1 Introduction

The present work provides the scientific community with a unified collection of developments in applied mathematical problems and as mathematical theory. First, we wish to acknowledge that this book is the result of the work of two doctoral students, Sabrine Arfaoui and Imen Rezgui, who were supervised by the third author until October 2014. These studies took place at the Computational Mathematics Laboratory UR11ES51 headed by Professor Samir Ben Ammou, a recent student of École Nationale des Ponts et Chaussées de Paris. The work focused on a special class of wavelets and their applications. It is proposed to develop special wavelet bases that are related to special functions in one part and adapted to spherical geometry in another.

Since their appearance, and especially over the last decades, wavelets have proved to be powerful bases for many domains, such as numerical analysis, signal/image processing, physics, biomaths, medicine, and data analysis. Their power stems from the fact that they do not require a large number of coefficients to accurately represent general functions and large data sets. This allows compression and efficient computations. Wavelets also offer both theoretical characterization of smoothness, insights into the structure of functions and operators, and practical numerical tools that lead to faster computational algorithms. Classical constructions have been limited to simple domains, such as intervals, cubes, Cartesian representations, tensor products, etc. So, one main challenge may be the construction of wavelets on general domains as they appear in graphics applications. In the present context, we aim to present wavelet constructions for functions defined on the sphere. We aim to show that using special functions, such as orthogonal polynomials, homogenous polynomials, and Bessel functions and their relatives, can be sources for well-adapted wavelets. Readers will notice that the constructed schemes lead to extremely easily implemented bases and allow fully adaptive algorithms.

In [14], a polynomial wavelet-type system adapted to the sphere is presented in order to expand continuous functions into wavelet series on the sphere. The method is characterized by an optimum order of growth of the degrees of polynomials. However, and as the authors themselves have already noticed and declared, the wavelet-type system presented is not suitable for implementations as no explicit formulas for coefficient functionals have been provided and the fact remains that the growth of the degrees of polynomials is too rapid.

In [142], a simple technique for constructing biorthogonal wavelets on the sphere with customized properties is developed. The construction is an incidence of a fairly general scheme compared to [152] and [153]. The authors mentioned an important task about wavelets on the sphere showing that efficient wavelet algorithms have practical applications since many computational problems are naturally stated on the sphere.

The first notion developed in this book is orthogonal polynomials. These are well-known because of their link to many mathematical, physical, engineering and com-

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puter sciences topics, such as scattering theory, automatic control, signal analysis, potential theory, approximation theory, and numerical analysis. Orthogonal polynomials are special as they are orthogonal with respect to some special weights allowing them to satisfy some properties that are not fulfilled with other polynomials. Such properties have made them useful candidates to resolve enormous problems in physics, probability, statistics, and other fields. In the present work, we aim to review orthogonal polynomials by recalling the original definitions, reproduce their properties, and develop some cases related to the most well-known method to reproduce some classes of them.

Next, as a natural extension of orthogonal polynomials, we present a review of homogenous polynomials and their interactions with harmonic analysis on the sphere. Specifically, we study the constructions of the spherical harmonics and develop the main results of the theory of harmonic analysis on the sphere, such as the addition theorem and the Fourier transformation. The link with some special features, such as ultra-spherical polynomials and Bessel functions are also reviewed.

As in all research studies where the track is unpredictable, this work uses many notions. As mentioned, we are exploring special wavelets. These are naturally related to special functions. That is why we immediately plunged into the context of special functions, that is, some particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. A detailed study of the most well-known types of these functions has been conducted. We detail the definitions, properties, and characterizations of Bessel, Hankel, and zonal functions. Proofs have been developed, sometimes in detail, relative to the base references and sometimes originally developed in the case of a lack of references. Graphic illustrations and some examples of applications are sometimes mentioned, such as differential equations, integro-differential equations, and time series.

Special functions are indispensable in many topics ranging from pure mathematics to applied fields. Thus, it is important to study their properties. Although many properties and characteristics of such functions appear in many mathematical documents, there is no unified treatment of the topic. With this book, we are filling this hole in the literature.

The last topic is spherical wavelets, which may be considered as a class of special functions. We made use of zonal, spherical harmonics, homogenous, as well as orthogonal polynomials. Recall that the spherical harmonics form the basis of the Hilbert space $L^2(S^n)$, where S^n is the unit sphere of R^n, $n \in \mathbb{N}$. Harmonic analysis on the sphere is the natural extension of Fourier series, which studies the expressibility of functions and generalized functions as sums of the fundamental exponential functions. The exponential functions are simpler functions, and are both eigenfunctions of the translation-invariant differential operator and group homomorphisms. Here also, spherical harmonics are simple and eigenfunctions of some differential operators.