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A Treatment of Generalized Fractional Differential Equations: Sumudu Transform Series Expansion Solutions, and Applications

Abstract: In this chapter we solve generalized fractional differential equations by using the Sumudu transform. The equations we treat include differential equations with generalized Riemann-Liouville fractional derivatives. The operators involved are very general in nature, and cover a wide range of fractional differential equations and their solutions in terms of various functions related to Mittag – Leffler functions.

Keywords: Caputo fractional derivative, fractional differential equations; generalized Riemann-Liouville fractional derivative; Hilfer fractional derivative, Mittag-Leffler functions; Sumudu transform

1 Introduction

In the last three decades, fractional calculus became an important tool for the development and analysis of accurate models of various phenomena of nature, in diverse fields of science such as engineering, economics, material sciences and aerospace sciences. This field of mathematical analysis - which deals with investigations and applications of integrals and derivatives of arbitrary order, received considerable attention by scientists and mathematicians in numerous theoretical and applied sciences. Recently, Special families of generalized fractional derivative operator $[D_{a\pm}^{\alpha;\beta}]$ of order α and type β, were introduced and investigated (Garra et al., 2014, Hilfer and Anton 1995; Hilfer 2000; Hilfer 2002; Hilfer 2008; Hilfer). Applications of fractional calculus may be based on fractional derivatives of different kinds, (Hilfer, Luchko, and Tomovski, 2009; Mainardi and Gorenflo, 2007; Sandev and Tomovski, 2010; Srivastava and Tomovski, 2009; Tomovski 2012). The solution of generalized differential equations of fractional order is quite involved. Some analytical methods were presented some using and numerous integral based transforms, such as the popular Laplace transform (Pod-

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lubny, 1994; Podlubny, 1999), the Fourier transform method (Miller and Ross, 1993), and more recently, the Sumudu transform (Belgacem et al., 2014, Bulut, Baskonus and Belgacem, 2013a, Bulut, Baskonus and Belgacem, 2013b; Chaurasia., Dubey, and Belgacem 2012; Dubey, Goswami and Belgacem, 2012; Gupta„ Sharma, and Belgacem, 2011 ; Katatbeh and Belgacem, 2011, Tuluce, Bulut and Belgacem, 2014) that treat fractional type equations and systems by means of the Sumudu transform method.

The prevalence and summary of the state of the art capabilities and main properties of the Sumudu transform including intricate developments, can be found in the works of Belgacem, 2006a, Belgacem 2009; and Belgacem, 2010. With more than two decades after its bringing back to circulation after Watugala Re-baptism (Watugala, 1993). With less than fifty articles in circulation, of which we recommend the following articles treating integer order types of equations of systems (Asiru, 2001; Asiru, 2002; Asiru, 2003; Belgacem, Karaballi and Kalla 2003; Belgacem 2006 b; Belgacem, 2009; Belgacem 2010; Husain and Belgacem, 2007; Kılıcman, Eltayeb, and Agarwal, 2010; Rana et al. 2007; Weerakoon, 1994; Weerakoon, 1999; Zang 2007).

In the recent published literature dedicated to Smudu transform, the only tables provided are those in the works of Belgacem, Karaballi et Kalla, 2003; Belgacem and Karaballi, 2006; Katatbeh and Belgacem, 2011; Belgacem and Silambarasan, 2012.

In this chapter, we apply the Sumudu transform of the generalized fractional derivative, and use the expansion coefficients of series to derive the explicit solution to homogeneous fractional differential equations.

2 Preamble

The Riemann – Liouvville fractional derivative is defined in the (Miller and Ross, 1993), as follows:

Definition 1: The Riemann – Liouvville fractional derivative of order, α, $m-1<\alpha<m, m\in N$, is defined as the left inverse of Riemann – Liouvville fractional integral, i.e.

$$D_t^{\alpha} f(t)=\frac{1}{\Gamma(m-\alpha)} D^m \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau. \tag{1}$$

In contrast, we find that Caputo gave an alternative definition for the fractional derivative, (Caputo,1969):

Definition 2: The Caputo fractional derivative of order $\alpha, m-1<\alpha<m, m\in N$ is given by

$$^{C}D_t^{\alpha} f(t)=I_t^{m-\alpha} D^m f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{1}{(t-x)^{\alpha-m+1}} D^m f(x)\, dx. \tag{2}$$

On the other hand, (Hilfer, 2002) stated yet another version of the fractional derivative, namely:

Definition 3: The Hilfer fractional derivative or composite fractional derivative of order $0 < \alpha < 1$, and type, $0 \le \beta \le 1$, is defined by

$$D_t^{\alpha,\beta} f(t) = I_t^{\beta(1-\alpha)} D I_t^{(1-\beta)(1-\alpha)} f(t) . \tag{3}$$

Furthermore, in 24 Hilfer R. (2008) extended his own definition, and termed it generalized Riemann-Liouville.

Definition 4: For $n-1 < \alpha < n, 0 \le \beta \le 1, n \in N$, The generalized Riemann-Liouville fractional derivative is defined by

$$D_t^{\alpha,\beta} f(t) = I_t^{\beta(n-\alpha)} D^n I_t^{(1-\beta)(n-\alpha)} f(t) . \tag{4}$$

In the definition above, type β allows $D_t^{\alpha,\beta}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative, and the Caputo fractional derivative. Clearly, the case $\beta = 0$, reverts back to yield the classical Riemann-Liouville fractional derivative:

$$D_t^{\alpha,0} f(t) = D^n I_t^{(n-\alpha)} f(t) = D_t^{\alpha} f(t) , \quad 0 < \alpha < 1. \tag{5}$$

Furthermore, in the case $\beta = 1$, the Caputo fractional derivative is obtained

$$D_t^{\alpha,1} f(t) = I_t^{(n-\alpha)} D^n f(t) = {}^{C}D_t^{\alpha} f(t) , \quad 0 < \alpha < 1. \tag{6}$$

Definition 5: A generalization of the Mittag – Leffler function $E_{\alpha,\beta}(z)$ is introduced by Prabhakar, 1971, as follows:

$$E_{\alpha,\beta}^{\gamma}(z) \quad = \quad \sum_{k=0}^{\infty} \frac{\gamma_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} . \tag{7}$$

where $\alpha, \beta, \gamma \in C; \Re(\alpha) > 0; \Re(\beta) > 0; z \in C$ and γ_k denotes the familiar Pochhammer symbol or the shifted factorial, since $(1)_k = \quad k!\,(k \in N_0)$, the set of whole numbers $(N_0 = N \cup \{0\} = \{0, 1, 2 \ldots\})$, and where

$$(\gamma)_k \quad = \quad \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \quad = \quad \begin{cases} 1 & (k = 0; \gamma \in C \setminus \{0\}) \\ \gamma(\gamma+1)\ldots(\gamma+k-1) & (k \in N; \gamma \in C) \end{cases} \tag{8}$$

Definition 6: The Sumudu transform is defined over the set of functions (See for instance Watugala, 1993):

$$A = \{f(t)\,|\exists M, \tau_1 , \tau_2 > 0, \left|f(t)\right| < Me^{|t|/\tau_j}, \;\; if \;\; \mathrm{t} \in (-1)^j \times [0, \infty)\},$$

by

$$\tilde{G}(u) = S[f(t)] = \int_0^{\infty} f(ut) e^{-t} dt, \;\; u \in (-\tau_1, \tau_2) \tag{9}$$

With the definitions above, we now recall some useful results which are directly applicable below:

$$S^{-1}\left[u^{\alpha-1}\left(1-wu^{\beta}\right)^{-\delta}\right]=t^{\alpha-1}E^{\delta}_{\beta,\alpha}\left(wt^{\beta}\right), \tag{10}$$

where, S^{-1} denotes the inverse Sumudu transform.

We can prove the result in the following way:

$$S\left[t^{\gamma-1}E^{\delta}_{\beta,\gamma}\left(\omega t^{\beta}\right)\right]=\int_{0}^{\infty}e^{-t}\left(ut\right)^{\gamma-1}E^{\delta}_{\beta,\gamma}\left(\omega\left(ut\right)^{\beta}\right)dt.$$

By using Eq. 7, we get,

$$u^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\delta)_n\left(\omega u^{\beta}\right)^{n}}{n!}=u^{\gamma-1}\left(1-\omega u^{\beta}\right)^{-\delta}.$$

By applying inverse Sumudu transform, we get our required result.

$$S^{-1}\left[\frac{u^{-2}}{\left(u^{-2}+au^{-\alpha}+b\right)}\right]=\sum_{k=0}^{\infty}(-b)^{k}\,t^{2k+1-1}E^{k+1}_{-\alpha+2,2k+1}\left[-at^{-\alpha+2}\right] \tag{11}$$

To find inverse Sumudu transform of this function we will use result 10

$$\frac{1}{u^{2}\left(u^{-2}+au^{-\alpha}+b\right)}=\frac{1}{\left(1+au^{-\alpha+2}\right)\left[1+\frac{bu^{2}}{\left(1+au^{-\alpha+2}\right)}\right]}=\sum_{k=0}^{\infty}\frac{\left(-bu^{2}\right)^{k}}{\left(1+au^{-\alpha+2}\right)^{k+1}}$$
$$=\sum_{kr=0}^{\infty}u^{2k+1-1}(-b)^{k}\left(1+au^{-\alpha+2}\right)^{-(k+1)} \tag{12}$$

Then by using result 10

$$S^{-1}\left[\frac{u^{-2}}{\left(u^{-2}+au^{-\alpha}+b\right)}\right]=\sum_{k=0}^{\infty}(-b)^{k}\,t^{2k+1-1}E^{k+1}_{-\alpha+2,2k+1}\left(-at^{-\alpha+2}\right) \tag{13}$$

(iii)

$$S^{-1}\left[\frac{u^{-1}}{u^{-2}+au^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}(-b)^{k}\,t^{2k+2-1}E^{k+1}_{-\alpha+2,2k+2}\left(-at^{-\alpha+2}\right) \tag{14}$$

(iv)

$$S^{-1}\left[\frac{au^{\beta(1-\alpha)-1}}{u^{-2}+au^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}a.(-b)^{k}\,t^{\beta(1-\alpha)+2k+1}E^{k+1}_{-\alpha+2,\beta(1-\alpha)+2k+2}\left(-at^{-\alpha+2}\right) \tag{15}$$

(v)

$$S^{-1}\left[\frac{au^{\beta(2-\alpha)-1}}{u^{-2}+au^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}a.(-b)^{k}\,t^{\beta(2-\alpha)+2k+1}E^{k+1}_{-\alpha+2,\beta(2-\alpha)+2k+2}\left(-at^{-\alpha+2}\right) \tag{16}$$

(vi)

$$S^{-1}\left[\frac{u^{-1}}{au^{-1}+u^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}(-b)^k\, t^{\alpha k+\alpha-1}E^{k+1}_{\alpha-1,\alpha k+\alpha}\left(-at^{\alpha-1}\right) \tag{17}$$

(vii)

$$S^{-1}\left[\frac{u^{\beta(1-\alpha)-1}}{au^{-1}+u^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}(-b)^k\, t^{\beta(1-\alpha)+\alpha k+\alpha-1}E^{k+1}_{\alpha-1,\beta(1-\alpha)+\alpha k+\alpha}\left(-at^{\alpha-1}\right) \tag{18}$$

$(viii)$

$$S^{-1}\left[\frac{u^{\beta(2-\alpha)-1}}{au^{-1}+u^{-\alpha}+b}\right]=\sum_{k=0}^{\infty}(-b)^k\, t^{\beta(2-\alpha)+\alpha k+\alpha-1}E^{k+1}_{\alpha-1,\beta(2-\alpha)+\alpha k+\alpha}\left(-at^{\alpha-1}\right) \tag{19}$$

Lemma 1: The Sumudu transform of generalized fractional derivative, $\left(D^{\alpha,\beta}_{a\pm}y\right)(x)$ is given as follows:

$$S\left(D^{\alpha,\beta}_{a\pm}y\right)(x)=\quad u^{-\alpha}S[y(x)](u)-\sum_{k=0}^{n-1}[u^{k-n+\beta(n-\alpha)}\lim_{x\to a+}\frac{d^k}{dx^k}(I^{(n-\alpha)(1-\beta)}_{a+}y)(x)]. \tag{20}$$

Proof. The generalized fractional differential operator is defined as

$$\left(D^{\alpha,\beta}_{a\pm}y\right)(x)=\left(\pm I^{\beta(1-\alpha)}_{a\pm}\left(D^{\alpha+\beta-\alpha\beta}_{a\pm}y\right)\right)(x)\,. \tag{21}$$

Applying the integral operator, (I^{α}_{a+}), on both side of 21, (see for instance Podlubny I. 1994).

$$\begin{aligned}I^{\alpha}_{a+}\left(D^{\alpha,\beta}_{a+}y\right)(x)&=\left(I^{\beta(1-\alpha)+\alpha}_{a+}\left(D^{\alpha+\beta-\alpha\beta}_{a+}y\right)\right)(x)\\&=y(x)-\sum_{k=0}^{n-1}\frac{(x-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1}\lim_{x\to a+}\frac{d^k}{dx^k}(I^{(n-\alpha)(1-\beta)}_{a+}y)(x)\end{aligned} \tag{22}$$

Then applying the Sumudu transform on each side of above equation we get

$$u^{\alpha}S\left(D^{\alpha,\beta}_{a+}y\right)(u)=S[y(x)](u)-\sum_{k=0}^{n-1}[\lim_{x\to a+}\frac{d^k}{dx^k}(I^{(n-\alpha)(1-\beta)}_{a+}y)(x)](u-a)^{k-(n-\alpha)(1-\beta)} \tag{23}$$

Multiplying both sides by, $u^{-\alpha}$, and taking $a=0$, we get required result.

$$S\left(D^{\alpha,\beta}_{0+}y\right)(u)=u^{-\alpha}S[y(x)](u)-\sum_{k=0}^{n-1}[u^{k-n+\beta(n-\alpha)}\lim_{x\to 0+}\frac{d^k}{dx^k}(I^{(n-\alpha)(1-\beta)}_{0+}y)(x)].$$

□

3 Main Results

In the foreword we establish the main findings of this chapter.
Theorem 1: The following generalized fractional differential equation

$$y''(t) + ay^{\alpha,\beta}(t) + by(t) = 0 \tag{24}$$
$$\text{where } n-1 < \alpha < n \text{ and } 0 < \beta \leq 1 \text{ and } a, b \in R$$

with the initial conditions, $y(0) = c_0 and y'(0) = c_1 \left(\mathrm{I}_0^{(1-\beta)(1-\alpha)} y\right)(0) = c_2$ and $\left(I_0^{(1-\beta)(2-\alpha)} y\right)(0) = c_3$, has a solution given by

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+1]r!}$$
$$+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+2]r!}$$
$$+ c_2 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(1-\alpha)+2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+\beta(1-\alpha)+2k+2]r!}$$
$$+ c_3 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(2-\alpha)+2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+\beta(2-\alpha)+2k+2]r!} \tag{25}$$

Proof. Applying the Sumudu Transform to Eq. 24, we obtain:

$$\begin{aligned} &u^{-2}S[y(t)] - u^{-2}y(0) - u^{-1}y'(0) + au^{-\alpha}S[y(t)]\,u \\ &-au^{\beta(1-\alpha)-1}(I_0^{(1-\beta)(1-\alpha)}y)(0) - au^{\beta(2-\alpha)-1}\left(I_0^{(1-\beta)(2-\alpha)}y'\right)(0) + bS[y(t)] = 0 \end{aligned} \tag{26}$$

Solving for S[y(t)] and using initial conditions we have

$$\begin{aligned} \mathrm{S}[\mathrm{y}(\mathrm{t})] &= \frac{\mathrm{u}^{-2}}{\mathrm{u}^{-2} + au^{-\alpha} + b} c_0 + \frac{u^{-1}}{u^{-2} + au^{-\alpha} + b} c_1 + \frac{au^{\beta(1-\alpha)-1}}{u^{-2} + au^{-\alpha} + b} c_2 + \frac{au^{\beta(2-\alpha)-1}}{u^{-2} + au^{-\alpha} + b} \\ &= 0. \end{aligned} \tag{27}$$

Now taking the inverse Sumudu transform on both sides and using results 11 to 16,

$$\begin{aligned} y(t) =& c_0 \sum_{k=0}^{\infty} (-b)^k \, t^{2k+1-1} E_{-\alpha+2,2k+1}^{k+1}\left(-at^{-\alpha+2}\right) \\ &+ c_1 \sum_{k=0}^{\infty} (-b)^k \, t^{2k+2-1_1} E_{-\alpha+2,2k+2}^{k+1}\left(at^{-\alpha+2}\right) \end{aligned}$$

$$+ c_2 \sum_{k=0}^{\infty} a.(-b)^k t^{\beta(1-\alpha)+2k+1} E^{k+1}_{-\alpha+2,\beta(1-\alpha)+2k+2}\left(at^{-\alpha+2}\right)$$
$$+ c_3 \sum_{k=0}^{\infty} a.(-b)^k t^{\beta(2-\alpha)+2k+1} E^{k+1}_{-\alpha+2,\beta(2-\alpha)+2k+2}\left(at^{-\alpha+2}\right) \quad (28)$$

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+1]\,r!}$$
$$+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+2]\,r!}$$
$$+ c_2 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(1-\alpha)+2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+\beta(1-\alpha)+2k+2]\,r!}$$
$$+ c_3 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(2-\alpha)+2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+\beta(2-\alpha)+2k+2]\,r!} \quad (29)$$

□

Theorem 2: The following generalized fractional differential equation

$$\mathrm{y}^{\alpha,\beta}(t) + ay'(t) + by(t) = 0 \quad \text{where } 1 < \alpha < 2 \text{ and } 0 < \beta \le 1 \text{ and } a, b \in R \quad (30)$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$, $\left(I_0^{(1-\beta)(1-\alpha)}y\right)(0) = c_2$ and $\left(I_0^{(1-\beta)(2-\alpha)}y\right)(0) = c_3$, has its solution given by:

$$y(t) = ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+\alpha-1}}{\Gamma[(\alpha-1)r+\alpha k+\alpha]\,r!}$$
$$+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{\beta(1-\alpha)+\alpha k+\alpha-1+(\alpha-1)r}}{\Gamma[(\alpha-1)r+\beta(1-\alpha)+\alpha k+\alpha]\,r!}$$
$$+ c_2 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{\beta(2-\alpha)+\alpha k+\alpha-1+(\alpha-1)r}}{\Gamma[(\alpha-1)r+\beta(2-\alpha)+\alpha k+\alpha]\,r!} \quad (31)$$

Proof. Applying the Sumudu transform of Eq. 30, we obtain:

$$u^{-\alpha}S[y(t)]\,u - u^{\beta(1-\alpha)-1}(I_0^{(1-\beta)(1-\alpha)}y)(0)$$
$$-u^{\beta(2-\alpha)-1}\left(I_0^{(1-\beta)(2-\alpha)}y'\right)(0) + au^{-1}S[y(t)] - au^{-1}y(0) + bS[y(t)] = 0 \quad (32)$$

Solving for $S[y(t)]$ and using the initial conditions above we get

$$\mathrm{S}[\mathrm{y}(\mathrm{t})] = a\frac{u^{-1}}{au^{-1}+u^{-\alpha}+b}c_0 + \frac{u^{\beta(1-\alpha)-1}}{au^{-1}+u^{-\alpha}+b}c_1 + \frac{u^{\beta(2-\alpha)-1}}{au^{-1}+u^{-\alpha}+b}c_2 = 0 \quad (33)$$

Now taking the inverse Sumudu transform on both sides and using results 17-19,

$$\begin{aligned} y(t) =& ac_0 \sum_{k=0}^{\infty} (-b)^k \, t^{\alpha k+\alpha-1} E^{k+1}_{\alpha-1,\alpha k+\alpha} \left(-at^{\alpha-1}\right) \\ &+ c_1 \sum_{k=0}^{\infty} (-b)^k \, t^{\beta(2-\alpha)+\alpha k+\alpha-1} E^{k+1}_{\alpha-1,\beta(2-\alpha)+\alpha k+\alpha} \left(-at^{\alpha-1}\right) \\ &+ c_2 \sum_{k=0}^{\infty} (-b)^k \, t^{\beta(1-\alpha)+\alpha k+\alpha-1} E^{k+1}_{\alpha-1,\beta(1-\alpha)+\alpha k+\alpha} \left(-at^{\alpha-1}\right) \end{aligned} \tag{34}$$

$$\begin{aligned} y(t) =& ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r \, t^{(\alpha-1)r+\alpha k+\alpha-1}}{\Gamma[(\alpha-1)r+\alpha k+\alpha]\, r!} \\ &+ c_2 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r \, t^{\beta(2-\alpha)+\alpha k+\alpha-1+(\alpha-1)r}}{\Gamma[(\alpha-1)r+\beta(2-\alpha)+\alpha k+\alpha]\, r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r \, t^{\beta(1-\alpha)+\alpha k+\alpha-1+(\alpha-1)r}}{\Gamma[(\alpha-1)r+\beta(1-\alpha)+\alpha k+\alpha]\, r!} \end{aligned} \tag{35}$$

□

Theorem 3: The following equation:

$$y^{\alpha,\beta}(t) - by(t) = 0, \text{ where } 0 < \alpha < 1 \quad \text{and } b \in R \tag{36}$$

with the initial condition $y(0) = c_0$ has its solution given by

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(1-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(1-\alpha)+\alpha+\alpha k}$$

Proof. We apply the Sumudu transform of 36, that is

$$u^{-\alpha} S[y(t)] - u^{\beta(1-\alpha)-1} I_0^{(1-\alpha)(1-\beta)} y(0) - bS[y(t)] \tag{37}$$

Solving for $S[y(t)]$

$$S[y(t)] = c_0 \frac{u^{\beta(1-\alpha)-1}}{(u^{-\alpha}-b)} = \frac{u^{\beta(1-\alpha)-1}}{u^{-\alpha}(1-bu^{\alpha})} = \sum_{k=0}^{\infty} \left(bu^{\alpha}\right)^k u^{\beta(1-\alpha)-1+\alpha} = \sum_{k=0}^{\infty} (b)^k u^{\beta(1-\alpha)-\alpha+\alpha k-1} \tag{38}$$

Now taking the inverse Sumudu of the equation above, we get:

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k \, t^{\beta(1-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(1-\alpha)+\alpha+\alpha k} \tag{39}$$

□

Remark 1 If $a = 0$ in Eq. 22, then the equation,

$$y^{\alpha,\beta}(t) - by(t) = 0 \qquad 1 < \alpha \leq 2 \tag{40}$$

with the initial conditions $y(0) = c_0 and y'(0) = c_1$ has its solution given by

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(1-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(1-\alpha)+\alpha+\alpha k} + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\beta(2-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(2-\alpha)+\alpha+\alpha k} \tag{41}$$

The next reult treats the generalized Vibration equation which is a particular case of the previous one.

Theorem 4: A nearly simple generalized harmonic vibration equation:

$$y^{\alpha,\beta}(t) - w^2 y(t) = 0, \qquad 1 < \alpha \leq 2 \tag{42}$$

with the initial conditions $y(0) = c_0 and y'(0) = c_1$ has its solution given by

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{\left(-w^2\right)^k t^{\beta(1-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(1-\alpha)+\alpha+\alpha k} + c_1 \sum_{k=0}^{\infty} \frac{\left(-w^2\right)^k t^{\beta(2-\alpha)+\alpha+\alpha k-1}}{\Gamma\beta(2-\alpha)+\alpha+\alpha k} \tag{43}$$

Proof. By putting, $b = w^2$ in Eq. 40, the proof of the statement of the theorem is completed. □

4 Conclusion

In this work we have derived solutions of some generalized and special type of fractional differential equations. These solutions are general in nature and to get them we have used the Sumudu transform technique. The Sumudu transform method turns out to be a very powerful and direct technique towards the elucidation of such applications. It is also very user friendly. Along with the references we provide below, we hope that our respected readers would find ample information about this multitasking tool, its properties, the obvious ones and the more intricate ones, as well as the various applications in fractional Calculus it was capable to successfully tackle. Any feedback or related communications would be highly welcome and taken into consideration.

Acknowledgment: F. B. M. Belgacem wishes to acknowledge the continued support of the Public Authority for Applied Education and Training Research Department, (PAAET RD), Kuwait.

The Authors wish to acknowledge the helpful support of Editor Prof. Carlo Cattani, and DGO Managing Editor Mathematics, Agnieszka Bednarczyk-Drag.

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