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!} \left(f^{(2j-1)}(n) - f^{(2j-1)}(0)\right) + R_k .$$

Letting n tend to $+\infty$, we will have

$$\frac{1}{2} + \underbrace{\int_0^{+\infty} (1+t)^{-z} dt}_{(z-1)^{-1}} - \sum_{l=1}^{p} \frac{b_{2l}}{2l!} f^{(2l-1)}(0) - \int_0^{\infty} \frac{B_{2p}(t)}{(2p)!} f^{(2p)}(t) dt ,$$

where

$$f^{(k)}(x) = \frac{-z(-z-1)\cdots(-z-k+1)}{(1+x)^{z+k}} = (-1)^k \frac{z(z+1)\cdots(z+k-1)}{(1+x)^{z+k}} .$$

So for $\mathrm{Re}(z) > 1$,

$$\zeta(z) = \underbrace{\frac{1}{2} + \frac{1}{z-1}}_{\text{meromorphic}} + \underbrace{\sum_{l=1}^{p} \frac{b_{2l}}{(2l)!} z \cdots (z + 2l - 2) + I_p}_{\text{holomorphic function}}$$

with

$$I_p(z) = -\int_0^{+\infty} \frac{z \cdots (z + 2p - 1)}{(1+t)^{z+2p}} B_{2p}(t) dt .$$

Now, the function $z \mapsto \frac{z\cdots(z+2p-1)}{(1+t)^{z+2p}} B_{2p}(t)$ is holomorphic and we have for all $\delta > 0$ and all z; $\mathrm{Re}(z) \geq 1 - 2p + \delta$,

$$\left|\frac{z\cdots(z+2p-1)}{(1+t)^{z+2p}} B_{2p}(t)\right| \leq \frac{|b_{2p}|z\cdots(z+2p-1)}{(1+t)^{1+\delta}} .$$

So, I_p is holomorphic on $\mathrm{Re}(z) > 1 - 2p$. □

Proposition 82. *The function ζ can be expressed in the integral form as follows.*

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^1 \frac{(-\ln u)^{z-1}}{1-u} \mathrm{d}u = \frac{1}{\Gamma(z)} \int_0^{+\infty} \frac{t^{z-1}}{\mathrm{e}^t - 1} \mathrm{d}t, \quad \mathrm{Re}(z) > 1 ,$$

where Γ is the Euler function.

Proof. We have

$$\zeta(z)\Gamma(z) = \sum_{n\geq 1} \frac{\Gamma(z)}{n^z} = \sum_{n\geq 1} \int_0^{+\infty} \mathrm{e}^{-u} \left(\frac{u}{n}\right)^{z-1} \frac{\mathrm{d}u}{n} .$$

By setting $u = nt$, we obtain

$$\int_0^{+\infty} \mathrm{e}^{-u} \left(\frac{u}{n}\right)^{z-1} \frac{\mathrm{d}u}{n} = \int_0^{+\infty} \mathrm{e}^{-nt} t^{z-1} \mathrm{d}t .$$

Hence, using the monotone convergence theorem, we obtain

$$\zeta(z)\Gamma(z) = \sum_{n\geq 1} \int_0^{+\infty} \mathrm{e}^{-nt} t^{z-1} \mathrm{d}t = \int_0^{+\infty} \mathrm{e}^{-t} \frac{1}{1-\mathrm{e}^{-t}} t^{z-1} \mathrm{d}t = \int_0^{+\infty} \frac{t^{z-1}}{\mathrm{e}^t - 1} \mathrm{d}t .$$ □

Proposition 83. *The function ζ satisfies the following quasi-induction rule:*

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x); \quad \forall x \in \mathbb{C} \setminus \{0, 1\} .$$

Proof. Let ε be such that $0 < \varepsilon < \pi$ and $n \in \mathbb{N}$. Consider the path C_ε^n represented in Figure 4.2 and the function

$$f_s(z) = \frac{(-z)^{s-1}}{e^z - 1}$$

with s being fixed. So, applying the residue theorem and letting $R \to +\infty$, $\varepsilon \to 0$, and next $n \to +\infty$, we get

$$2i\pi(2\pi)^{s-1}\zeta(1-s)2\sin\left(\frac{\pi x}{2}\right) = \Gamma(s)\zeta(s)2i\sin(s\pi) .$$

Next, using Proposition 63, we get

$$(2\pi)^{s-1}\zeta(1-s)2\sin\left(\frac{\pi x}{2}\right)=\zeta(s)\frac{1}{\Gamma(1-s)}\,.$$

Or equivalently

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi x}{2}\right)\zeta(1-s)\Gamma(1-s)\,.$$

□

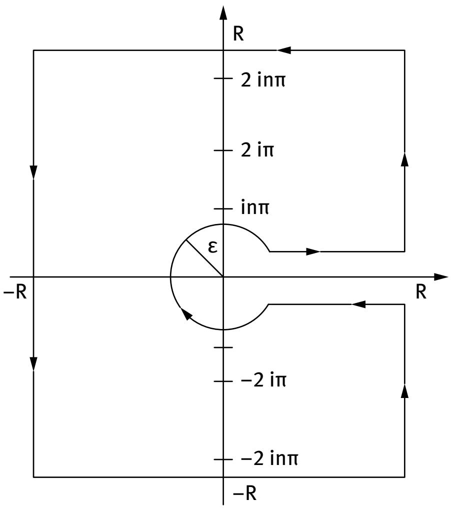

Fig. 4.2: The path C_ε^n.

Finally, Figure 4.3 graphically illustrates the ζ function.

4.2.5 Hypergeometric function

The origin of hypergeometric functions goes back to the early 19th century, when Gauss studied the second-order ordinary differential equation

$$x(1-x)y''+[c-(a+b+1)x]y'-aby=0 \tag{4.11}$$

with some constants a, b, and c in $\mathbb{R}$. Next, by developing a solution of (4.11) on a series of form $\sum_n \alpha_n x^x$, we obtain for c, $a-b$, and $c-a-b$ not integers, a general solution given by

$$y=F(a,b,c,x)+Bx^{1-c}F(a-c+1,b-c+1,2-c,x) \tag{4.12}$$

where F is the series

$$F(a,b,c,x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}\,,$$

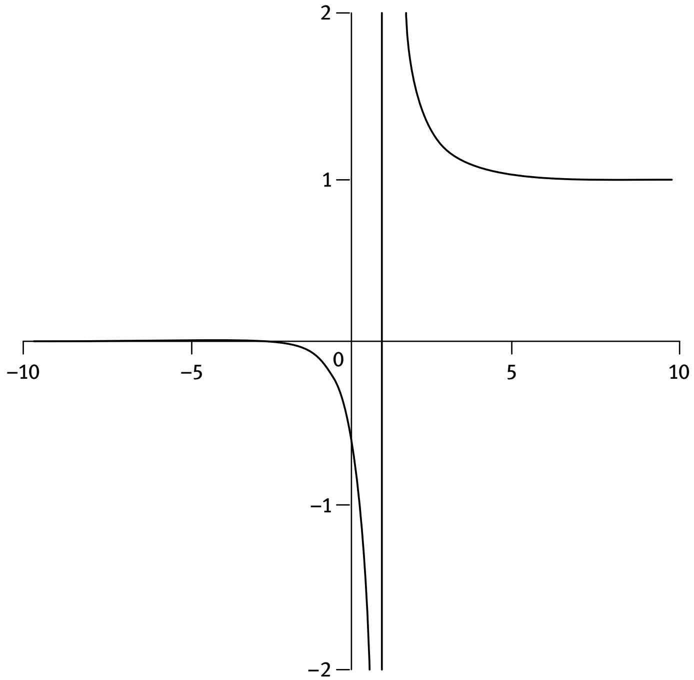

Fig. 4.3: General shape of the zeta function for (−10) to +10.

which is often denoted by ${}_2F_1(a, b, c, x)$, converges uniformly inside the unit disk and is known as the hypergeometric function.

When a, b, and c are integers, the hypergeometric function can be reduced to a transcendental function such as

$$ {}_2F_1(1, 1; 2; x) = -x^{-1} \ln(1 - x) \,. $$

Theorem 84. *F is differentiable with respect to x and*

$$ \frac{\partial F}{\partial x}(a, b, c, x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x) \,. $$

Proof. Write

$$ F(a, b, c, x) = \sum_{n=0}^{\infty} \alpha_n(a, b, c) x^n \,, $$

where

$$ \alpha_n = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)\Gamma(n + 1)} \,. $$

Inside its convergence domain, we have

$$\frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} (n+1)\alpha_{n+1} x^n \,.$$

Observe next that

$$\begin{aligned}(n+1)\alpha_{n+1}(a,b,c) &= \frac{a(a+1)\cdots(a+n)b(b+1)\cdots(b+n)}{n!c(c+1)\cdots(c+n)} \\ &= \frac{ab}{c}\alpha_n(a+1,b+1,c+1)\,.\end{aligned}$$

Hence,

$$\frac{\partial F}{\partial x} = \frac{ab}{c}\sum_{n=0}^{\infty} \alpha_n(a+1,b+1,c+1)x^n = \frac{ab}{c}F(a+1,b+1,c+1,x)\,. \qquad \square$$

Theorem 85. *For* $0 < \operatorname{Re} b < \operatorname{Re} c$, $\operatorname{Re} a < \operatorname{Re} c - \operatorname{Re} b$, *and* $|x| \le 1$, *it holds that*

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt\,.$$

Proof. Let, for $|x| < 1$,

$$I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt\,.$$

It is straightforward that I is a convergent integral. Next, we have

$$\begin{aligned}(1-tx)^{-a} &= \sum_{n=0}^{\infty} \frac{(-a)(-a-1)\cdots(-a-n+1)}{n!}(-tx)^n \\ &= \sum_{n=0}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)}{n!}(tx)^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} t^n x^n\,.\end{aligned}$$

Hence,

$$I = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} x^n \int_0^1 t^{b+n-1}(1-t)^{c-b-1} dt$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} x^n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$
$$= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)\Gamma(n+1)\Gamma(b)} x^n$$
$$= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(a)} F(a, b; c; x) .$$ □

Theorem 86.

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} .$$

Proof. Taking $x = 1$ in the integral expression of the hypergeometric function in Theorem 85, one obtains

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; 1) = \int_0^1 t^{b-1}(1-t)^{c-a-b-1} dt = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} .$$

Therefore,

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} .$$ □

Proposition 87. *We have*

(1) $F(n, 1, 1, x) = (1-x)^{-n}$.
(2) $xF(1, 1, 2, x) = -\log(1-x)$.
(3) $\lim_{\beta \leftrightarrow \infty} F(1, \beta, 1, \frac{x}{\beta}) = e^x$.
(4) $\lim_{\beta \to \infty} xF(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}) = \sin x$.
(5) $\lim_{\beta \to \infty} xF(\alpha, \beta, \frac{1}{2}, \frac{-x^2}{4\alpha\beta}) = \cos x$.
(6) $xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) = \arcsin x$.
(7) $xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = \arccos x$.

Proof. (1) Denote $y(x) = (1-x)^{-n}$. It is straightforward that y is a solution of (4.11) for $a = n$, and $b = c = 1$, and with $y(0) = 1$. So, (4.12) says that

$$y(x) = F(n, 1, 1, x) + Bx^{1-1}F(n-1+1, 1-1+1, 2-1, x) = CF(n, 1, 1, x)$$

with B and thus C being constants. Observing next that $F(n, 1, 1, 0) = 1$, we get $C = 1$ or equivalently $B = 0$.
(2) Again the function $y(x) = -x^{-1}\log(1-x)$ is a solution of (4.11) for $a = b = 1$, and $c = 2$. Hence, it is of the form

$$y(x) = F(1, 1, 2, x) + Bx^{1-2}F(0, 0, 0, x) ,$$

or equivalently,

$$-\log(1-x) = xF(1, 1, 2, x) + Be^x ,$$

which by setting $x = 0$ gives $B = 0$.
(3) Recall firstly that

$$F\left(1, \beta, 1, \frac{x}{\beta}\right) = \frac{\Gamma(1)}{\Gamma(1)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma(1+n)} \frac{x^n}{\beta^n n!} ,$$

which means that

$$F\left(1, \beta, 1, \frac{x}{\beta}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)}{\beta^n\Gamma(\beta)} \frac{x^n}{n!} .$$

So, let $K \in \mathbb{N}$ be fixed such that $2|x| \le K$ and denote

$$u_n(\beta) = \frac{\Gamma(\beta+n)}{\beta^n\Gamma(\beta)} \frac{x^n}{n!} .$$

It is straightforward that for $\beta \ge K$, we have

$$|u_n(\beta)| \le v_n = |u_n(K)| = \frac{\Gamma(K+n)}{K^n\Gamma(K)} \frac{|x|^n}{n!} .$$

Next, observe that

$$\lim_{n\to+\infty} \frac{v_{n+1}}{v_n} = \frac{|x|}{K} < 1 .$$

Hence, the D'Alembert rule affirms that the series $F(1, \beta, 1, \frac{x}{\beta})$ converges uniformly in β in the interval $[K, +\infty[$. Observing now that

$$\lim_{\beta\to+\infty} u_n(\beta) = 1 ,$$

we get

$$\lim_{\beta\to+\infty} F\left(1, \beta, 1, \frac{x}{\beta}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x .$$

(4) Recall that

$$F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\frac{3}{2}+n)} \frac{(-1)^n x^{2n}}{4^n\alpha^n\beta^n n!} .$$

Denote

$$\sigma_n(\alpha) = \frac{\Gamma(\alpha+n)}{\alpha^n\Gamma(\alpha)} \frac{x^n}{n!} .$$

We get

$$F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \sum_{n=0}^{\infty} \sigma_n(\alpha)\sigma_n(\beta) \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+n)} \frac{(-1)^n x^{2n}}{4^n n!} .$$

Using similar arguments as for (3) above and the properties of Euler's Γ function, we get

$$\lim_{\alpha,\beta\to+\infty} F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{\sin x}{x} .$$

(5) Follows by quite the same techniques as the previous assertion.
(6) Observe that

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) = \frac{1}{2\Gamma(\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)}{\frac{1}{2}+n} \frac{x^{2n}}{n!} .$$

Next, using the well-known relation $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$, we obtain

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{n+1}(2n+1)(n!)^2} x^{2n} = \frac{\arcsin x}{x} .$$

(7) Follows by the same arguments as assertion (6). □

Definition 88. The hypergeometric function may be generalized for $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$, $p, q \in \mathbb{N}$ by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q, x) = \sum_{n=0}^{\infty} \alpha_n x^n ,$$

where

$$\alpha_0 = 1 \quad \text{and} \quad \frac{\alpha_{n+1}}{\alpha_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)}{(n+b_1)(n+b_2)\cdots(n+b_q)} \frac{1}{n+1} ,$$

or differently by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_k (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!} ,$$

where $(a)_n$ is the *increasing factorial* or the *Pochhammer symbol* given by

$$(a)_n = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1) .$$

4.2.6 Legendre function

Legendre functions are fundamental solutions of the Laplace equation on the sphere. There are two classes of solutions that are related to the parameters λ and μ, which will be explained later. In the following, we denote the first kind by P_λ and the second kind by Q_λ. The associated Legendre functions corresponding to P_λ and Q_λ are denoted by P_λ^μ and Q_λ^μ, respectively. These are respective generalizations of Legendre polynomials $P_\ell(x)$ and associated Legendre polynomials $P_\ell^m(x)$, to noninteger values of ℓ and m.

Definition 89. The Legendre functions are solutions of the general Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right] y = 0\,,$$

where λ and μ are generally complex numbers called, respectively, the degree and the order of the associated Legendre function.

The case of Legendre functions corresponding to $\mu = 0$ and $\lambda \in \mathbb{N}$ reduces to *orthogonal Legendre polynomials.*

Proposition 90.
(1) *For $\mu = 0$, the following integral form is a Legendre function:*

$$F_\lambda(z) = \frac{1}{2\pi i} \int_C \frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+1}} dt\,,$$

for $|z-1| < 2$ where $\mathcal{C}$ is a circle surrounding the points 1 and z and not -1.
(2) *For $\lambda \in \mathbb{C}$ and $|x| > 1$, $x \in \mathbb{R}$, we get*

$$F_\lambda(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x + \sqrt{x^2-1}\cos\theta\right)^\lambda d\theta$$
$$= \frac{1}{\pi} \int_0^1 \left(x + \sqrt{x^2-1}(2t-1)\right)^\lambda \frac{dt}{\sqrt{t(1-t)}}\,.$$

Proof. (1) Applying the derivatives of F_λ, we get

$$(1-z^2)F_\lambda''(z) - 2zF_\lambda'(z) + \lambda(\lambda+1)F_\lambda(z)$$
$$= \frac{\lambda+1}{2\pi i} \int_C \frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+3}} (\lambda t^2 - 2(\lambda+1)zt + \lambda + 2) dt$$
$$= \frac{\lambda+1}{2\pi i} \int_C \frac{d}{dt} \frac{(t^2-1)^{\lambda+1}}{2^\lambda (t-z)^{\lambda+2}} dt = 0\,.$$

So, F_λ satisfies the Legendre equation.
(2) Consider for the integral form the circle $\mathcal{C}$ centered at x with radius $r = \sqrt{x^2-1}$. We first obtain

$$t^2 - 1 = \sqrt{x^2-1}e^{i\theta}2(x + \sqrt{x^2-1}\cos\theta), \;\; \theta \in [-\pi, \pi]\,.$$

Hence,

$$\frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+1}}dt = \frac{\sqrt{x^2-1}^{\,\lambda} e^{i\lambda\theta} 2^\lambda (x + \sqrt{x^2-1}\cos\theta)^\lambda}{2^\lambda \sqrt{x^2-1}^{\,\lambda+1} e^{i(\lambda+1)\theta}} i\sqrt{x^2-1}e^{i\theta} d\theta\,.$$

As a result,

$$F_\lambda(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(x + \sqrt{x^2-1}\cos\theta\right)^\lambda d\theta\,.$$

Next, setting $t = \frac{1+\cos\theta}{2}$, we obtain

$$F_\lambda(x) = \frac{1}{\pi}\int_0^1 \left(x + \sqrt{x^2-1}(2t-1)\right)^\lambda \frac{dt}{\sqrt{t(1-t)}}\,.$$

□

Proposition 91. *The following are Legendre functions:*
- *The first-kind function P_λ^μ defined for $|1-z| < 2$ by*

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right),$$

  where Γ is Euler's Gamma function.
- *The second-kind function $Q_\lambda^\mu(z)$ defined for $|z| > 1$ by*

$$Q_\lambda^\mu(z) = C_\mu^\lambda \frac{(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda + \frac{3}{2}; \frac{1}{z^2}\right),$$

where $C_\mu^\lambda = \frac{\sqrt{\pi}\Gamma(\lambda+\mu+1)}{2^{\lambda+1}\Gamma(\lambda+3/2)}e^{i\mu\pi}$, and ${}_2F_1$ is the hypergeometric function.

Proof. It suffices to show that the functions

$$F(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right), \quad |1-z| < 2$$

and for $|z| > 1$,

$$F(z) = C_\mu^\lambda \frac{(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda + \frac{3}{2}; \frac{1}{z^2}\right)$$

are solutions of the general Legendre Definition 89. We will develop the first part. The second follows by similar techniques. So, for simplicity denote

$$H(z) = \frac{1}{\Gamma(1-\mu)}\,{}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right),$$

$$A(z) = \frac{1}{1-z^2}, \quad B(z) = \left[\frac{1+z}{1-z}\right]^{\mu/2}$$

and $Z = \frac{1-z}{2}$. Standard calculus yields that

$$(1-z^2)H''(z) + 2(\mu_z)H'(z) + \lambda(\lambda+1)H(z) = 0\,, \tag{4.13}$$

$$F'(z) = \mu A(z)B(z)H(z) + B(z)H'(z)\,,$$

and

$$F''(z) = \mu(\mu+2z)A^2(z)B(z)H(z) + 2\mu A(z)B(z)H'(z) + B(z)H''(z)\,. \qquad \square$$

Now, recall the Legendre equation

$$(1-z^2)y'' - 2zy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-z^2}\right]y = 0\,.$$

Replacing y by F and taking into account equation (4.13), we show that F is a Legendre function.

Proposition 92. *The Legendre function F_λ satisfies the following three-level induction rule:*

$$(\lambda+1)F_{\lambda+1}(z) - (2\lambda+1)zF_\lambda(z) + \lambda F_{\lambda-1}(z) = 0\,.$$

Proof. Let C be the contour as above. We have

$$F_\lambda(z) = \frac{1}{2^{\lambda+1}\pi i}\int_C \frac{(t^2-1)^\lambda}{(t-z)^{\lambda+1}}\,dt\,.$$

Classical arguments show that F_λ is holomorphic and

$$F'_\lambda(z) = \frac{(\lambda+1)}{2^{\lambda+1}\pi i}\int_C \frac{(t^2-1)^\lambda}{(t-z)^{\lambda+2}}\,dt\,.$$

On the other hand,

$$\frac{d}{dt}\frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+1}} = \frac{2(\lambda+1)t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} - \frac{(\lambda+1)(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}\,.$$

Hence,

$$0 = \int_C \left(\frac{2t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} - \frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}\right)dt\,.$$

Consequently,

$$\begin{aligned}\frac{1}{2^{\lambda+1}\pi i}\int_C \frac{(t^2-1)^\lambda}{(t-z)^\lambda} &= \frac{1}{2^{\lambda+1}\pi i}\int_C \frac{t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &\quad - \frac{1}{2^{\lambda+1}\pi i}\int_C \frac{z(t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &= F_{\lambda+1} - zF_\lambda(z)\,. \end{aligned} \tag{4.14}$$

Differentiating with respect to z, we obtain

$$F'_{\lambda+1}(z) - zF'_\lambda(z) = (\lambda+1)F_\lambda(z)\,.$$

Thus,

$$\begin{aligned} 0 &= \int_C \frac{d}{dt}\left[\frac{t(t^2-1)^\lambda}{(t-z)^\lambda}\right] dt \\ &= \int_C \left[\frac{(t^2-1)^\lambda}{(t-z)^\lambda} + \frac{2\lambda t^2(t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \frac{\lambda t(t^2-1)^\lambda}{(t-z)^{\lambda+1}}\right] dt \\ &= \int_C \frac{(t^2-1)^\lambda + 2\lambda[(t^2-1)+1](t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \frac{\lambda[(t-z)+z](t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &= \int_C \left[(\lambda+1)\frac{(t^2-1)^\lambda}{(t-z)^\lambda} + 2\lambda\frac{(t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \lambda z\frac{(t^2-1)^\lambda}{(t-z)^{\lambda+1}}\right] dt\,. \end{aligned}$$

Finally using (4.14), we deduce that

$$\begin{aligned} 0 &= (\lambda+1)[F_{\lambda+1}(z) - zF_\lambda(z)] + 2\lambda F_{\lambda-1}(z) - \lambda z F_\lambda(z) \\ &= (\lambda+1)F_{\lambda+1}(z) - (2\lambda+1)zF_\lambda(z) + 2\lambda F_{\lambda-1}(z)\,. \end{aligned}$$

□

4.2.7 Bessel function

Bessel functions form an important class of special functions and are applied almost everywhere in mathematical physics. They are also known as cylindrical functions, or cylindrical harmonics, because they are part of the solutions of the Laplace equation in cylindrical coordinates met in heat propagation along a cylinder. In pure mathematics, Bessel functions can be introduced in three ways: as solutions of second-order differential equations, through a recurrent procedure as solutions of a three-level recurrent functional equation, and via the Rodrigues derivation formula.

Definition 93. The Bessel equation is a linear differential equation of second order written in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{v^2}{x^2}\right)y = 0\,,$$

where v is a positive constant.

Remark 94.

(1) Any solution of Bessel's equation is called the *Bessel function*.

(2) Given two linearly independent solutions y_1 and y_2 of Bessel's differential equation, the general solution is expressed as a linear combination

$$y = C_1 y_1 + C_2 y_2\,,$$

where C_1 and C_2 are two constants.

Theorem and Definition 95. *Bessel's differential equation has a solution of the form*

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k}\,. \tag{4.15}$$

The function J_v is called the Bessel function of the first kind of the order v.

Proof of the Theorem. We will search a nontrivial solution of the form

$$y = x^p \sum_{i\geq 0} a_i x^i = \sum_{i\geq 0} a_i x^{i+p}\,,$$

where p is a real parameter. By replacing y and its derivatives in Definition 93, we get

$$\sum_{i\geq 0} a_i(i+p)(i+p-1)x^{i+p} + \sum_{i\geq 0} a_i(i+p)x^{i+p} + (x^2 - v^2)\sum_{i\geq 0} a_i x^{i+p} = 0\,.$$

Or equivalently,

$$\sum_{i\geq 0}\left[(i+p)(i+p-1) + (i+p) - v^2\right] a_i x^{i+p} + \sum_{i\geq 0} a_i x^{i+p-2} = 0\,,$$

which means that

$$\sum_{i\geq 0}\left[(i+p)^2 - v^2\right] a_i x^{i+p} + \sum_{j\geq 2} a_{j-2} x^{j+p} = 0\,.$$

Therefore,

$$a_0(p^2 - v^2) = a_1((p+1)^2 - v^2) = 0$$

and

$$a_i((i+\nu)^2-\nu^2)+a_{i-2}=0, \quad \forall i \geq 2\,.$$

For $p=\nu$, we get $a_1=0$ and

$$i(i+2\nu)a_i=-a_{i-2}, \quad \forall i \geq 2\,.$$

Thus,

$$a_i=-\frac{a_{i-2}}{i(2\nu+i)}, \quad \forall i \geq 2\,.$$

Hence, the coefficients a_{2k+1}, and

$$a_{2k}=(-1)^k\frac{a_0}{2^{2k}k!(\nu+k)(\nu+k-1)\cdots(\nu+1)}, \quad \forall k \geq 0\,.$$

Taking $a_0=\frac{1}{2^\nu\Gamma(\nu+1)}$, and observing that

$$\Gamma(\nu+k+1)=(\nu+k)(\nu+k-1)\cdots(\nu+1)\Gamma(\nu+1)\,,$$

we get

$$a_{2k}=(-1)^k\frac{1}{2^{2k+\nu}k!\Gamma(\nu+k+1)}, \quad k \geq 0.$$

As a result, the solution of the equation will be

$$y=\left(\frac{x}{2}\right)^\nu\sum_{k\geq 0}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\,.$$

□

Remark 96.

(1) For $p=-\nu$, the solution of Bessel's equation in Definition 93 is called Bessel's function of the first kind with the order $-\nu$ and is denoted by $J_{-\nu}(x)$ with

$$J_{-\nu}(x)=\left(\frac{x}{2}\right)^{-\nu}\sum_{k\geq 0}\frac{(-1)^k}{k!\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{2k}\,.$$

(2) For ν, noninteger J_ν and $J_{-\nu}$ are linearly independent and therefore the general solution of the Bessel equation is of the form

$$y(x)=C_1J_\nu(x)+C_2J_{-\nu}(x)\,.$$

(3) The same solution can be obtained by choosing $p+1=\nu$ in the proof of Theorem 95.

Proposition 97. *For $\nu=n\in\mathbb{N}$, we have*

$$J_n=(-1)^nJ_{-n}\,.$$

Proof. We have

$$\begin{aligned} J_{-n}(x) &= \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{2k-n} \\ &= \sum_{m\geq 0} \frac{(-1)^{m+n}}{(m+n)!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n} \\ &= (-1)^n \sum_{m\geq 0} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \\ &= (-1)^n J_n(x) . \end{aligned}$$

□

Example 4.1. For $v = 0$,

$$J_0(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k\geq 0} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k},$$

which is an even function. Else, $J_0(0) = 1$. For $v = 1$, we obtain

$$J_1(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+2)} \left(\frac{x}{2}\right)^{2k+1} = \sum_{k\geq 0} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1},$$

which is an odd function and satisfies $J_1(0) = 0$.

Definition 98. The Bessel function of the second kind of the order α denoted usually by Y_α and is given by

$$Y_\alpha(x) = \begin{cases} \frac{\cos(\pi\alpha)J_\alpha(x)-J_{-\alpha}(x)}{\sin(\pi\alpha)}, & \text{for } \alpha \notin \mathbb{Z} \\ \lim_{v\to\alpha} \frac{\cos(\pi v)J_v(x)-J_{-v}(x)}{\sin(\pi v)}, & \text{for } \alpha \in \mathbb{Z} . \end{cases}$$

Proposition 99. *For $\alpha \in \mathbb{Z}$, Y_α is a solution of Bessel's differential equation, singular at* 0 *and satisfying precisely* $\lim_{x\to 0} Y_0(x) = \infty$.

Proof. For $\alpha \notin \mathbb{Z}$, Y_α is a linear combination of J_α and $J_{-\alpha}$. Hence it is a solution of the Bessel's differential equation. We now prove this for $\alpha \in \mathbb{Z}$. It holds for all $v \notin \mathbb{Z}$ and all x that

$$x^2 Y_v''(x) + xY_v'(x) + (x^2 - v^2)Y_v(x) = 0 .$$

Letting $\nu \to \alpha \in \mathbb{Z}$, we obtain

$$x^2 Y_\alpha''(x) + xY_\alpha'(x) + (x^2 - \alpha^2)Y_\alpha(x) = 0\,.$$

Next, we show that $\lim_{x\longrightarrow 0} Y_\alpha(x) = +\infty$. Indeed, for $\alpha \notin \mathbb{Z}$, Y_α is a linear combination of J_α and $J_{-\alpha}$. So it is a solution of Bessel's differential equation. Next, substituting Y_ν for $\nu \notin \mathbb{Z}$ in the differential equation and letting ν tend to α we get a solution Y_α for $\alpha \in \mathbb{Z}$. Y_α is singular at 0 because of the powers $(\frac{x}{2})^\alpha$ and $(\frac{x}{2})^{-\alpha}$. We now prove the remaining part. Recall that

$$Y_\alpha(x) = \lim_{\nu\to\alpha} \frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)}\,.$$

We have for $\nu = \alpha$, $\sin(\pi\nu) = 0$, $\cos(\pi\nu) = (-1)^\alpha$, $(-1)^\alpha J_\alpha(x) = J_{-\alpha}(x)$. By applying L'Hôpital's rule, we obtain

$$\begin{aligned} Y_\alpha(x) &= \lim_{\nu\longrightarrow\alpha} \frac{\frac{\partial}{\partial\nu}\,[\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)]}{\frac{\partial}{\partial\nu}\sin(\pi\nu)} \\ &= \frac{2}{\pi}J_\alpha(x)\left[\ln\frac{x}{2} + C\right] - \frac{1}{\pi}\sum_{k=0}^{\alpha-1}\frac{\Gamma(\alpha-k)}{k!}\left(\frac{x}{2}\right)^{2k-\alpha} \\ &\quad - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k-\alpha}}{k!\Gamma(\alpha+k+1)}\left[\sum_{m=1}^{\alpha+k}\frac{1}{m} + \sum_{m=1}^{k}\frac{1}{m}\right], \end{aligned}$$

where C is Euler's constant. For $\alpha = 0$, we obtain

$$Y_0(x) = \frac{2}{\pi}J_0(x)\left[\ln\frac{x}{2} + C\right] - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\sum_{m=1}^{k}\left(\frac{1}{m}\right)\left(\frac{x}{2}\right)^{2k-\alpha}\,.$$

Thus $\lim_{x\longrightarrow 0} Y_0(x) = +\infty$. □

Definition 100. The Bessel generating function of the first kind is given by

$$u(x,t) = \sum_{n=-\infty}^{+\infty} J_n(x)t^n\,.$$

Lemma 101. *For all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$, we have*

$$u(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\,.$$

Proof. We have

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{k\geq 0} \frac{\left(\frac{xt}{2}\right)^k}{k!} \sum_{m\geq 0} \frac{\left(-\frac{x}{2t}\right)^m}{m!}$$
$$= \sum_{k\geq 0} \sum_{m\geq 0} (-1)^m \frac{t^{k-m}}{m!k!} \left(\frac{x}{2}\right)^{k+m} .$$

Setting $k = m + n$, we get

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{m\geq 0} \sum_{n+m\geq 0} (-1)^m \frac{t^n}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n}$$
$$= \sum_{m\geq 0} \sum_{n\geq -m} (-1)^m \frac{t^n}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}$$
$$= \sum_{m\geq 0} \sum_{n=-\infty}^{+\infty} (-1)^m \frac{\left(\frac{x}{2}\right)^{2m+n}}{m!\Gamma(n+m+1)} t^n$$
$$= \sum_{n=-\infty}^{+\infty} \sum_{m\geq 0} (-1)^m \frac{\left(\frac{x}{2}\right)^{2m+n}}{m!\Gamma(n+m+1)} t^n$$
$$= \sum_{n=-\infty}^{+\infty} J_n(x) t^n$$
$$= u(x,t) .$$

□

Theorem 102. *The Bessel function J_n satisfies*

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \quad \forall n \in \mathbb{N} .$$

Proof. Differentiating the generating function u with respect to the variable t we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\sum_{n=-\infty}^{+\infty} J_n(x)t^n\right) = \sum_{n=-\infty}^{+\infty} nJ_n(x)t^{n-1} = \sum_{n=-\infty}^{+\infty} (n+1)J_{n+1}(x)t^n .$$

On the other hand, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right) = \frac{x}{2}e^{\frac{x}{2}t} - \frac{x}{2t^2}e^{\frac{x}{2t}}\\ &= \frac{x}{2}e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\left(1-\frac{1}{t^2}\right)\\ &= \frac{x}{2}u(x,t)\left(1-\frac{1}{t^2}\right)\\ &= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n\left(1-\frac{1}{t^2}\right)\\ &= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n - \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^{n-2}\\ &= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n - \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_{n+2}(x)t^n\,.\end{aligned}$$

By identification, we obtain

$$nJ_n(x) = \frac{x}{2}J_{n-1}(x) + \frac{x}{2}J_{n+1}(x), \quad \forall n \geq 0\,.$$

Therefore

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x), \quad \forall n \geq 0\,.$$

□

Theorem 103. *The Bessel function J_n is differentiable and its derivative satisfies*

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\,.$$

Proof. Differentiating the generating function u with respect to x, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \frac{1}{2}\left[\sum_{n=-\infty}^{+\infty} J_n(x)t^{n+1} - \sum_{n=-\infty}^{+\infty} J_n(x)t^{n-1}\right]\,.$$

On the other hand,

$$\frac{\partial u}{\partial x} = \sum_{n=-\infty}^{+\infty} J_n'(x)t^n, \quad \forall n \geq 1\,.$$

Consequently

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\,.$$

□

Remark 104. In the particular case $n = 0$, we obtain

$$J_0'(x) = -J_1(x)\,.$$

Theorem 105. *The first-kind Bessel function can be expressed by the integral form*

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\varphi - n\varphi)d\varphi\ . \tag{4.16}$$

In particular, for $n = 0$*, we have*

$$J_0(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x\sin\varphi)d\varphi\ .$$

Proof. Recall that

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{k=-\infty}^{+\infty} J_k(x)t^k\ .$$

Setting $t = e^{i\varphi}$, we get

$$e^{ix\sin\varphi} = \sum_{k=-\infty}^{+\infty} J_k(x)e^{ik\varphi}\ , \tag{4.17}$$

which is the Fourier series of the 2π-periodic function $f(\varphi) = e^{ix\sin\varphi}$. Therefore,

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\varphi}e^{-in\varphi}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\sin\varphi - n\varphi\right)d\varphi\ .$$

In particular, for $n = 0$, we have

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\sin\varphi\right)d\varphi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\left(x\sin\varphi\right)d\varphi\ .$$

□

Proposition 106. *Let* λ *and* μ *be two different roots of the Bessel function* $J_\nu(x)$*. The Bessel functions* $J_\nu(x)$ *satisfy the following orthogonality property:*

$$\int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)dx = 0\ .$$

Proof. Denote

$$y_{\nu,\lambda}(x) = J_\nu(\lambda x) \qquad \text{and} \qquad y_{\nu,\mu}(x) = J_\nu(\mu x)\ .$$

Then, $y_{\nu,\lambda}$ and $y_{\nu,\mu}$ are solutions of the following Bessel-type differential equations:

$$(xy'_{\nu,\lambda})'(x) + \left(\lambda^2 x - \frac{\nu^2}{x}\right) y_{\nu,\lambda}(x) = 0 \tag{4.18}$$

$$(xy'_{\nu,\mu})'(x) + \left(\mu^2 x - \frac{\nu^2}{x}\right) y_{\nu,\mu}(x) = 0\,. \tag{4.19}$$

Multiplying the first one by $y_{\nu,\mu}$ and the second by $y_{\nu,\lambda}$ and integrating on $(0, 1)$, we get

$$(\lambda^2 - \mu^2) \int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)dx = 0\,.$$

Therefore, since $\lambda \neq \mu$, we get

$$\int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)dx = 0\,. \qquad \square$$

Figures 4.4 and 4.5 illustrate the graphs of the first and second kind Bessel functions.

4.2.8 Hankel function

Hankel functions are applied as physical solutions for incoming or outgoing waves in cylindrical geometry. These are linearly independent solutions of the complex-parameter Bessel equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0\,, \tag{4.20}$$

where α is an arbitrary complex number.

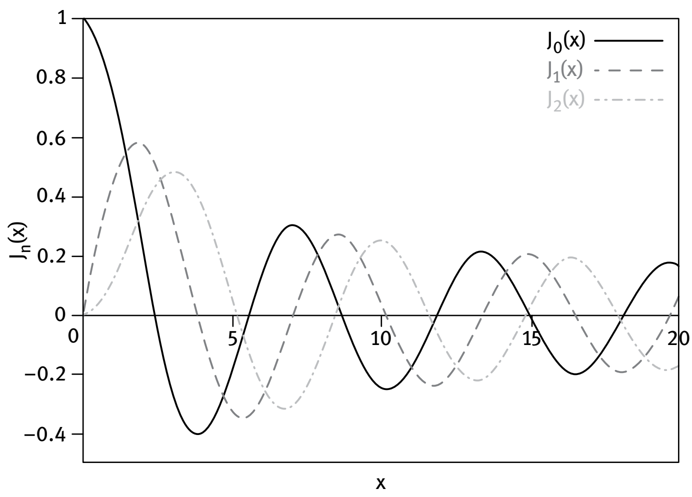

Fig. 4.4: Graphs of the first three first-kind Bessel functions.

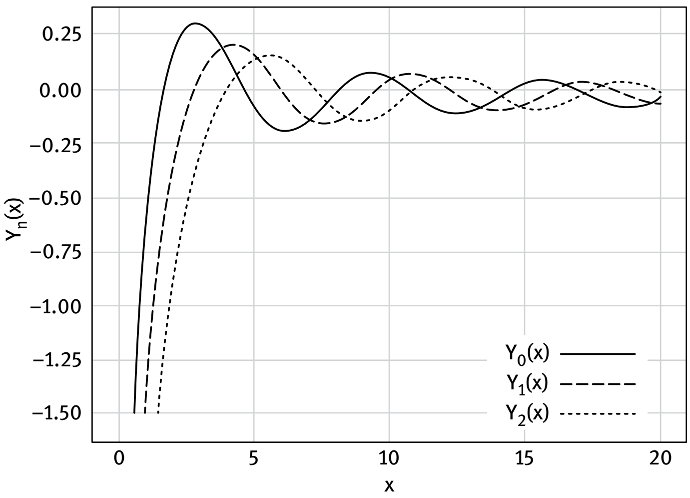

Fig. 4.5: Graphs of the first three second-kind Bessel functions.

Definition 107. Hankel functions of the first and second kind are defined, respectively, by

$$H^1_\alpha(x) = J_\alpha(x) + iY_\alpha(x) \quad \text{and} \quad H^2_\alpha(x) = J_\alpha(x) - iY_\alpha(x)\,,$$

where J_α and Y_α are the Bessel functions of the first and second kind, respectively.

Proposition 108. *The following assertions are true.*

(1) $H^1_\alpha(x) = \frac{J_{-\alpha}(x) - e^{-i\alpha\pi}J_\alpha(x)}{i\sin(\alpha\pi)}$.

(2) $H^2_\alpha(x) = \frac{J_{-\alpha}(x) - e^{i\alpha\pi}J_\alpha(x)}{-i\sin(\alpha\pi)}$.

(3) $H^1_{-\alpha}(x) = e^{i\alpha x}H^1_\alpha(x)$.

(4) $H^2_{-\alpha}(x) = e^{-i\alpha x}H^2_\alpha(x)$.

Proof. (1) Recall that the second-kind Bessel function is

$$Y_\alpha(x) = \frac{\cos(\pi v)J_v(x) - J_{-v}(x)}{\sin(\pi v)}\,.$$

Therefore,

$$\begin{aligned}
H^1_\alpha(x) &= J_\alpha(x) + i\frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)} \\
&= \frac{J_\alpha(x)[\sin(\pi\nu) + i\cos(\pi\nu)] - iJ_{-\nu}(x)}{\sin(\pi\nu)} \\
&= \frac{J_\alpha(x)[i\sin(\pi\nu) - \cos(\pi\nu)] + J_{-\nu}(x)}{i\sin(\pi\nu)} \\
&= \frac{J_{-\nu}(x) - J_\alpha(x)[\cos(\pi\nu) - i\sin(\pi\nu)]}{i\sin(\pi\nu)} \\
&= \frac{J_{-\alpha}(x) - e^{-i\alpha\pi}J_\alpha(x)}{i\sin(\alpha\pi)} .
\end{aligned}$$

(2) Similarly to (1), we have

$$\begin{aligned}
H^2_\alpha(x) &= J_\alpha(x) - i\frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)} \\
&= \frac{J_\alpha(x)[\sin(\pi\nu) - i\cos(\pi\nu)] + iJ_{-\nu}(x)}{\sin(\pi\nu)} \\
&= \frac{J_\alpha(x)[-i\sin(\pi\nu) - \cos(\pi\nu)] + J_{-\nu}(x)}{i\sin(\pi\nu)} \\
&= \frac{J_{-\nu}(x) - J_\alpha(x)[\cos(\pi\nu) + i\sin(\pi\nu)]}{-i\sin(\pi\nu)} \\
&= \frac{J_{-\alpha}(x) - e^{i\alpha\pi}J_\alpha(x)}{-i\sin(\alpha\pi)} .
\end{aligned}$$

(3) It follows from (1) that

$$\begin{aligned}
H^1_{-\alpha}(x) &= \frac{J_\alpha(x) - e^{i\alpha\pi}J_{-\alpha}(x)}{-i\sin(\alpha\pi)} \\
&= -e^{i\alpha\pi}\frac{e^{-i\alpha\pi}J_\alpha(x) - J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{i\alpha\pi}\frac{J_{-\alpha}(x) - e^{-i\alpha\pi}J_\alpha(x)}{i\sin(\alpha\pi)} \\
&= e^{i\alpha\pi}H^1_\alpha(x) .
\end{aligned}$$

(4) Similarly to (3), we have

$$\begin{aligned}
H^2_{-\alpha}(x) &= \frac{J_\alpha(x) - e^{-i\alpha\pi}J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}\frac{e^{i\alpha\pi}J_\alpha(x) - J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}\frac{J_{-\alpha}(x) - e^{i\alpha\pi}J_\alpha(x)}{-i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}H^2_\alpha(x) .
\end{aligned}$$

□

Proposition 109. *The first-kind Hankel function H_n^1, $n \in \mathbb{Z}$ can be expressed in the integral form as*

$$H_n^1(x) = \frac{1}{i\pi}\int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}}dt\,. \tag{4.21}$$

Proof. It follows from Proposition 97 and Definition 93 that

$$H_n^1(z) = \frac{1}{i\pi}J_n'(t)\,.$$

On the other hand, from Theorem 105, equation (4.16), we have that

$$J_n'(x) = -\frac{1}{2\pi}\int_0^{2\pi} \sin(x\sin\varphi - n\varphi)\sin\varphi d\varphi\,.$$

Now, standard computations as in Theorem 105 yield that

$$\int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}}dt = J_n'(x)\,.$$

Hence,

$$H_n^1(z) = \frac{1}{i\pi}\int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}}dt\,. \qquad \square$$

Theorem 110. *Hankel functions H_α^i are differentiable and we have*

$$\frac{d}{dz}H_\alpha^i(z) = \frac{1}{2}\left(H_{\alpha-1}^i(z) - H_{\alpha+1}^i(z)\right), \quad i = 1, 2\,,$$

and

$$\frac{2\alpha}{z}H_\alpha^i(z) = H_{\alpha-1}^i(z) + H_{\alpha+1}^i(z), \quad i = 1, 2\,.$$

Proof. We have

$$H_n^1(z) = \frac{1}{i\pi}J_n'(t) = \frac{1}{2}\left[\frac{1}{i\pi}J_{n-1}(x) - \frac{1}{i\pi}J_{n+1}(x)\right]\,.$$

Hence,

$$\frac{d}{dz}H_n^1(z) = \frac{1}{2}\left[\frac{1}{i\pi}J_{n-1}'(z) - \frac{1}{i\pi}J_{n+1}'(z)\right] = \frac{1}{2}\left[H_{n-1}(z) - H_{n+1}(z)\right]\,.$$

We now prove the next part. To do this, we recall the explicit form of Bessel function J_v from (4.15), which states that

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} .$$

Now, for $i = 1$, we get

$$H^1_{\alpha-1}(z) + H^1_{\alpha+1}(z) = \frac{(J_{-\alpha+1} - J_{-\alpha-1}) + e^{-i\alpha\pi}(J_{\alpha-1} + J_{\alpha+1})}{-i \sin \alpha\pi} .$$

Next, it suffices to evaluate the quantities in the numerator. We evaluate one quantity and leave The others for readers. Using the above expression, we get

$$\begin{aligned} J_{\alpha-1} + J_{\alpha+1} &= \left(\frac{x}{2}\right)^\alpha \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\alpha+k)} \left(\frac{x}{2}\right)^{2k-1} \\ &\quad + \left(\frac{x}{2}\right)^\alpha \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\alpha+k+2)} \left(\frac{x}{2}\right)^{2k-1} \\ &= \frac{\alpha+1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha-1} - \frac{2\alpha}{x} J_\alpha(x) . \end{aligned}$$

Using the same techniques and next substituting into the equality above, we get the desired result. □

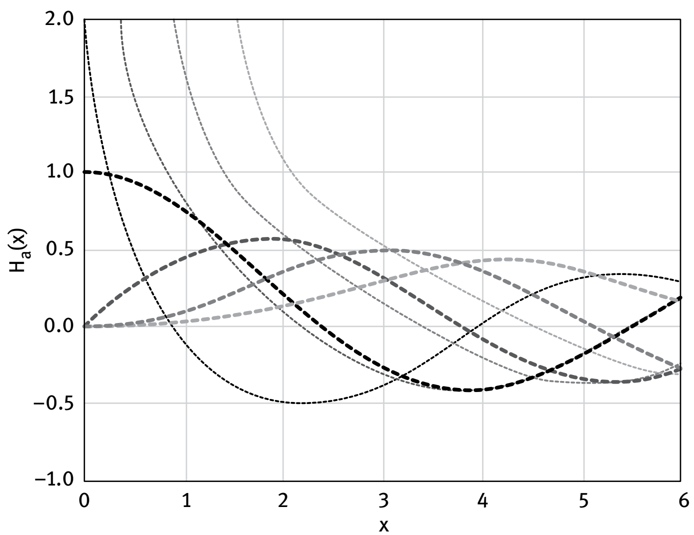

Fig. 4.6: Hankel function.

4.2.9 Mathieu function

Mathieu functions were originally introduced as solutions of the Mathieu differential equation

$$\frac{d^2x}{dt^2} + \omega^2(t)x = 0; \quad \text{or} \quad \omega^2(t) = \omega_0^2[1 - \xi_0 \cos(t)] . \tag{4.22}$$

It is a special case of the general Hill equation given by

$$\frac{d^2y}{dt^2} + f^2(t)y = 0 ,$$

where f is a periodic function.

The Mathieu differential equation has in fact many variants. One variant may be obtained by a scaling modification by setting $y(t) = x(2t)$, which therefore satisfies the equation

$$\frac{d^2y}{dt^2} + [a - 2q\cos(2t)]\, y = 0 , \tag{4.23}$$

where a and q are constant coefficients. By setting $u = it$ in (4.23), we get the Mathieu modified differential equation

$$\frac{d^2y}{du^2} - [a - 2q\cosh(2u)]\, y = 0 . \tag{4.24}$$

By setting $x = \cos(t)$, we obtain a second Mathieu modified differential equation

$$(1-t)^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} + (a + 2q(1 - 2t^2))y = 0 .$$

As in the theory of the Schrödinger equation, we can guess stationary solutions of the form

$$F(a, q, x) = e^{i\mu x}P(a, q, x) , \tag{4.25}$$

where μ is a complex number called the Mathieu exponent and P is a periodic complex valued function. The following graph is illustrated with $a = 1$, $q = \frac{1}{5}$, and $\mu = 1 + 0.0995i$.

Definition 111. For fixed a, q we define

– The Mathieu cosine $C(a, q, x)$ by

$$C(a, q, x) = \frac{F(a, q, x) + F(a, q, -x)}{2F(a, q, 0)} .$$

– The Mathieu sine $S(a, q, x)$ by

$$S(a, q, x) = \frac{F(a, q, x) - F(a, q, -x)}{2F'(a, q, 0)} .$$

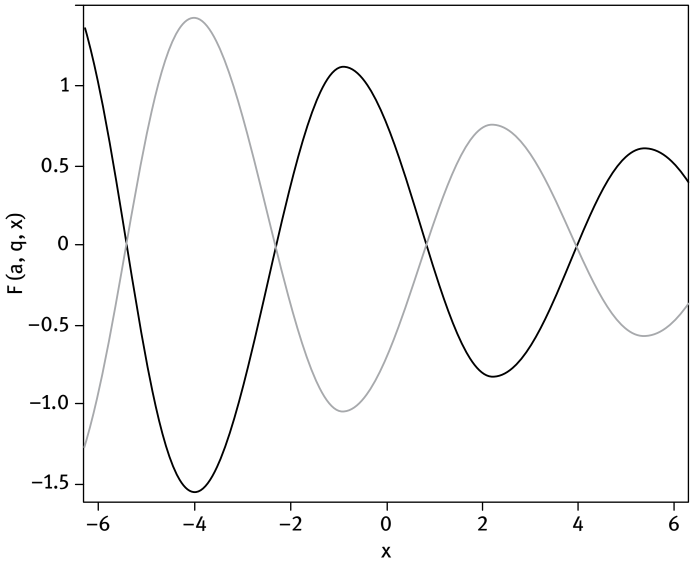

Fig. 4.7: Mathieu function: Real part and imaginary part, $a = \mu = 1$ and $q = 0.2$.

Properties 112. The following assertions hold:
(1) $C(a, q, 0) = 1$ and $S(a, q, 0) = 0$.
(2) $C'(a, q, 0) = 0$ and $S'(a, q, 0) = 1$.
(3) $C(a, q, -x) = C(a, q, x)$: The Mathieu cosine is an even function.
(4) $S(a, q, -x) = -S(a, q, x)$: The Mathieu sine is an odd function.
(5) $C(a, 0, x) = \cos(\sqrt{a}x)$ and $S(a, 0, x) = \frac{\sin(\sqrt{a}x)}{\sqrt{a}}$.

Proof. (1) We have

$$C(a, q, 0) = \frac{F(a, q, 0) + F(a, q, 0)}{2F(a, q, 0)} = \frac{2F(a, q, 0)}{2F(a, q, 0)} = 1 .$$

Similarly, for the sine function, we have

$$S(a, q, 0) = \frac{F(a, q, 0) - F(a, q, 0)}{2F'(a, q, 0)} = \frac{0}{2F'(a, q, 0)} = 0 .$$

(2) We have

$$C'(a, q, 0) = \frac{F'(a, q, 0) - F'(a, q, 0)}{2F(a, q, 0)} = \frac{0}{2F(a, q, 0)} = 0 \,,$$

and similarly,

$$S'(a, q, 0) = \frac{F'(a, q, 0) + F'(a, q, 0)}{2F'(a, q, 0)} = \frac{2F'(a, q, 0)}{2F'(a, q, 0)} = 1 \,.$$

(3) We have

$$C(a, q, -x) = \frac{F(a, q, -x) + F(a, q, x)}{2F(a, q, 0)} = \frac{F(a, q, x) + F(a, q, -x)}{2F(a, q, 0)} = C(a, q, x) \,.$$

Then, the Mathieu cosine is an even function.
(4) Similarly,

$$S(a, q, -x) = \frac{F(a, q, -x) - F(a, q, x)}{2F'(a, q, 0)} = -\frac{F(a, q, x) - F(a, q, -x)}{2F'(a, q, 0)} = -S(a, q, x) \,.$$

Then, the Mathieu sine is an odd function.
(5) Follows from the fact that $S(a, 0, .)$ and $C(a, 0, .)$ are solutions of the Mathieu equation

$$\frac{d^2y}{dx^2} + ay = 0$$

and the assertions (1) and (2). □

Remark 113.
- The general solution of the Mathieu equation (for fixed a and q) is a linear combination of the Mathieu cosine and sine.
- In general, the Mathieu cosine and sine are not periodic. However, for small values of q we have

$$C(a, q, x) \sim \cos(\sqrt{a}x) \quad \text{and} \quad S(a, q, x) \sim \frac{\sin(\sqrt{a}x)}{\sqrt{a}} .$$

The Mathieu cosine is illustrated graphically in Figure 4.8.

4.2.10 Airy function

The Airy function was introduced by the astronomer George Biddell Airy in optical calculations. These are solutions of the second-order differential equation known as the Airy differential equation

$$y'' - xy = 0 . \tag{4.26}$$

One idea to resolve such an equation is to use the well-known Fourier Transform, which leads formally to a set of solutions called Airy functions based on the following

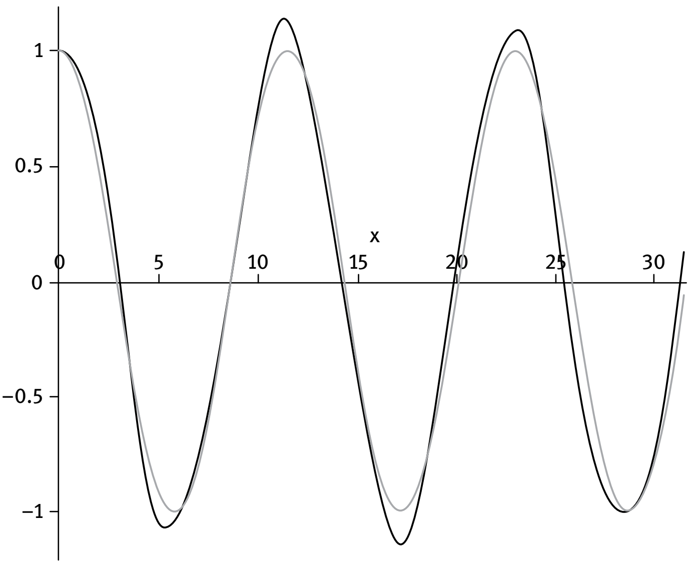

Fig. 4.8: Mathieu cosine: C(0.3;0.1;x) (Grey).

integral representation:

$$A(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\xi x + \frac{\xi^3}{3} \right) d\xi \,,$$

which is in fact a divergent integral. In fact, the integral is a semi-convergent integral. Indeed, for $0 < a < L < +\infty$, an integration by parts yields that

$$\int_a^L \cos\left(\xi x + \frac{\xi^3}{3} \right) d\xi = 2 \int_a^L \sin\left(\xi x + \frac{\xi^3}{3} \right) \frac{\xi}{(x + \xi^2)^2} d\xi + \left[\frac{\sin\left(\xi x + \frac{\xi^3}{3} \right)}{x + \xi^2} \right]_a^L \,.$$

As the integral $\int_a^\infty \sin(\xi x + \frac{\xi^3}{3}) \frac{\xi}{(x+\xi^2)^2} d\xi$ is absolutely convergent, the desired result follows.

Definition 114. For $\eta > 0$, we define the Airy function Ai by means of the following integral:

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{\mathbb{R}+i\eta} e^{i\xi x} e^{i\frac{\xi^3}{3}} d\xi \,.$$

Furthermore, applying classical techniques of parameter-depending integrals, we can prove that

(1) Ai is continuous on $\mathbb{R}$.

(2) $\lim_{x\to+\infty} \mathrm{Ai}(x) = 0$.

Indeed, note that $\mathrm{Ai}(x)$ may be written in the form

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}} d\xi \,.$$

Next, as for $\eta > 0$, we get

$$\mathrm{Re}\left(ix(\xi + i\eta) + i\frac{(\xi + i\eta)^3}{3} \right) = -x\eta - \xi^2\eta + \frac{\eta^3}{3} \,,$$

the last integral is then absolutely convergent. Furthermore, it is uniformly convergent on any compact set in $\mathbb{R}$. So, since the function $x \mapsto e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}}$ is continuous for all η and ξ, the function Ai is then continuous on $\mathbb{R}$. In fact, we may prove that Ai is $\mathcal{C}^\infty$ and that for all $k \in \mathbb{N}$,

$$\mathrm{Ai}^{(k)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (i(\xi + i\eta))^k e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}} d\xi \,.$$

We prove further that Ai is independent of the parameter η. Indeed,

$$\frac{d\mathrm{Ai}}{d\eta} = \frac{d}{d\eta}\frac{1}{2\pi}\int_{\mathbb{R}} \left(e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}\right) d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{d}{d\xi}\left\{e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}\right\} d\xi = 0\,,$$

as the function

$$\xi \mapsto e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}$$

is in the Schwartz class.

Properties 115. The following properties of the Airy function Ai hold:
(1) The function Ai satisfies the Airy differential equation (4.26).
(2) $\mathrm{Ai}(j.)$ is a solution of the Airy differential equation (4.26), whenever $j^3 = 1$.
(3) The function Ai is an entire function of x.
(4) For all $x \in \mathbb{R}$, $\mathrm{Ai}(x) \in \mathbb{R}$.
(5) $A_i(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}$ and $A_i'(0) = \frac{-1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}$.

Proof. (1) As noted above, the Airy function Ai is twice differentiable and

$$\begin{aligned}\mathrm{Ai}''(x) &= \frac{1}{2\pi}\int_{\mathbb{R}+i\eta} (i\xi)^2 e^{ix\xi}e^{i\frac{\xi^3}{3}}\,d\xi \\ &= \frac{i}{2\pi}\int_{\mathbb{R}+i\eta} e^{ix\xi}\frac{d}{d\xi}\left(e^{i\frac{\xi^3}{3}}\right) d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}+i\eta} \xi e^{ix\xi}e^{i\frac{\xi^3}{3}}\,d\xi = x\mathrm{Ai}(x)\,.\end{aligned}$$

(2) Let $\widetilde{\mathrm{Ai}}(x) = \mathrm{Ai}(jx)$. We have

$$\widetilde{\mathrm{Ai}}''(x) = j^2\mathrm{Ai}''(jx) = j^2(jx\mathrm{Ai}(jx)) = j^3x\widetilde{\mathrm{Ai}}(x) = x\widetilde{\mathrm{Ai}}(x)\,.$$

(3) The function f_η defined by $f_\eta(x,\xi)) = e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}$ is analytic as a function of x for all ξ. Furthermore, for all $R > 0$ and $|x| \le R$, we have

$$|f_\eta(x,\xi)| \le e^{-R\eta}e^{-R\xi}e^{-\eta\xi^2}\,.$$

The last function is integrable according to ξ. So, Ai is analytic.

(4) For $x \in \mathbb{R}$ we have

$$\begin{aligned}\overline{\mathrm{Ai}(x)} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix(\xi-i\eta)} e^{-i\frac{(\xi-i\eta)^3}{3}}\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix(-\xi+i\eta)} e^{i\frac{(-\xi+i\eta)^3}{3}}\,d\xi\;. \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix(\omega+i\eta)} e^{i\frac{(\omega+i\eta)^3}{3}}\,d\omega\;. \\ &= \mathrm{Ai}(x)\;.\end{aligned}$$

(5) As Ai is independent of $\eta > 0$, and Ai(0) is real, we can write

$$\mathrm{Ai}(0) = \frac{1}{2\pi}\,\mathrm{Re}\left(\int_{\mathbb{R}} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\right)\;.$$

Denote I as the last integral and $J = \frac{1}{2}I$. Simple computations yield that

$$I = 2\int_0^{+\infty} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\;,$$

which means that

$$\mathrm{Ai}(0) = \frac{1}{\pi}\,\mathrm{Re}\left(\int_0^{+\infty} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\right) = \frac{1}{\pi}\,\mathrm{Re}(J)\;.$$

Next, for $R > 0$ large enough consider the points $O(z_0 = 0)$, $A(z_A = R)$, $B = (z_B = \mathrm{Re}^{i\frac{\pi}{6}})$, $C(z_C = i + z_B)$, and $D = (z_D = i)$ and the contours γ_R composed of the juxtaposition of the segment $[o, A]$, the arc (AB) and the segment BO in the positive sense, and δ_R the parallelogram contour $OBCDO$ countered also in the positive sense. So, applying the residues theory on the function $f(z) = e^{i\frac{(z+i)^3}{3}}$ and the contour γ_R, we get

$$\pi\mathrm{Ai}(0) = \mathrm{Re}\left(\lim_{R\to+\infty} K_R\right)\;,$$

where K_R is the integral given by

$$K_R = \int_{[B,O]} f(z)dz = \int_{[C,D]} e^{i\frac{z^3}{3}}\,dz\;.$$

Now, applying again the residues theory with the function $g(z) = e^{i\frac{z^3}{3}}$ on the parallelogram contour $OBCDO$, we obtain

$$\lim_{R\to+\infty} K_R = \int_0^{+\infty} g\left(te^{i\frac{\pi}{6}}\right) e^{i\frac{\pi}{6}}\,dt = e^{i\frac{\pi}{6}}\int_0^{+\infty} e^{-\frac{t^3}{3}}\,dt\;.$$

Hence,

$$\pi \mathrm{Ai}(0) = \frac{\sqrt{3}}{2} \int_0^{+\infty} e^{-\frac{t^3}{3}} dt = \frac{\sqrt{3}}{2} 3^{-2/3} \int_0^{+\infty} x^{-2/3} e^{-x} dx = \frac{\Gamma(\frac{1}{3})}{2.3^{1/6}} ,$$

which means that

$$\mathrm{Ai}(0) = \frac{\Gamma(\frac{1}{3})}{2\pi 3^{1/6}} .$$

Analogous techniques may be applied to obtain $\mathrm{Ai}'(0)$. □

Now, we introduce the second-kind Airy function ([18]).

Definition 116. The second-kind Airy function is defined by

$$\mathrm{Bi}(x) = e^{i\pi/6} \mathrm{Ai}(jx) + e^{-i\pi/6} \mathrm{Ai}(j^2 x) , \tag{4.27}$$

where $j = e^{i2\pi/3}$.

Proposition 117. *The second-kind Airy function Bi is a solution of the Airy differential equation* (4.26) *and satisfies*

$$Bi(0) = \frac{1}{3^{\frac{1}{6}} \Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad B_i'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)} .$$

Furthermore, Bi is real on the real axis $\mathbb{R}$.

Proof. We have

$$\begin{aligned} \mathrm{Bi}''(x) &= j^2 e^{i\pi/6} \mathrm{Ai}''(jx) + j^4 e^{-i\pi/6} \mathrm{Ai}''(j^2 x) \\ &= j^2 e^{i\pi/6} jx \mathrm{Ai}(jx) + j^4 e^{-i\pi/6} j^2 x \mathrm{Ai}(j^2 x) \\ &= x(e^{i\pi/6} \mathrm{Ai}(jx) + e^{-i\pi/6} \mathrm{Ai}(j^2 x)) \\ &= x \mathrm{Bi}(x) . \end{aligned}$$

Hence, Bi satisfies (4.26). Next,

$$\mathrm{Bi}(0) = e^{i\pi/6} \mathrm{Ai}(0) + e^{-i\pi/6} \mathrm{Ai}(0) = \sqrt{3} \mathrm{Ai}(0) = \sqrt{3} \frac{\Gamma(\frac{1}{3})}{2\pi 3^{1/6}} .$$

Now, observing that

$$\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}} ,$$

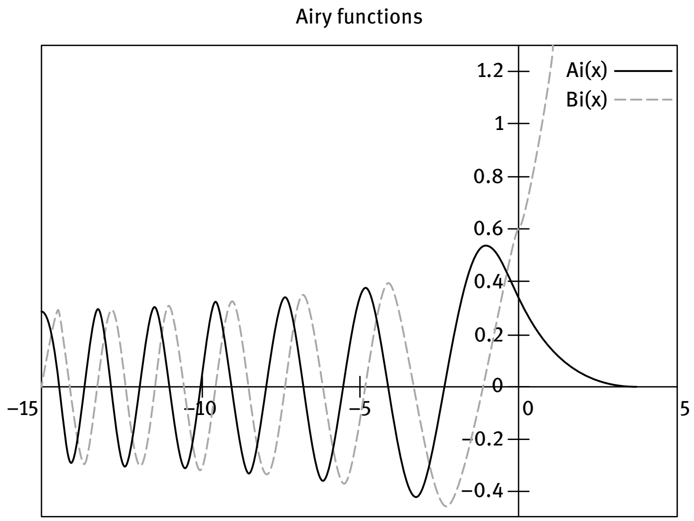

Fig. 4.9: Airy function Ai and Bi.

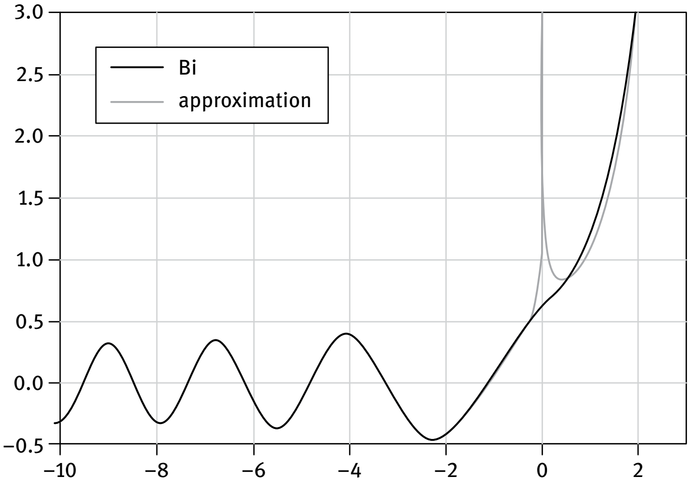

Fig. 4.10: The Airy function Bi and its approximation.

we get

$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} \, .$$

The same techniques yield $\mathrm{Bi}'(0)$. Finally, for $x \in \mathbb{R}$, we have

$$\overline{\mathrm{Bi}(x)} = e^{-i\pi/6}\mathrm{Ai}(\bar{j}x) + e^{i\pi/6}\mathrm{Ai}(\bar{j}^2 x) = e^{-i\pi/6}\mathrm{Ai}(j^2 x) + e^{i\pi/6}\mathrm{Ai}(jx) = \mathrm{Bi}(x) \, . \qquad \square$$

Airy functions Ai and Bi are illustrated in Figure 4.9. Furthermore, Figure 4.10 illustrates the Airy function Bi and its approximation.

4.3 Hankel–Bessel transform

In this section, we focus on the most known transform associated with the special functions developed previously. We will review the Hankel–Bessel transform of functions. Readers are referred to [53] for more details. We denote the inner product in $L^2(\mathbb{R}^+, dx)$ by

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g}(x)dx$$

and the associated norm by $\|.\|_2$. Similarly, we denote the inner product in $L^2(\mathbb{R}^+, \xi d\xi)$ by

$$\langle f, g\rangle_\xi = \int_0^\infty f(\xi)\overline{g}(\xi)\xi d\xi$$

and the associated norm by $\|.\|_{\xi,2}$.

Definition 118. Let $f \in L^2(\mathbb{R}^+, dx)$. The Bessel transform of f is defined by

$$\mathcal{B}(f)(\xi) = \int_0^{+\infty} f(x)\sqrt{x}J_\nu(x\xi)dx, \quad \forall \xi > 0\,,$$

where J_ν is the Bessel function of first kind and index ν.

We immediately have the following characteristics:

Proposition 119.
(1) *For all $f \in L^2(\mathbb{R}^+, dx)$, $\mathcal{B}(f) \in L^2(\mathbb{R}^+, \xi d\xi)$.*
(2) *The Bessel transform $\mathcal{B}$ is invertible and its inverse is*

$$\mathcal{B}^{-1}(g)(x) = \int_0^{+\infty} g(\xi)\sqrt{x}J_\nu(x\xi)\xi d\xi, \quad \forall g \in L^2(\mathbb{R}^+, \xi d\xi)\,.$$

Proof. (1) Let f and g be in $L^2(\mathbb{R}^+, dx)$. We have

$$\begin{aligned}\langle \mathcal{B}(f), \mathcal{B}(g)\rangle_\xi &= \int_0^{+\infty} \mathcal{B}(f)(\xi)\mathcal{B}(g)(\xi)\xi d\xi \\ &= \int_{\mathbb{R}_+^3} \sqrt{x}\sqrt{y} f(x)g(y)J_\nu(x\xi)J_\nu(y\xi)\xi dx dy d\xi \\ &= \int_{\mathbb{R}_+^2} \sqrt{x}\sqrt{y} f(x)g(y)\frac{\delta(x-y)}{x} dx dy \\ &= \int_{\mathbb{R}_+} \sqrt{x}\sqrt{x} f(x)g(x)\frac{1}{x} dx \\ &= \langle f, g\rangle \ .\end{aligned}$$

So, taking $g = f$, we get

$$\|\mathcal{B}(f)\|_{\xi,2} = \|f\|_2$$

which means that $\mathcal{B}$ is an isometry.

(2) Denote $\widetilde{\mathcal{B}}(f)$ the right-hand quantity. We will prove that $\mathcal{B}(\widetilde{\mathcal{B}}(f)) = f$. Indeed,

$$\begin{aligned}\mathcal{B}(\widetilde{\mathcal{B}}(f))(\xi) &= \int_0^{+\infty} \widetilde{\mathcal{B}}(f)(x)\sqrt{x}J_\nu(x\xi)dx \\ &= \int_0^{+\infty}\int_0^{+\infty} f(\eta)\sqrt{x}J_\nu(x\eta)\eta\sqrt{x}J_\nu(x\xi)d\eta dx \\ &= \int_0^{+\infty} f(\eta)\eta\frac{\delta(\eta-\xi)}{\eta} d\eta \\ &= f(\xi) \ .\end{aligned}$$

□

Definition 120. The Hankel transform, also called Fourier–Bessel transform of the order ν, is defined by

$$\mathcal{H}(f)(\xi) = \int_0^{\infty} f(x)J_\nu(x\xi)x dx; \quad \forall f \ . \tag{4.28}$$

Remark 121. Hankel transform $\mathcal{H}$ and Bessel one $\mathcal{B}$ are related via the equality

$$\mathcal{H}(f)(\xi) = \mathcal{B}(\sqrt{.}f)(\xi) \ .$$