

FROM DESCRIPTIVE FUNCTIONS TO SETS OF ORDERED PAIRS¹

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In this paper I respond to one of the familiar objections to the project in *Principia Mathematica* of reducing mathematics to logic. In particular, I want to refute the notion that *PM* is inferior to a non-logical axiomatic theory such as Zermelo-Fraenkel set theory as a foundation for mathematics because it relies on the obscure notion of “propositional function” in contrast with set theory which provides a simple account of functions as sets of ordered pairs, themselves reduced to certain other sets. Below I will present the theory of “descriptive” functions presented in *PM* and suggest that it was Russell’s view that the account of descriptive functions provides a logicist account which is superior to both the Frege’s account of functions, and the notion of functions as sets of sets. There was a deliberate choice made in *PM* not to found the theory of functions in set theory, and not to identify functions with sets of ordered pairs. In tracing the history of this topic, I will show that current treatments of functions in logic are more sensitive to these issues than one might at first think. Below I will first review the account of “descriptive functions” in *PM*, and then compare this with Frege’s analysis of functions, then show how functions are treated in contemporary logic. Next, I will describe Norbert Wiener’s reduction of relations to sets and then review the evidence that Russell proposed his own account deliberately as an improvement over Frege’s, and finally conclude with some discussion of what this reveals about propositional functions.

¹ An earlier version of this paper was presented at the 31st International Ludwig Wittgenstein Symposium. Thanks to Allen Hazen, James Levine, Paul Oppenheimer and Ed Zalta for discussion of this topic and George Bealer for comments at the conference. A companion to this paper, “Russell and Frege on the Logic of Functions” was presented at the 4th International Symposium for Cognition, Logic and Communication: 200 Years of Analytic Philosophy, at the University of Latvia in Riga, August 29, 2008, and will appear in the 4th issue of the *Baltic International Yearbook for Cognition, Logic and Communication*.

The notion of “descriptive function” in *PM* makes essential use of the theory of definite descriptions and, indeed, that use seems to be the sole technical function of the theory of descriptions in the work. The theory of descriptions in *12 is based on a pair of contextual definitions, which allow the elimination of expressions for definite descriptions from the contexts in which they occur. The primary definition is:

$$*14.01. [(\zeta x)\phi x] . \psi(\zeta x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b : \psi b \quad \text{Df}$$

This can be paraphrased as saying that ‘the ϕ is ψ ’ means the same as ‘There is a b such that anything x is ϕ if and only if that x is identical with b , and that b is ψ ’. Here ‘ ψ ’ is the context from which the description ‘ $(\zeta x)(\phi x)$ ’ is to be eliminated. That this is the scope of the description is indicated by the prefixed occurrence of the description in square brackets: ‘ $[(\zeta x)(\phi x)]$ ’. This definition allows the replacement of formulas in which definite descriptions appear in subject position. A further contextual definition is provided for the occurrence of descriptions as, ‘ $E!(\zeta x)(\phi x)$ ’, which expresses the assertion that a description is *proper*, that is, that there is exactly one ϕ .

Just as the definitions of *14 allow for the elimination of definite descriptions from different contexts, so the theory of classes in *20 is based on a series of contextual definitions. Occurrences of class expressions ‘ $\hat{z}\psi z$ ’ read as ‘the class of z which are ψ ’, can be eliminated from contexts ‘ f ’ via the primary definition:

$$*20.01. f\{\hat{z}(\psi z)\} . = : (\exists \phi) : \phi!x . \equiv_x . \psi x : f\{\phi!\hat{z}\} \quad \text{Df}$$

To say that the class $\hat{z}(\psi z)$ is f is to say that there is some (predicative) function ϕ which is coextensive with ψ and that ϕ is f . There is no explicit mention of scope, but in all regards this definition closely copies that of definite descriptions. The definition of class expressions is completed by a series of other definitions, including those which use variables that range over classes, the “Greek letters” such as ‘ α ’, which are used both as bound (apparent) and free (real) variables for classes. Together, the definitions of *20 provide a reduction of the theory of classes to the theory of propositional functions. One immediate consequence of this definition is that a solution for Russell’s paradox is provided by the restrictions of the theory of types. The “class of all classes that are not members of themselves”, upon

analysis, requires a function to apply to another function of the same type, which is prohibited by the theory of types.² While this “no-classes” theory of classes succeeds in resolving the paradoxes via the elimination of talk of classes in favor of talk about propositional functions, it is precisely at this point that we part ways with the now standard, alternative, project of founding mathematics on axiomatic set theory.

The next section of *Principia Mathematica*, “*20 General Theory of Relations” presents the extension of the “no-classes” theory to the corresponding notion for binary relations, the theory of so-called “relations in extension”. By analogy with the way the no-classes theory of *20 which defines a class expression ‘ $\hat{z}(\psi z)$ ’ using a contextual definition, in *20 we are given contextual definitions for eliminating expressions of the form ‘ $\hat{x} \hat{y} \psi(x,y)$ ’, which represents the “relation in extension” which holds between x and y when $\psi(x,y)$ obtains:

$$*21\cdot01. f\{\hat{x} \hat{y} \psi(x,y)\} . = : (\exists \phi) : \phi! (x,y) . \equiv_{x,y} . \psi(x,y) : f\{\phi! (\hat{u}, \hat{v})\} \text{ Df}$$

The relation of x bearing ψ to y has the property f just in case some predicative function ϕ , which is coextensive with ψ has the property f . From *21 onwards, “Capital Latin Letters”, i.e. ‘R’, ‘S’, ‘T’, etc., are reserved for these relations in extension. They are variables, replaced by such expressions as ‘ $\hat{x} \hat{y} \psi!(x,y)$ ’, as Whitehead and Russell say, “just as we used Greek letters for variable expressions of the form $\hat{z}\phi!z$.” ([PM] 201). These new symbols for relations in extension are written between variables, as in ‘ xRy ’ or ‘ uSv ’. A propositional function would precede the variables, as in ‘ $\phi(x,y)$ ’. (It is not clear how this notation for relations in extension would be extended to three or four place relations. Indeed in general below, as in discussion of the analysis of relations in terms of sets of ordered pairs, the discussion will always be restricted to binary relations.) It should be noted, as Quine has observed, that the intensional propositional functions represented by ‘ ϕ ’ and ‘ ψ ’, etc., drop out here from the development of *Principia Mathematica*, and that from this point on we only encounter relations in extension, symbolized by ‘R’, ‘S’, ‘T’, etc.³

Definite descriptions, though of course very important to the later development of the philosophy of language, do not appear in the later sections of *PM* where the work of reducing mathematics to logic is really carried out. In fact after *30·01 descriptions disappear from the symbolism,

² See my Linsky [2002].

³ See Quine ([1963], p.251).

having performed their most important function. We are now ready for the notion of “descriptive functions.” This takes the form of yet another definition, in this case of the expression ‘R’y’, read as “the R of y”:

$$*30.01. \quad R'y = (\exists x) (xRy) \quad \text{Df}$$

The expression ‘R’y’ is defined by the definite description ‘ $(\exists x)(xRy)$ ’. If ‘ xRy ’ means ‘ x is father of y ’ then ‘R’y’ is ‘the x such that x is father of y ’, or ‘the father of y ’. As Whitehead and Russell point out, this definition is not a contextual definition that shows how expression ‘R’y’ is to be eliminated from a context, such as ‘ $f\{R'y\}$ ’, but rather is simply an explicit instruction about to replace the symbols ‘R’, wherever they occur.

The notion of “descriptive function” provides an analysis of the ubiquitous “mathematical functions” of arithmetic and analysis that are reduced to logical notions in later numbers of *Principia Mathematica*. Whitehead and Russell say:

The functions hitherto considered, with the exception of a few particular functions such as $\alpha \cap \beta$ have been propositional, *i.e.* have had propositions for their values. But the ordinary functions of mathematics, such as x^2 , $\sin x$, $\log x$, are not propositional. Functions of this kind always mean “the term having such and such a relation to x .” For this reason they may be called *descriptive* functions, because they *describe* a certain term by means of its relation to their argument. ([PM], p.231)

Descriptive functions provide *Principia Mathematica*’s analysis of mathematical functions. It is a logicist analysis of mathematical functions in terms of the logical notions of relation in extension and definite descriptions. It has been said that Frege “mathematicized” logic in preparation for his analysis of arithmetic.⁴ That mathematization involved not only the invention of symbolic logic, but also relied on the mathematical notion of function as a primitive notion in his logic. Concepts are functions from objects to truth values. Frege’s notion of the extension of a concept is its course of values, which is a notion that applies to all functions. The notion of course of values is centrally implicated in Russell’s paradox, and so is seen, like Whitehead and Russell’s theory, as one of the unsuccessful logicist attempts to avoid postulating sets as primitive, mathematical, entities. The account of descriptive functions in *30 thus brings out clearly, some might think, the primary objections to Whitehead and Russell’s version of

⁴ By Burton Dreben for one, according to Peter Hylton, ([1993], n.28).

logicism. It relies on notions much better understood within the mathematical theory of sets. A function, on this account, is simply a set of ordered pairs, ordered pairs themselves being sets of a certain sort, and a propositional function would be a function from arguments to propositions. As propositions are not needed for the extensional, first order, logic in which axiomatic set theory is formulated, *30 thus epitomizes the wrong path taken by Whitehead and Russell's version of logicism.

I would like to suggest that an examination of the development of the idea of function in logic from Frege and Russell on into the early part of the twentieth century will defend the notion of descriptive function as a successful way of reducing the mathematical notion of function to logical notions alone.

While it is correct to say that Frege relies on the notion of mathematical function as a primitive, that is not to say that he did not provide a famously original and ground breaking logical analysis of function expressions and variables. Frege's 1891 paper "Function and Concept" and most explicitly his 1904 paper "What is a Function?" talk about the mathematical notion of function, of which concepts are a special case. Frege explains the nature of variables as linguistic entities which may be assigned different values and not as signs of "variable quantities" as many had confusedly described them to be. Frege's further notion of concepts as "unsaturated entities" which are completed by objects and yield truth-values is well known. A function expression in general, and those for mathematical functions among them, will also refer to unsaturated entities which yield objects as values. A function expression, then, such as ' $\sin x$ ', ' x^2 ', and ' $\log x$ ' will have as its *Bedeutung*, or reference an unsaturated entity which, when applied to a number as argument, yields a number as value. The logical status of expressions for functions is that they are "incomplete" names for numbers. Just as Frege had problems in even naming concepts such as "the concept horse", similarly there is a difficulty with naming functions. In fact the sine function ought to be expressed somehow as ' $\sin()$ ' with a blank or hole to indicate its unsaturated nature. The expression ' $\sin x$ ', on the other hand, expresses a given number, the value of the function, for each assignment of a number to the variable ' x '. It is clear from the discussion of the problem of naming concepts that Frege would have rejected Church's lambda notation as a way of naming functions, for example, with ' $\lambda x \sin x$ ' as naming the sine function.

In his *Introduction to Mathematical Logic*, Alonzo Church manages to turn Frege's view into the current standard current view on the logical

syntax of function expressions and terms. Church avoids Frege's talk of function expressions having as a reference (*Bedeutung*) some unsaturated (and unnameable) entity, which, when saturated by an argument, gives a value. Instead we find:

If we suppose the language fixed, every singular form (function expression) has corresponding to it a function f (which we will call the *associated function* of the form) by the rule that the value of f for an argument x is the same as the value of the form for the value x of the free variable of the form ... (Church [1956], p.19)

This account avoids the expressions “denoted” or “designates”, instead using the neutral “corresponding to” and “associated with.” Church wishes to explain the semantics of function expressions without running afoul of Frege’s “concept horse” problem by saying that functional expressions *name* functions.⁵ But this is Frege’s account of the semantics. Church, and those after him for some time, took the difference in kind between functions and objects as a difference of logical type. It was only in the late 1930s that, following Quine, it became standard to view logic as first order logic, and relations and functions, via their reduction to sets of ordered pairs, as themselves just objects.⁶

In contemporary logic texts one finds this tentative reformulation of Frege’s view fully transformed into the now standard account of the role of functions in logic. In the definition of the syntax of formal languages there is a notion of a *term*, and the semantics, based on the notion of the satisfaction of a formula by a sequence, there is a clause for terms. The following, then, is typical. First we specify an set A of variables and logical symbols, and a set S of non-logical relation symbols, function symbols and constants, and the notion of the set of strings of elements of that alphabet, $(A \cup S)^*$.⁷

Definition. *Terms* are those strings in $(A \cup S)^*$ obtained by finitely many applications of the following rules:

- (1) Every variable is a term.
- (2) Every constant is a term.

⁵ Frege introduces this problem in “Function and Concept” ([1891], p.196).

⁶ See Mancuso ([2005], pp.335–339).

⁷ Following Ebbinghaus *et.al.* ([1993], p.15), but Enderton [2001] and others are almost identical. This is a standard account.

- (3) If $t_1, t_2, \dots, t_n$ are terms and f is an n -ary function symbol, then $f t_1, t_2, \dots, t_n$ is a term.

The semantics is based on the notion of a *structure* A for the language, which includes a set D as its domain, and individual c^A in D for each constant c and an n -ary function f^A for each n -ary function symbol f . An *assignment* β in a structure A is a function which maps the variables into the domain D of the structure A . An *interpretation* I is a pair $\langle A, \beta \rangle$ consisting of a structure A and an assignment β in A . The notion of the interpretation of a term, $I(t)$ is defined as follows:

- (a) For a variable x , let $I(x) = \beta(x)$
- (b) For a constant c , let $I(c) = c^A$
- (c) For a function symbol f , let $I(f) = f^A$
- (d) For any n -ary function symbol f and terms $t_1, t_2, \dots, t_n$,

$$I(f t_1, t_2, \dots, t_n) = f^A(I(t_1), I(t_2), \dots, I(t_n))$$

The notion $I \models \phi$ of truth of a formula ϕ on an interpretation I is then defined in the familiar way, and given that sentences are formulas without free variables, the notion $A \models \phi$ of the truth of ϕ in the structure A is defined as truth of ϕ on all interpretations in A .

The notion that functions and relations are sets of ordered pairs is often included in logic texts, but it is either in a separate first chapter on set theoretic “preliminaries” or in a later chapter on set theory as an example of a first order theory, formulable with only one non-logical constant, the relation symbol ‘ $\in$ ’.⁸ As mentioned above, this change came late in the development of logic, and was only finally settled on by Tarski and Quine around 1940. The resulting separation of this theory of functions and relations in set theory from the treatment of functions in logic is universal. One simply won’t find an account by which the value of ‘ $f(t)$ ’ is the “second member of the ordered pair that has the interpretation of t as its first element, from the set of ordered pairs that is the interpretation of f .”

The notion that relations are sets of ordered pairs and of the ordered pair $\langle x, y \rangle$ in turn as a certain cleverly selected set containing sets contain-

⁸ Enderton [2001] has a “Chapter 0: Useful Facts about Sets” and in it we find: “A *function* is a relation F with the property of being *single-valued*: for each x in $\text{dom } F$ there is only one y such that $\langle x, y \rangle \in F$. As usual, this unique y is said to be the value $F(x)$ that F assumes at x .” (Enderton [2001], p.5). Suppes ([1957], pp.229ff.) includes this material in “Part II: Elementary intuitive Set Theory”.

ing x and y was introduced as an improvement to *Principia Mathematica* by Norbert Wiener [1914]. In the introductory material in ([1967] 224-226), van Heijenoort credits the origin of the idea to Hausdorff and Kuratowski, and Wiener himself says that “... what we have done is practically to revert to Schröder’s treatment of a relation as a class of ordered couples.”

In *PM*, however the ordered pair of x and y , $x \downarrow y$, is defined using the relation between elements of α and elements of β in extension, and $\alpha \uparrow \beta$ which is already defined by:

$$*35.04. \alpha \uparrow \beta = \hat{x} \hat{y} (x \in \alpha \cdot y \in \beta) \quad \text{Df}$$

The $\downarrow$ relation is then defined by:

$$*55.01. x \downarrow y = \iota'x \uparrow \iota'y \quad \text{Df}$$

In other words, $x \downarrow y$ holds if x stands in the relation in extension to y that holds just in case $x \in \{x\}$ and $y \in \{y\}$.⁹ This definition should be contrasted with that which Norbert Wiener [1914] proposed, by which the pair is defined as:

$$\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y$$

In modern notation this is $\{\{x\}, \Lambda\}, \{\{y\}\}$. The definition from Kuratowski [1921] is used more commonly now to define ordered pairs as $\{\{x, y\}, x\}$.¹⁰

In his autobiography, *Ex-Prodigy*, Wiener describes this paper as arising out of his reading course on mathematical logic with Russell, and constituted his introduction to writing:

⁹ The ι' descriptive function, which maps a class α onto the class with α as its unique member, the “unit class” of α , began in Peano’s notation as the converse of an operation which mapped a unit class onto its unique member. That converse operation was indicated with the rotated, or “inverted” iota which later symbolized the definite description operator.

¹⁰ It is interesting to note that our now current account of functions as sets of ordered pairs, no two of which have the same *second* member, is in fact a reversal of the order in Russell. Thus “the R of y ” is defined as “the x such that xRy ”, and thus the first member of the relation in extension is the value. One will find a page long discussion (p.24) in Quine [1963], in which he defends this older practice, which he attributes to Peano and Russell, against the later development.

Nevertheless, in connection with the course I did one little piece of work which I later published; and although it excited neither any particular approval on the part of Russell nor any great interest at the time, the paper which I wrote on the reduction of the theory of relations to the theory of classes has come to occupy a certain modest permanent position in mathematical logic. (Wiener [1953] p.191)

While remarking that Russell did not react to the paper, Wiener describes this as a now standard part of mathematical logic. If what we have seen above is correct, of course Russell would find the analysis of relations in terms of classes of ordered pairs, and ordered pairs themselves as further constructions from classes, as getting things front to back. Russell's analysis of functions in terms of relations in extension, and those in terms of propositional functions, is in the opposite direction. Wiener is here claiming credit for the alternative set theory approach to these aspects of the foundations of mathematics, which was eventually to prevail over Whitehead and Russell's logicist account. It is no wonder that Russell did not particularly "approve" of this as an important contribution to mathematical logic.

Given this discussion of mathematical functions, it is possible to shed light on a major issue for the interpretation of Russell's logic, the nature of propositional functions. It is standard to approach the topic of propositional functions by explaining how they differ from the mathematical functions with which we are now familiar, these arbitrary sets of ordered pairs, no two of which have the same second member. How do they differ from mathematical functions?

Peter Hylton [1993] has tried to understand Bertrand Russell's notion of propositional functions by first distinguishing them from the more familiar mathematical functions on which Frege's work are thus based. Hylton's interest is in contrasting Frege's notion of "three-stage" semantics, with its names, *Sinn*, or sense, and *Bedeutung*, or reference, with Russell's more direct "two-stage semantics" of names and referents. Hylton points out that propositional functions "yield propositions as values" which propositions contain their arguments as constituents. A mathematical function simply yields an object as value, in which there is no trace of the argument. The number 4 does not contain any trace of its being the value of the squaring function applied to 2. Instead, if anything, it is the sense of a function expression that embodies the mapping of argument onto value, and preserves the sense of the name of the argument in the sense of the expression for the value. Thus the sense of ' 2^2 ', will record that is the value

of the function for a particular argument, even though the reference of the expression is simply the number 4.

Hylton is right to point out this important aspect of propositional functions. Propositional functions, for Russell, are certainly not mathematical functions from objects to propositions that need not include the argument as a constituent. In fact it was Ramsey, in his 1925 paper, “Mathematical Logic” (Ramsey [1931]), who was the first to propose that propositional functions should be treated as such arbitrary mathematical functions from objects to propositions. Russell reviewed Ramsey’s papers twice (Russell [1931] and [1932]). In the review in *Mind* of 1931 he credits Ramsey with three main objections to *Principia*; “supposing that all classes and relations in extension are definable by finite propositional functions”, then the criticism for which Ramsey is best known, “a failure to distinguish two kinds among the contradictions, of which only one requires the theory of types, which can accordingly be much simplified” and the third, “the treatment of identity.” The second review describes Ramsey’s notion of extensional functions, but expresses some qualms:

If a valid objection exists – as to which I feel uncertain – it must be derived from inquiry into the meaning of “correlation.” A correlation, interpreted in a purely extensional manner, means a collection of ordered pairs. Now such a collection exists if somebody collects it, or if something logical or empirical brings it about. But, if not, in what sense is there such a collection? (Russell [1931], p.117)

Russell seems not to accept the idea of an arbitrary function in extension, which is not determined by some relation, at least for the special case of functions from objects to propositions. On the other hand, Russell clearly understood the consequences of Cantor’s theorem about the cardinality of the set of subsets of a given set, for, as he says later in *My Philosophical Development*, it was Cantor’s theorem, applied to the “class” of everything, that led him to the paradox in the first place.¹¹ So, Russell would certainly have held that there are more classes than expressions for them, and so more functions than there are definable relations.¹² But the further step is to accept that there are more classes than relations, that is, more sets than

¹¹ See Russell, ([MPD], p.58).

¹² It is interesting to note that Hintikka and Sandu [1992] have charged that Frege also did not have the idea of an arbitrary function from objects to truth values, and so did not have the notion of arbitrary set needed to have a “standard” second order logic. Burgess [1995] suggests that the evidence is not so clear.

there are extensions of propositional functions or relations. When Russell speaks of “something logical” which “brings about” a collection, this need not be anything that can be grasped by the human mind, or expressed in a language, as propositional functions, which are part of logic, may go beyond anything definable in language.

Beyond having values that record the identity of their arguments, propositional functions differ even further from the mathematical functions with arguments and values that Frege used throughout his logical theory. The most appropriate semantic theory for a system with propositional functions may not interpret their semantic values as functions at all. This point can be put precisely, as it has by Paul Oppenheimer and Ed Zalta [ms], with their distinction between “relational type theory” and “functional type theory”. They show how in a theory with relations (and propositions) such as Zalta’s “Object Theory”, it is possible to interpret propositional functions as relations, but not as functions. Their point can be illustrated by looking at Alonzo Church’s [1976] notation for the types of propositional functions. His notation does not require a type for propositions. Instead there is a type for individuals, ι and monadic propositional functions with individuals as arguments (ι), but there is no type for propositions. Some type theories, including that of Church himself for other purposes, represent types with symbols for arguments and values. Thus one might use o as a type for propositions, and $\iota \rightarrow o$ as the type for functions from individuals to propositions. But Church’s notation uses the formulation of an empty pair of parentheses $()$ to indicate the type of propositions. This notation indicates the arguments that a propositional function takes, but does not require a semantics of functions from arguments to values as its interpretation. Instead one might simply give truth conditions for the result of “applying” a propositional function $\phi \hat{x}$ to an argument, rather than assigning it a proposition as value in the semantics. Indeed this fits well with Russell’s official abandonment of propositions in the Introduction to *PM*, following the problematic “multiple relation theory of judgment.” Officially, at least, there are no propositions in the type theory of *PM*, even though the logic is a system of propositional functions. Oppenheimer and Zalta show that in a particular formal type theory, that of Zalta’s “Object Theory” there are relational expressions, the counterpart of propositional functions, which simply cannot be treated as denoting functions. Instead they will denote relations, where those are intensional entities, built up with operations on primitive relations using the analogues of operators from algebraic semantics. So, while Ramsey explicitly proposed that propositional functions be

treated as mathematical functions from objects to propositions, and discussions of propositional functions use the language of functional application, there is still no need to interpret propositional functions as a species of mathematical functions.

George Bealer has been pointed out that the notion of mathematical function does at least provide an account of predication.¹³ If the meaning of a predicate F is a function, then the meaning of the result of the application of the predicate F to a subject term t can be seen as the result of the application of the function that interprets F , that is $\| F \|$, to the object that interprets a , that is, $\| t \|$. But a propositional function $F\hat{x}$ is not supposed to be a mathematical function. How then do we explain what happens when F is predicated of t ? We get a proposition Ft , of which $\| t \|$ is a constituent. The propositional function $F\hat{x}$ is not a constituent, so then what relates the function to the proposition which is its value?¹⁴ I prefer to think that Russell saw this as a primitive notion, best not explained in terms of the derivative notion of mathematical function.

Russell's views about the relation between mathematical functions and propositional functions, or relations, are not primarily driven by a reaction to Frege. They seem to be independently motivated. Consider the following from "On Meaning and Denotation" [1903]:

If we take denoting to be fundamental, the natural way to assert a many-one relation will not be xRy but $y = \phi x$. This, of course, is the usual mathematical way; and there is much to be said for it. All the ordinary functions, such as x^2 , $\sin x$, $\log x$, etc., seem to occur more naturally in this form than as $\hat{R}|x$. Again, in ordinary language, "y is the father of x" clearly states an identity, not a relation: it is "y = the father of x". ([CP4], p.340)

But if we take *propositional* functions to be fundamental – as I have always done, first consciously and then unconsciously – we must proceed through relations to get to ordinary functions. For then we start with ordinary functions such as "x is a man"; these are originally the only functions of one variable. To get at functions of another sort, we have to pass through xRy ; but then, with 7, we get all the problems of denoting. And, as we have seen, a form of denoting more difficult than 7 is involved in the use of variables to start with. Thus denoting seems impossible to escape from. ([CP4], p.340)

¹³ At the conference from which this paper derives.

¹⁴ At ([PM] 38) we find: "Thus for example, the proposition "Socrates is human" can be perfectly apprehended without regarding it as a value of the function "x is human." But, famously, Russell held that we must be acquainted with the constituents of a proposition in order to understand, or apprehend it. This requires that functions are not constituents of propositions.

So, Russell does see propositional functions, or rather, relations, as more fundamental than mathematical functions. However, he sees the move to them as problematic, infected with the problems of denoting. Although Russell may have found propositional functions to be more basic than mathematical functions, until he solved the problem of denoting in “On Denoting” [1905] he was not justified in thinking that he had explained the less obvious in terms of the more basic, instead the reduction of mathematical functions led directly to his big problem that concerned him in those days, the problem of denoting.

With a proper theory of denoting, in particular, the theory of descriptions of *12 of *Principia Mathematica*, in hand, Whitehead and Russell are then ready to complete the logicist analysis of mathematical functions as “descriptive functions” in *30. This, one might conclude, is probably the most important role for the theory of definite descriptions in the logicist project of *PM*.

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