Preface

This book presents classical mechanics and quantum mechanics in an almost completely algebraic setting, thereby introducing mathematicians, physicists, and engineers to the ideas relating classical and quantum mechanics with Lie algebras and Lie groups. We shall mostly be concerned with systems described by a finite-dimensional configuration space; the infinite-dimensional case corresponds to classical and quantum field theory, and is beyond the scope of this book. However, we present the concepts in such a way that they are valid even in infinite dimensions, and present the material so that it also provides some insight into the infinite-dimensional case.

Much of the material covered here is not part of standard textbook treatments of classical or quantum mechanics (or is only superficially treated there). For physics students who want to get a broader view of the subject, this book may therefore serve as a useful complement to standard treatments of quantum mechanics. On the other hand, we do not attempt to cover topics amply discussed in the standard textbooks. Our goal is different, namely to show the advantage that may be offered by a point of view that treats quantum mechanics in a manner that emphasizes symmetry concepts (and hence Lie algebras and Lie groups), and that looks like classical mechanics as closely as possible.

We present everything as far as possible based on classical mechanics. This forced an approach to quantum mechanics close to Heisenberg's matrix mechanics, rather than the usual approach dominated by Schrödinger's wave mechanics. Indeed, although both approaches are formally almost equivalent, only the Heisenberg approach to quantum mechanics has a close similarity with classical mechanics. The book emphasizes the closeness of classical and quantum mechanics; effort is made to make this closeness as apparent as possible.

Written by a mathematician and a physicist,[1] this book is, almost without exception, about precise concepts and exact results in classical mechanics and quantum mechanics, but motivated and discussed in terms of their physical meaning. The structural properties are presented independent of computational techniques for obtaining quantitatively correct numbers from the assumptions made. This allows us to focus attention on the simplicity and beauty of theoretical physics, which is often hidden in a jungle of techniques for estimating or calculating quantities of interests. The standard approximation machinery for calculating from first principles physical properties, such as ex-

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plicit cross sections for high energy experiments, can be found in many textbooks and is not repeated here.

How to read this book. Quantum physics is not a subject naturally amenable to a fully sequential presentation. Understanding grows in levels—deeper understanding of a topic at some level requires the mastery of this topic and many related topics at some lower level of understanding. This is therefore an open-ended book, a book to which you will want to return again and again, since it covers material at different levels of sophistication, and there is a mix of introductory material and advanced topics.

The material is logically arranged such that, before they are used, concepts are defined and statements are proved (except in informal summaries or motivations). Nevertheless, this book is not intended for a strictly linear reading: Start reading where your interest is, and use the equation references, the symbol index, and the subject index to look up the things you need to know to understand your subject of interest or attention. If you get stuck, and especially on first reading, simply skip what you don't understand and resume reading in the next subsection, section, chapter, or part.

The book should stimulate appetite for more, and lead the reader into going deeper into the subject. For this reason, many more advanced topics are deliberately touched, but their discussion is often far too short for a comprehensive treatment, and only the surface is scratched.

Where the account given here is incomplete, which happens for some of the more advanced topics, references to the literature are given. Some general references for further reading are the following:

- Barut & Raczka [28], Cornwell [74], Gilmore [111], and Sternberg [260], for the general theory of Lie algebras, Lie groups, and their representations from a physics point of view;
- Wybourne [301] and Fuchs & Schweigert [102] for a more application oriented view of Lie algebras;
- Kac [150] and Neeb [201] for infinite-dimensional Lie algebras;
- Papoušek & Aliev [213] for quantum mechanics and spectroscopy;
- van der Waerden [275] for the history of quantum mechanics;
- Talagrand [268], Zeidler [303], and Weinberg [289] for treatments of quantum field theory from three quite different complementary perspectives.

Background needed. A first version of the book originated as course notes from a course given by the first author, written up by the second author, and considerably expanded and polished by combined efforts, resulting in a balanced and uniform presentation. Part II is based on unpublished work by the first author (Neumaier [206]). Thanks go to Phillip Bachler, Roger Balian, Daniel Ciobotu, Clemens Elster, Martin Fuchs, Johann Kim, Morteza Kimiaei, Thomas Klimpel, Mihaly Markot, Mike Mowbray, Karl-Hermann Neeb, Hermann Schichl, Peter Schodl, and Tapio Schneider, who contributed through their comments on earlier versions of parts of the book.

Though we added a considerable amount of additional material, we would have liked to be more complete in many respects. But a term has only this many hours, and our time to extend and polish the lectures after they were given was also limited. Nevertheless, we believe that the topics treated are the fundamental ones, whose understanding gives a solid foundation to assess the wealth of material on other topics.

The audience of the course consisted mainly of mathematics students shortly before finishing their diploma or doctorate degree and a few postgraduates, mostly with only a high school level background knowledge in physics.

Thus we assume some mathematical background knowledge, but only a superficial acquaintance with physics, at the level of what, for example, is available to readers of the Scientific American. It is assumed that the reader has a good command of matrix algebra (including complex numbers) and knows basic properties of vector spaces, linear algebra (including eigenvalues and bilinear forms), groups (up to quotient groups), ordinary differential equations, elementary topology, and Hilbert spaces. However, the relevant definition are repeated (though somewhat tersely) to fix the (not always standardized) notation and terminology used. No background in Lie algebras, Lie groups, or differential geometry is assumed.

While we give precise definitions of almost all mathematical concepts and notation encountered, we avoid the deeper use of functional analysis and differential geometry without being mathematically inaccurate. This is achieved by concentrating on situations that have no special topological difficulties and only need a single chart. We use (the little we need about) measures, distributions, manifolds, surface integrals, and smoothness in a rigorous way, though without giving an explanation from scratch. But we mention where one would have to be more careful about existence or convergence issues when generalizing to infinite dimensions.

Notation and terminology. We made effort to choose display, notation, and terminology such that formulas and text are close to physics usage, and easy to read, browse, and digest. In particular, we use far fewer indices than in typical physics books, and far less abstraction than in typical mathematics books. There are generous author, symbol, and subject indices and a detailed table of contents. We put terms in bold face when referred to in the subject index; this includes all newly defined terms, and, on their first occurrence, also alternative names under which certain concepts are known. We also list the (up to first two) authors at each citation, and use small caps to display them for easy search. We use the symbol := to define the formula on its left as being equal to the formula on its right. We use the mathematicians' abbreviations **iff** for "if and only if" (frequently) and **w. l. o. g.** for "without loss of generality" (sparingly). Complex numbers will figure prominently in this book. We write i for the imaginary unit $\sqrt{-1}$, and ι for the purely imaginary number $\iota := i/\hbar$, where $\hbar$ is **Planck's constant** in the form introduced by Dirac [83]. (Planck had used instead the constant $h := 2\pi\hbar$, whose use would cause many additional factors of 2π in our formulas.) In this book, $\hbar$ is simply a fixed positive number, so that ι is purely imaginary with a positive imaginary part. But in SI units, the value of Planck's constant is

$$\hbar := 1.054571817 \cdot 10^{-34}\,\mathrm{Js};$$

quantum physicists often use units in which $\hbar = 1$.

Physics vocabulary is different from mathematical vocabulatory. Mathematically inclined readers of physics texts can encounter difficulties through the way physicists use terms that have a mathematical meaning. Mathematically identical objects might bear different names in the physics literature, depending on the context, the aspect under scrutiny, or the interpretation. Even worse, the same concept or notation may mean slightly different things to mathematicians and physicists. In particular, this forced us to distinguish between mathematicians' Lie algebras with the commutator $X \angle Y := [X, Y]$ as Lie product and physicists' Lie algebras with the scaled commutator $X \angle Y := \iota[X, Y]$ as Lie product. Also, physicists are accustomed to many words that mathematicians do not use. We hope to have given an account that enables the mathematically inclined reader to overcome such difficulties.

Basic ideas and principles. Quantum mechanics cannot be understood without some background in physics, especially in classical mechanics. Indeed, as we shall see, although classical mechanics and quantum mechanics differ in some respects, they are very closely related in others, and share many concepts.

For the mathematicians, most of the folklore vocabulary of physicists may not be familiar. Besides reviewing the required background knowledge from linear algebra, Part I therefore gives, as a prelude, a general overview of basic ideas and principles in physics, providing the background for the later systematic exposition of the material of this book. On a first reading of the prelude, there is no need to understand things in depth. We merely introduce informally names and formulas for certain concepts from physics, and try to convey the impression that these have important applications to reality, and that there are many interesting solved and unsolved mathematical problems in many areas of theoretical physics.

On the physics side, we usually first present the mathematical models for a physical theory before relating these models to reality. This is adequate both for mathematically-minded readers without much physics knowledge and for physicists who know already on a more elementary level how to interpret the basic settings in terms of real-world examples.

Unlike the mathematical parts, which are developed rigorously, in full generality (within the limitations mentioned before), and in a strictly logical order, we usually introduce physical concepts by means of informal historical interludes, and only discuss simple physical situations in which the relevant concepts can be illustrated. We refer to the general situation only by means of remarks; however, after reading the book, the reader should be able to go deeper into the original literature that treats these topics in greater physical depth.

Probabilities in quantum mechanics are usually introduced in terms of an abstract concept without an intuitive meaning, namely that of a state vector (in simple cases the wave function), which has no classical analogue. In contrast, in this book, our goal is to emphasize a view that shows that classical mechanics and quantum mechanics are very similar. So we shall hardly make use of state vectors and wave functions. Rather than postulating state vectors and probabilities as fundamental, as in the usual approach, we derive the probabilistic view in Chapter 4 in a manner exhibiting the conditions under which this view is appropriate. Therefore, from the point of view of this book, the state vector is only a mathematical artifice to simplify certain calculations of physical interest.

In physics, the where, when, and what of physical situations in general is encoded in terms of fields. The *where* is given by specifying a **position x** in **physical space**, and the *when* is given by specifying a **moment** $t = x_0$ in **physical time**. The allowed combinations $x = (x_0, \mathbf{x})$ determine the possible points in a 4-dimensional **manifold** called **spacetime**. And *what* can be observed at a given position $\mathbf{x}$ at a given moment x_0—temperature, color, matter flow, etc.—is determined by the **value** $\phi(x)$ of a physical **field** ϕ at the point $x = (x_0, \mathbf{x})$, with one field for every observable property.

Different but physically equivalent views of the same objective situation are related by transformations forming a **group** of **motions** on the sets of objects being transformed. As **symmetry groups**, they constitute the basic organizational principle of modern physics. If the motions are continuously differentiable, the group is called a **Lie group**, and the infinitesimal symmetries form a **Lie algebra**. The theory of the latter is, in a sense, the basic workhorse of this book, applied nearly everywhere. We shall see that physics is, in essence, applied Lie algebra.

Later on in the book, definitions will be given in precise mathematical language; the outline in Chapter 1 and the subject index show where. The only exception is that, beyond the overview, given in Part I, we present in this book very little about classical or quantum field theory and statistical mechanics. This effectively restricts spacetime to the 1-dimensional time manifold, and allows us to dispense with the need for differential geometry. However, with this restriction one can still cover everything in physics, where the spatial dependence is either absent, or so restricted that motion proceeds only along predefined paths in space, hence is effectively 1-dimensional.

For the topics not discussed in sufficient detail, we encourage readers to investigate for themselves some of the literature—cited in abundance—to get a better feeling and a deeper understanding than we can offer here.

We hope that you enjoy reading this book!

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