Regular Pullbacks

Jason G. Boynton and Sean Sather-Wagstaff

Abstract. We investigate transference of ring-theoretic properties in certain pullback constructions, focusing on the Noetherian property, Prüfer conditions, coherence, the n-generator property, and factorization. This paper contains both new results on the subject and a survey of some of the literature.

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Dedicated to the memory of Jim Brewer, with gratitude and respect.

1 Introduction

Let R be a commutative ring with identity. A classical theorem in dimension theory states that the Krull dimension of the polynomial ring $R[X]$ is between $n+1$ and $2n+1$ where $n = \dim(R)$. Moreover, it was shown by Seidenberg [35, Theorem 3] that for every pair of nonnegative integers m, n such that $n + 1 \leq m \leq 2n + 1$, there exists an integrally closed quasilocal domain R such that $\dim(R) = n$ and $\dim(R[x]) = m$. The ring R is constructed using what we now know of as the "classical" $D + M$ construction, introduced (as best we know) by Krull [30] and popularized by Gilmer [26, Appendix 2].

For the classical construction, start with a valuation domain V containing a retract field K, meaning that $V = K + M$ where M is the unique maximal ideal of V. Let D be a subring of K, and form the subring $D + M \subset V$. This is the situation considered by Dobbs and Papick [16]. A more general version of this is introduced by Brewer and Rutter [10] where the valuation condition in the ring $T = K + M$ is dropped. Brewer and Rutter lay much of the foundation for this subject, focusing on the transference of properties between $D + M$ and the rings D and $K + M$. A sample of their results is contained in the next theorem.

Theorem 1.1. *Given an integral domain of the form $T = K + M$ where K is a field and D is a subring of K, the following statements hold for the ring $R = D + M$:*

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(i) [10, Theorem 4] *The ring R is Noetherian if and only if T is Noetherian, D is a field and $[K : D] < \infty$.*

(ii) [10, Theorem 3] *The ring R is coherent if and only if T is coherent and either (M is T-finite, D is a field, and $[K : D] < \infty$) or (T_M is a valuation ring, D is coherent, and $Q(D) = K$).*

(iii) [10, Theorem 5] *The ring R is a Prüfer domain if and only if T and D are Prüfer domains and $Q(D) = K$.*

(iv) [10, Theorem 10] *The ring R is a Prüfer domain with the n-generator property if and only if T and D are Prüfer domains with the n-generator property.*

The $D + M$ construction is a special case of a pullback. Moreover, it is a particular pullback coming from a *conductor square*. Since this construction is the focus of this paper, we describe it here explicitly. Start with a ring surjection $\eta_1 : T \twoheadrightarrow B$ and an inclusion of rings $\iota_1 : A \hookrightarrow B$ with $B \neq 0$, hence $A \neq 0$. Let R denote the pullback of these maps, that is, the subring of $A \times T$ consisting of all elements (a, t) such that $\iota_1(a) = \eta_1(t)$. The natural maps $\eta_2 : R \twoheadrightarrow A$ and $\iota_2 : R \hookrightarrow T$ yield a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xhookrightarrow{\iota_2} & T \\ {\scriptstyle \eta_2}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ A & \xhookrightarrow{\iota_1} & B \end{array} \qquad (\square)$$

such that $\operatorname{Ker}(\eta_2)$ and $\operatorname{Ker}(\eta_1)$ are isomorphic via ι_2. (We abuse notation in the sequel, viewing R as a subring of T, and writing $\operatorname{Ker}(\eta_2) = \operatorname{Ker}(\eta_1)$.) The common ideal $\operatorname{Ker}(\eta_i)$ is the largest common ideal of R and T; it is denoted C and called the *conductor* of T into R. When C contains a T-regular element, we say that the conductor square ($\square$) is *regular*.

Conductor squares can also be built as follows. Let T be a commutative ring with subring R, and suppose that R and T have a common, non-zero ideal. We call the largest common ideal C the *conductor* of T into R. Setting $A = R/C$ and $B = T/C$, we obtain a commutative diagram ($\square$) which is a conductor square. For additional information on pullbacks, see Fontana, Huckaba, and Papick [21, Chapter I].

It is common in the study of pullback constructions to assume that T is an integral domain and that C is a maximal ideal of T. However, important examples are obtained by allowing zero-divisors in the pullback square. For example, let D be an integral domain with field of fractions K, and let $E = \{e_1, \dots, e_r\} \subset D$. Setting $T = K[X]$ and $C = (X - e_1) \cdots (X - e_r)K[X]$, we have $B = T/C \cong \prod_{i=1}^{r} K$. Using $A = \prod_{i=1}^{r} D$ in the conductor square, we get $R = \operatorname{Int}(E, D) = \{g \in K[X] \mid g(E) \subset D\}$, the ring of integer-valued polynomials on D determined by the subset E. Observe that

the rings A and B are not integral domains. It is worth noting that McQuillan [33, Proposition 5] explicitly identifies $\mathrm{Int}(E, D)$ as $C + \sum_{i=1}^{r} D\phi_i$ where $\phi_1, \ldots, \phi_r$ are the Lagrange interpolation polynomials of degree $r - 1$. Other important examples using pullbacks are collected by Lucas [32].

The point of this paper is to investigate the following question of Chapman and Glaz [12, Open Problem 50]: What ring-theoretic properties transfer in the conductor square ($\Box$) when C is not a maximal (or even a prime) ideal of T? We take our motivation from Theorem 1.1, and from other similar results, e.g., [1, 9, 27, 33].

In this paper, we survey some of the results in the literature for conductor squares, and we include some results that are (as best we know) new. We include specific references for the older results, not necessarily to the original article where they appeared, but we only include proofs for these results in a few cases. Given the wealth of research in this area, we cannot hope to survey every known result. Our choices reflect our current research interests. The articles of Gabelli and Houston [25] and Kabbaj [29] contain excellent surveys of other aspects of this area.

The new results focus on regular conductor squares. Our perspective is that the regularity condition implies that the rings R and T are not too far apart. (For instance, see Proposition 2.5.) This is akin to Glaz's assumption in [27] that the map $R \hookrightarrow T$ be a "flat epimorphism." It is worth noting that Sections 2–5 contain both new and old results, while Sections 6 and 7 consist entirely of survey material.

2 Some Background

We begin with some preliminary results regarding general pullback constructions. Recall that the *total quotient ring* of a commutative ring U is the localization $Q(U) := V^{-1}U$ where V is the set of non-zero-divisors of U. An *overring* of U is a U-algebra W that is isomorphic (as a U-algebra) to a subring of $Q(U)$. Also, given a ring homomorphism $f: U \to W$ and a multiplicatively closed subset $S \subseteq W$, the localization of W as a U-module $S^{-1}W$ is a U-algebra under the natural operations; moreover, it is an $S^{-1}U$-algebra that is isomorphic to the localization $f(S)^{-1}W \simeq (S^{-1}U) \otimes_U W$.

Lemma 2.1. *Consider the conductor square* ($\Box$).

(i) [27, p. 149] *There is an isomorphism* $B \simeq A \otimes_R T$.

(ii) [21, Lemma 1.1.4(3)] *If* $P \in \mathrm{Spec}\,(R)$ *and* $C \not\subseteq P$, *then there is a unique* $Q \in \mathrm{Spec}\,(T)$ *such that* $Q \cap R = P$; *moreover, the induced map* $R_P \to T_Q$ *is an isomorphism.*

(iii) *If* $T \simeq S^{-1}R$ *for some multiplicatively closed set* $S \subset R$, *then* $B \simeq S^{-1}A$; *moreover,* B *is an overring of* A.

(iv) *If C is finitely generated over R, then it is finitely generated over T. The converse holds if $R \hookrightarrow T$ is finite.*

(v) *The extension $A \hookrightarrow B$ is of finite type (resp. integral, finite) if and only if $R \hookrightarrow T$ is of finite type (resp. integral, finite).*

Proof. (iii) If $T \simeq S^{-1}R$, then $B \simeq A \otimes_R T \simeq A \otimes_R (S^{-1}R) \simeq S^{-1}A$. To see that $B \simeq S^{-1}A$ is an overring of A, use the fact that the map $A \hookrightarrow B \simeq S^{-1}A \simeq \eta_2(S)^{-1}A$ is a monomorphism to conclude that $\eta_2(S)$ consists of non-zero-divisors for A, so B is naturally a subring of $Q(A)$.

(iv) For one implication, assume that $C = Rc_1 + \cdots + Rc_n$. Since C is an ideal of R and an ideal of T, we have $C = TC = Tc_1 + \cdots + Tc_n$, so C is finitely generated over T.

For the converse, suppose that $T = Rt_1 + \cdots + Rt_m$ and that $C = Tc_1 + \cdots + Tc_n$. Then for each $c \in C$ we have $c = \sum_{i=1}^{n} s_i c_i = \sum_{i \le n}(\sum_{j \le m} r_{ij} t_j)c_i = \sum_{i,j} r_{ij}(t_j c_i)$. It follows that $\{t_j c_i\}$ is a set of generators for C over R.

(v) If $R \hookrightarrow T$ is of finite type (resp. integral, finite), then $A \hookrightarrow B$ is of finite type (resp. integral, finite) by part (i). The converse holds by [21, Lemma 1.1.4 (7)]. □

It is reasonable to ask if "finitely presented" can be added to the list of finiteness conditions in Lemma 2.1 (v) above. In the result that follows, we find that under certain conditions, this is indeed the case.

Lemma 2.2. *Consider the conductor square* (□).

(i) *If T is finitely presented over R, then B is finitely presented over A.*

(ii) *If T is finitely generated over R, B is finitely presented over A, and C is finitely presented (over R or T), then T is finitely presented over R.*

Proof. (i) Assume that T is finitely presented over R, and consider an exact sequence $R^n \to R^m \to T \to 0$ over R. It follows that T is finitely generated over R. The right-exactness of $- \otimes_R A$ provides an exact sequence

$$R^n \otimes_R A \to R^m \otimes_R A \to T \otimes_R A \to 0$$

over A. From the isomorphism $T \otimes_R A \cong B$, this yields an exact sequence $A^n \to A^m \to B \to 0$ over A, so B is finitely presented over A.

(ii) Assume that T is finitely generated over R, B is finitely presented over A, and C is finitely presented (over R or T). Since T is finitely generated over R, there is an R-module epimorphism $\alpha\colon R^m \twoheadrightarrow T$. To show that T is finitely presented over R, it suffices to show that $\mathrm{Ker}(\alpha)$ is finitely generated over R.

As in the proof of part (i), the right-exactness of $- \otimes_R A$ provides an A-module epimorphism $\overline{\alpha}\colon A^m \twoheadrightarrow B$. The maps α and $\overline{\alpha}$ fit into the following commutative

diagram with exact rows and columns:

$$\begin{array}{ccccc} R^m & \xrightarrow{f^m} & A^m & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \\ T & \xrightarrow{f'} & B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0. & & \end{array}$$

Here, the maps $f: R \to A$ and $f': T \to B$ are the natural ones from the conductor square ($\square$). In particular, we have $\mathrm{Ker}(f^m) = C^m$ and $\mathrm{Ker}(f') = C$. Given this commutative diagram, we conclude that $\alpha(\mathrm{Ker}(f^m)) \subseteq \mathrm{Ker}(f')$, yielding the next commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^m & \xrightarrow{\epsilon} & R^m & \xrightarrow{f^m} & A^m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \\ 0 & \longrightarrow & C & \xrightarrow{i} & T & \xrightarrow{f'} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

Here, the maps i and ϵ are the inclusions, and α' is the restriction of α to $\mathrm{Ker}(f^m) = C^m$.

Claim 1. The map α' is T-linear. The map $\alpha: R^m \to T$ is R-linear, so there are elements $t_1, \ldots, t_m \in T$ such that $\alpha(r_1, \ldots, r_m) = \sum_{j=1}^m r_j t_j$. It follows that $\alpha'(c_1, \ldots, c_m) = \sum_{j=1}^m c_j t_j$, and that α' respects T-scalar multiplication.

Claim 2. The map α' is surjective. (This is a fairly routine diagram chase with a twist.) Let $c \in C$. Since f^m is surjective, there is a vector $\mathbf{r} = (r_1, \ldots, r_m) \in R^m$ such that $\sum_{j=1}^m r_j t_j = \alpha(\mathbf{r}) = 1$. Hence, $c\mathbf{r} = (cr_1, \ldots, cr_m) \in C^m$ is an element such that $\alpha'(c\mathbf{r}) = \sum_{j=1}^m cr_j t_j = c(\sum_{j=1}^m r_j t_j) = c(1) = c$.

Claim 3. The module $\mathrm{Ker}(\alpha')$ is finitely generated over T and over R. Since α' is R-linear and T-linear, we know that $\mathrm{Ker}(\alpha')$ is an R-module and a T-module.

Assume first that C is finitely presented over T. It follows that C^m is finitely generated over T, and the exact sequence

$$0 \to \mathrm{Ker}(\alpha') \to C^m \xrightarrow{\alpha'} C \to 0$$

implies that $\mathrm{Ker}(\alpha')$ is finitely generated over T; see [34, Corollary 3.63]. Since T is finitely generated over R, this implies that $\mathrm{Ker}(\alpha')$ is finitely generated over R.

Assume next that C is finitely presented over R. The argument of the previous paragraph implies that $\mathrm{Ker}(\alpha')$ is finitely generated over R. Since $\mathrm{Ker}(\alpha')$ is a T-module and T is an R-algebra, it follows that $\mathrm{Ker}(\alpha')$ is finitely generated over T. This completes the proof of Claim 3.

The Snake Lemma provides the following exact sequence:

$$0 \to \mathrm{Ker}(\alpha') \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\overline{\alpha}) \to 0.$$

Since B is finitely presented over A, we know that $\mathrm{Ker}(\overline{\alpha})$ is finitely generated over A, hence over R. Claim 3 implies that $\mathrm{Ker}(\alpha')$ is finitely generated over R, so the above exact sequence implies that $\mathrm{Ker}(\alpha)$ is finitely generated over R, as desired. □

The next result concerns local rings and pullbacks. Here, we observe how the local property transfers in a general pullback of the type (□).

Proposition 2.3. *Consider the conductor square* (□).

(i) [21, Lemma 1.1.5] *If R is local, then C is contained in the Jacobson radical of T and there is a* 1-1 *correspondence between the maximal ideals of B and the maximal ideals of T.*

(ii) *If A and T are local rings, then R is a local ring.*

(iii) *The rings R and B are local if and only if A and T are local.*

Proof. (ii) Since A is local, it has a unique maximal ideal $\overline{M}$ for some maximal ideal M of R.

We claim that $C \subseteq N$ for each maximal ideal N of R. By way of contradiction, suppose that $C \not\subseteq N$. Lemma 2.1 (ii) provides a unique prime ideal N' of T such that $N' \cap R = N$. The uniqueness of N', with the fact that N is maximal, implies that N' is maximal. Since T is local, it follows that N' is the unique maximal ideal of T. The fact that T is local and $C \neq T$ implies that $C \subset N'$; but the condition $N' \cap R = N$ contradicts the assumption $C \not\subseteq N$.

Using the claim with the prime correspondence for quotient rings, we conclude that R is local with unique maximal ideal M

(iii) ($\Rightarrow$) As R is local, so is A. As B is local, part (i) implies that T is local.

($\Leftarrow$) As T is local, so is B; and R is local by part (ii). □

Notation 2.4. Given the conductor square (□) and a prime ideal P of R that contains C, we may use the R-flatness of R_P to build a new square ($\square_P$) with conductor ideal C_P displayed below.

$$\begin{array}{ccc} R_P & \hookrightarrow & T_P \\ \downarrow & & \downarrow \\ A_P & \hookrightarrow & B_P \end{array} \qquad (\square_P)$$

Our next result is particular to regular conductor squares of the type (□). Part (i) is from the folklore of the subject; as best we know, parts (ii) and (iii) are new. Although the proofs of parts (i) and (ii) are very straightforward, they prove to be extremely useful in the sequal. In a sense, part (ii) says that T is very close to R. Part (iii) is a generalization of [10, Lemma 1].

Proposition 2.5. *Consider the regular conductor square* (□).

(i) *T is an overring of R.*

(ii) *C contains an isomorphic copy of the R-module T.*

(iii) *If C is finitely generated over R, then every maximal ideal in B contracts to a maximal ideal in R.*

Proof. First we select any T-regular element $c \in C$

(i) One readily checks that the map $T \to Q(R)$ given by $t \mapsto \frac{ct}{c}$ is a well-defined monomorphism.

(ii) Since c is T-regular, there is an R-module isomorphism $T \simeq Tc \subseteq C \subseteq R$.

(iii) Fix a maximal ideal $\mathfrak{n} \subset B$, and let $\mathfrak{N}$ denote the contraction of $\mathfrak{n}$ in T along the surjection $T \twoheadrightarrow B$. Then $\mathfrak{N}$ is a maximal ideal of T containing C. Set $\mathfrak{p} = A \cap \mathfrak{n}$.

Claim. $C/\mathfrak{N}C \neq 0$. Since c is not annihilated by any element of T, we have $0 \neq c/1 \in C_{\mathfrak{N}}$. In particular, the module $C_{\mathfrak{N}}$ is non-zero. Also, since C is finitely generated over R, it is finitely generated over T. Hence $C_{\mathfrak{N}}$ is finitely generated over $T_{\mathfrak{N}}$. Nakayama's Lemma implies that $0 \neq C_{\mathfrak{N}}/\mathfrak{N}C_{\mathfrak{N}} \cong C/\mathfrak{N}C$.

Claim. $C/\mathfrak{N}C$ is a finitely generated $A/\mathfrak{p}$-module. Let $\mathfrak{P}$ denote the contraction of $\mathfrak{p}$ in R along the surjection $R \twoheadrightarrow A$. Via the composition $R \to A \to B$, we have $\mathfrak{P}B = \mathfrak{p}B \subseteq \mathfrak{n}$. It follows that $C/\mathfrak{N}C$ is a module over $R/\mathfrak{P} \cong A/\mathfrak{p}$ via the structure $\overline{a} \cdot \overline{c} = \overline{a \cdot c}$. Since C is finitely generated over R, it follows that $C/\mathfrak{N}C$ is also finitely generated over R, hence over $R/\mathfrak{P} \cong A/\mathfrak{p}$.

It remains to show that $\mathfrak{p}$ is maximal. The quotient $C/\mathfrak{N}C$ is a non-zero vector space over the field $T/\mathfrak{N} \cong B/\mathfrak{n}$, so there is a $B/\mathfrak{n}$-module epimorphism $C/\mathfrak{n}C \twoheadrightarrow B/\mathfrak{n}$. By construction, this is an $A/\mathfrak{p}$-module epimorphism. Since $C/\mathfrak{n}C$ is finitely generated over $A/\mathfrak{p}$, it follows that $B/\mathfrak{n}$ is finitely generated over $A/\mathfrak{p}$. That is, the extension $A/\mathfrak{p} \hookrightarrow B/\mathfrak{n}$ is module-finite. Thus, we have $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{n}) = 0$, so $\mathfrak{p}$ is maximal. □

3 Pullbacks of Noetherian Rings

In this section, we investigate the transference of Noetherianicity in a conductor square of the type (□). We begin by recalling the next result which relates the Noetherianicity of rings in the most general setting for pullbacks.

Theorem 3.1 ([21, Proposition 1.1.7]). *For the conductor square* (□)*, the rings R and B are Noetherian and $R \hookrightarrow T$ is finite if and only if A and T are Noetherian rings and $A \hookrightarrow B$ is finite.*

Theorem 1.1 (i) above suggests the following analogous result for conductor squares (□) in which C contains a T-regular element.

Theorem 3.2. *Consider the conductor square* (□) *and the following conditions:*

(i) *The ring R is Noetherian.*

(ii) *The rings A, T, and B are Noetherian and the extensions $A \hookrightarrow B$ and $R \hookrightarrow T$ are finite.*

(iii) *The rings A and T are Noetherian and the extension $R \hookrightarrow T$ is finite.*

(iv) *The rings A and T are Noetherian and the extension $A \hookrightarrow B$ is finite.*

The implications (iv) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) *always hold. If the conductor square* (□) *is regular, then the conditions* (i)–(iv) *are equivalent.*

Proof. The implication (ii) $\Rightarrow$ (iii) is trivial, and the equivalence (iv) $\Rightarrow$ (iii) is from Lemma 2.1 (v).

(iii) $\Rightarrow$ (ii) Since $A \hookrightarrow B$ is finite, Lemma 2.1 (v) asserts that the map $R \hookrightarrow T$ is finite. Also, the fact that $A \hookrightarrow B$ is finite and A is Noetherian implies that B is Noetherian.

(ii) $\Rightarrow$ (i) Since $R \hookrightarrow T$ is finite and T is Noetherian, Eakin's Theorem [18, Theorem 2] implies that R is Noetherian.

(i) $\Rightarrow$ (iii) Assume that the conductor square (□) is regular. Since R is Noetherian, the ideal C is finitely generated over R and A is Noetherian. Since (□) is regular, Proposition 2.5 (ii) says that C contains an R-submodule that is isomorphic to T. Hence, T is a submodule of a finitely generated module over the Noetherian ring R and so, it too is a finitely generated R-module. It follows that the extension $R \hookrightarrow T$ is finite, which in turn ensures that T is Noetherian. □

The next three examples show why we need to assume that C is regular as an ideal of T in the implications (i) $\Rightarrow$ (n) of Theorem 3.2.

Example 3.3. Let F be a field, and let S be a commutative F-algebra. Consider the rings $R = F \times F$ and $T = F \times S$ with the common ideal $C = F \times 0$. The quotient rings are $A = R/C \cong F$ and $B = T/C \cong S$; under these isomorphisms, the induced map $A \to B$ is the same as the map $F \to S$ giving S its F-algebra structure. In particular, the ring R is Noetherian, but the rings T and B need not be Noetherian. (They are Noetherian if and only if S is Noetherian.) Also, the maps $R \to T$ and $A \to B$ are not necessarily finite. (They are finite if and only if S is finite over F.)

Since pathologies are often easy to construct using products, we present the next examples which do not decompose as products.

Example 3.4. Let F be a field. Consider the rings $R = F[X,Y]/(XY, Y^2)$ and $S = F[\![X,Y]\!]/(XY, Y^2)$ with the natural inclusion $R \to S$. The ideal $C = (Y)R = (Y)S$ is isomorphic to F as an R-module and as an S-module since $XY = 0 = Y^2$. For quotients, we have $A = R/C \cong F[X]$ and $B = T/C \cong F[\![X]\!]$; under these isomorphisms, the induced map $A \to B$ is the same as the natural inclusion $F[X] \to F[\![X]\!]$. The ring $F[\![X]\!]$ is not finitely generated as an $F[X]$-module.[1] It follows that T is not finitely generated as an R-module.

Of course, in the previous example, the rings T and B are Noetherian. This is not the case in the next example.

Example 3.5. Let F be a field. Consider the rings $R = F[X,Y]/(XY, Y^2)$ and $S = F[X, Y, Z_1, Z_2, \ldots]/(XY, Y^2, YZ_1, YZ_2, \ldots)$ with the natural inclusion $R \to S$. The ideal $C = (Y)R = (Y)S$ is isomorphic to F as an R-module and as an S-module since $XY = 0 = Y^2 = Z_i Y$. For quotients, we have $A = R/C \cong F[X]$ and $B = T/C \cong F[X, Z_1, Z_2, \ldots]$; under these isomorphisms, the induced map $A \to B$ is the same as the natural inclusion $F[X] \to F[X, Z_1, Z_2, \ldots]$. The ring $F[X, Z_1, Z_2, \ldots]$ is not Noetherian and is not finitely generated as an $F[X]$-module. It follows that T is not Noetherian and is not finitely generated as an R-module.

4 Pullbacks of Prüfer Rings

In this section we consider the following six extensions of the Prüfer condition to commutative rings with zero-divisors and investigate their behavior in the conductor square ($\Box$).

Definition 4.1. A *fractional ideal* of a commutative ring R is an R-submodule of the total quotient ring $Q(R)$, possibly zero and possibly non-finitely generated. An ideal $I \subseteq R$ is *invertible* if there is a fractional ideal K such that $IK = R$.

(i) R is *semihereditary* if every finitely generated ideal of R is projective.

(ii) R has *weak global dimension* ≤ 1 if every finitely generated ideal of R is flat.

(iii) R is *arithmetical* if its lattice of ideals is distributive.

(iv) R is *Gaussian* if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$.

[1] This is well known, but we do not know of a proper reference. To explain this fact, consider the induced ring homomorphism $F[X]_{(X)} \to F[\![X]\!]$. Since $F[X]_{(X)}$ is not complete (with respect to the ideal-adic topology determined by its maximal ideal), we conclude from [22, Theorem B] that $F[\![X]\!]$ is not finitely generated over $F[X]_{(X)}$, so it is not finitely generated over the subring $F[X]$. Alternately, suppose that $F[\![X]\!]$ were finitely generated over $F[X]_{(X)}$. Since $F[\![X]\!]$ is flat over the local ring $F[X]_{(X)}$, it is free, so there is an $F[X]_{(X)}$-module epimorphism $F[\![X]\!] \twoheadrightarrow F[X]_{(X)}$. Since $F[\![X]\!]$ is finitely generated over $F[X]_{(X)}$, it follows that $F[X]_{(X)}$ is complete, a contradiction.

(v) R is *locally Prüfer* if R_P is a *Prüfer ring* (see condition (vi)) for every prime ideal M of R.

(vi) R is *Prüfer* if every finitely generated regular ideal is invertible.

We say that R satisfies *Prüfer condition* (n) when R satisfies condition (n) from the above list.

It is worth noting that the definitions above are equivalent when R is a domain. Also, for non-domains, to verify locally Prüfer, it is not enough to check localizations at maximal ideals; see [5, Example2.4].

The following characterizations of Prüfer rings will be quite useful for us. The proof is straightforward.

Lemma 4.2. *Let R be any commutative ring.*

(i) *If I is finitely generated and regular then: I is invertible if and only if I is projective if and only if I is locally principal.*

(ii) *If every 2-generated ideal of R is locally principal, then every finitely generated ideal of R is locally principal.*

(iii) *If R is local, then R is Prüfer if and only if every 2-generated regular ideal is principal.*

(iv) *The ring R is Prüfer if and only if every 2-generated regular ideal is locally principal.*

The papers [2, 6] also show that the implications in the next result are strict.

Theorem 4.3 ([2, for $n = \mathrm{i, ii, iii, iv}$] and [6, Theorem 2.2 for $n = \mathrm{iv, v, vi}$]). *For any commutative ring, we have the following implications for Prüfer condition (n):* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi).

The next result relates the Prüfer condition of a ring with its total quotient ring. It is crucial for the main theorem of this section.

Theorem 4.4 ([3, Theorems 3.7 and 3.12 for $n = \mathrm{i, ii, iii, iv, v, vi}$] and [6, Theorem 3.4 for $n = 5$]). *Let R be any commutative ring. Then R has Prüfer condition (n) if and only if R is a Prüfer ring and $Q(R)$ has Prüfer condition (n).*

Use the fact that every overring of a Prüfer ring is again a Prüfer ring (see for example [31, Theorem 10.19]) together with Theorem 4.4 to obtain the following.

Lemma 4.5. *Let R be any commutative ring. If R has Prüfer condition (n) and if T is an overring of R, then T has the same Prüfer condition (n).*

The next result gives more information about overrings in the local case.

Lemma 4.6 ([5, Theorem 3.6]). *Let R be any local commutative ring. If R has Prüfer condition (n) and if T is an overring of R, then T is a local ring with Prüfer condition (n). Moreover, $T = R_P$ for some prime ideal P of R.*

The next result shows that all of the Prüfer conditions are in fact well behaved in $(\square)$ in the local case. Note that the cases $n = \text{v}$ and $n = \text{vi}$ are equivalent in the local case.

Theorem 4.7 ([5, Theorem 4.1]). *Consider the regular conductor square* $(\square)$. *The commutative ring R is a local ring with Prüfer condition (n) if and only if T is a local ring with Prüfer condition (n), A is a local Prüfer ring, and B is an overring of A.*

In the next result, we say that B is *locally an overring of A* if for every prime $P \in \operatorname{Spec}(A)$ the localization B_P is an overring of A_P. Note that the cases $n = \text{i, ii, ii, iv}$ are from [5, Theorem 4.2], but our proof works equally well for all cases.

Theorem 4.8. *Consider the regular conductor square* $(\square)$. *For $n = \text{i, ii, ii, iv, v}$, the ring R has Prüfer condition (n) if and only if T has Prüfer condition (n), A is locally Prüfer, and B is locally an overring of A.*

Proof. $(\Rightarrow)$ Assume that R has Prüfer condition (n). The fact that T has Prüfer condition (n) follows from Proposition 2.5 (i) and Lemma 4.5. Since $n \in \{\text{i, ii, iii, iv, v}\}$, we conclude that R is locally Prüfer by the implication $(n) \Rightarrow$ (v) in Theorem 4.3. To complete this implication, let $P \in \operatorname{Spec}(A)$, and let $\mathfrak{P}$ be the contraction of P in R along the surjection $R \twoheadrightarrow A$. It follows that the localization $(\square_{\mathfrak{P}})$ is a regular conductor square such that $R_{\mathfrak{P}}$ is Prüfer. From Theorem 4.7, it follows that $A_P = A_{\mathfrak{P}}$ is Prüfer and B_P is an overring of A_P. We conclude that A is locally Prüfer and B is locally an overring of A.

$(\Leftarrow)$ Assume that T has Prüfer condition (n), A is locally Prüfer, and B is locally an overring of A. Since $n \in \{\text{i, ii, iii, iv, v}\}$, we conclude that T is locally Prüfer by the implication $(n) \Rightarrow$ (v) in Theorem 4.3.

We claim that R is locally Prüfer. To see this, let $\mathfrak{P} \in \operatorname{Spec}(R)$. If $C \not\subseteq \mathfrak{P}$ then, by Lemma 2.1 (ii), there is a prime ideal $\mathfrak{Q} \subset T$ such that $R_{\mathfrak{P}} \simeq T_{\mathfrak{Q}}$; since T is locally Prüfer, the ring $R_{\mathfrak{P}} \simeq T_{\mathfrak{Q}}$ is Prüfer. Assume that $C \subseteq \mathfrak{P}$. In this case, we have the regular conductor square $\left(\square_{\mathfrak{P}}\right)$. Since $A_{\mathfrak{P}}$ is a local Prüfer ring and $B_{\mathfrak{P}}$ is an overring of $A_{\mathfrak{P}}$, Lemma 4.6 implies that $B_{\mathfrak{P}}$ is a local Prüfer ring. Thus, Proposition 2.3 (i) implies that $T_{\mathfrak{P}}$ is local. Since T is locally Prüfer, the ring $T_{\mathfrak{P}}$ is Prüfer. An application of Theorem 4.7 to the conductor square $(\square_{\mathfrak{P}})$ implies that $R_{\mathfrak{P}}$ is Prüfer. This establishes the claim.

The claim implies that R is Prüfer because of the implication (v) $\Rightarrow$ (vi) in Theorem 4.3. The ring T is an overring of R by Proposition 2.5 (i), so we have $Q(R) = Q(T)$. Since T has Prüfer condition (n), we conclude from Theorem 4.4 that $Q(R) = Q(T)$ has Prüfer condition (n). The fact that R is Prüfer then implies that R has Prüfer condition (n) by another application of Theorem 4.4. □

Question 4.9. Does the conclusion of Theorem 4.8 hold for $n = \text{vi}$?

5 Pullbacks of Coherent Rings

We refer the reader to [27] for more background information on coherent rings, including unspecified terminology.

Definition 5.1. Let R be any commutative ring.

(i) An R-module M is *coherent* if it is finitely generated and if every finitely generated R-submodule of M is finitely presented.

(ii) The ring R is *coherent* if it is coherent as an R-module.

It is well known that every Noetherian ring is coherent, as is every Prüfer domain. Moreover, every semihereditary ring is coherent.

To discuss the behavior of coherence in conductor squares, we recall some facts about coherent rings.

Theorem 5.2 ([27, Theorem 4.1.1]). *Let $\phi\colon R \to T$ be any homomorphism of commutative rings making T into a finitely presented R-module. (For instance, this is the case when $T \cong R/I$ where I is a finitely generated ideal of R.) If R is coherent, then so is S. The converse holds when ϕ is injective.*

The next result represents the first progress on the transference of the coherent property for conductor squares outside of the $D + M$ setting. Note that the term "epimorphism" is used in a category-theory sense, as in [27] ; in particular, an epimorphism of commutative rings need not be surjective.

Theorem 5.3 ([27, Theorems 4.1.4 and 5.1.3]). *Given the conductor square* ($\square$), *suppose that $R \hookrightarrow T$ is a flat epimorphism and that C is flat as an R-module.*

(i) *If R is coherent, then so is T.*

(ii) *If A, T are coherent and C is a maximal ideal of T, then R is coherent.*

(iii) *If A is coherent such that wk.gl.dim.$(A) \leq \infty$ and if T is semihereditary, then R is coherent.*

(iv) *If A is Noetherian and T is coherent, then R is coherent.*

To continue our survey of coherence, we need a few more definitions.

Definition 5.4. Let D be any integral domain with quotient field K.

(i) The *inverse* of a fractional ideal I is the fractional ideal $I^{-1} = (D : I) = \{x \in K \mid xI \subseteq D\}$.

(ii) A fractional ideal I is *divisorial* if $I = (I^{-1})^{-1}$.

(iii) The *v-closure* of a fractional ideal I is $I_v = (I^{-1})^{-1}$. (This is also called the "v divisorial closure" of I.)

(iv) A fractional ideal I is *v-invertible* if $(II^{-1})_v = D$.

(v) A fractional ideal I is *v-finite* if $I^{-1} = J^{-1}$ for some finitely generated fractional ideal J of D.

(vi) The *t-closure* of a fractional ideal I is $I_t = \bigcup\{J_v \mid J \text{ is a finitely generated fractional subideal of } I\}$.

(vii) A fractional ideal I is *t-invertible* if $(II^{-1})_t = D$.

We now recall several coherent-like properties studied in [24].

Definition 5.5. An integral domain D is

(i) *quasicoherent* if every finitely generated ideal I of D has the property that I^{-1} is finitely generated.

(ii) a *v-coherent* if every finitely generated ideal I of D has the property that I^{-1} is v-finite.

(iii) a *finite conductor domain* if the intersection of any two principal ideals of D is finitely generated.

(iv) a *Prüfer v-multiplication domain* (PVMD) if every finitely generated ideal of D is t-invertible.

(v) a *v-domain* if every finitely generated ideal of D is v-invertible.

(vi) a *DVF domain* if every divisorial ideal of D is v-finite.

(vii) a *Mori domain* if it satisfies the ascending chain condition on divisorial ideals.

We summarize the relations between these conditions (from [24]) next:

$$\begin{array}{ccccccc} & & \text{coherent} & \Longrightarrow & \text{quasicoherent} & \Longrightarrow & \text{finite conductor} \\ & & & & \Downarrow & & \\ v\text{-domain} & \Longleftarrow & \text{PVMD} & \Longrightarrow & v\text{-coherent} & \Longleftarrow & \text{DVF} \Longleftarrow \text{Mori.} \end{array}$$

The transference of these coherent-like properties in a special case of ($\square$) is well-studied in [24]. We list the main results of that paper in the three theorems that follow.

Theorem 5.6. *Consider the conductor square* ($\square$) *such that R and T are domains, $Q(A) = B$, and C is a maximal ideal of T.*

(i) [24, Theorem 3.4] *The ring R is v-coherent if and only if A and T are v-coherent and C is a t-ideal of T.*

(ii) [24, Theorem 4.7] *The ring R is (quasi)coherent if and only if A and T are (quasi)coherent and T_C is a valuation domain.*

(iii) [24, Theorem 4.8] *The ring R is a finite conductor domain if and only if A and T are finite conductor domains and T_C is a valuation domain.*

(iv) [24, Theorem 4.20 (1)] *If T is local, then R is a DVF domain if and only if A and T are DVF domains and C is a nonprincipal v-finite divisorial ideal of T.*

Theorem 5.7. *Consider the conductor square* (□) *such that R and T are domains, $Q(A) \neq B$, and C is a maximal ideal of T, so B is a field.*

(i) [24, Theorem 3.5] *The ring R is v-coherent if and only if A and T are v-coherent and either C is not a t-ideal of T or C is a v-finite divisorial ideal of T.*

(ii) [24, Theorems 4.9 and 4.11] *The ring R is (quasi)coherent if and only if T is (quasi)coherent, A is a field with $[B : A] < \infty$, and C is a finitely generated ideal of T.*

(iii) [24, Theorem 4.10] *The ring R is a finite conductor domain if and only if T is a finite conductor domain, A is a field with $[B : A] < \infty$, and C is a finitely generated ideal of T.*

(iv) [24, Theorem 4.20(2)] *If T is local, then R is a DVF domain if and only if A and T are DVF domains and either C is not a t-ideal of T or C is a v-finite divisorial ideal of T.*

Theorem 5.8. *Consider the conductor square* (□) *where R and T are domains and C is a maximal ideal of T.*

(i) [20, Theorem 4.1] *The ring R is a PVMD if and only if A and T are PVMDs, $Q(A) = B$, and T_C is a valuation domain.*

(ii) [24, Theorem 4.15] *The ring R is a v-domain if and only if A and T are v-domains, $Q(A) = B$, and T_C is a valuation domain.*

(iii) [24, Theorem 4.18] *The ring R is a Mori domain if and only if T is a Mori domain and A is a field.*

The final result of this section characterizes the coherency of R in a regular conductor square (□).

Theorem 5.9. *Consider the conductor square* (□) *and the following conditions:*

(i) *The ring R is coherent and the extension $R \hookrightarrow T$ is finite.*

(ii) *The rings A, T, and B are coherent, B is finitely presented over A, and T is finitely presented over R.*

(iii) *The rings A and T are coherent and T is finitely presented over R.*

(iv) *The rings A and T are coherent and B is finitely presented over A.*

The following hold:

(a) *The implications* (iv) $\Leftarrow$ (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) *always hold.*

(b) *If C is finitely generated over R or over T, then we have* (iv) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i).

(c) *If the conductor square* ($\square$) *is regular and C is finitely generated over R or over T, then the conditions* (i)–(iv) *are equivalent.*

Proof. (a) The implication (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (iv) is from Lemma 2.2 (i). For (iii) $\Rightarrow$ (ii), use Lemma 2.2 (i) to conclude that B is finitely presented over A, and use Theorem 5.2 to show that B is coherent. The implication (ii) $\Rightarrow$ (i) also follows from Theorem 5.2, using the fact that T finitely presented over R implies that T is finitely generated over R, by definition.

(b) Assume that C is finitely generated over R or T. We need to show (iv) $\Rightarrow$ (iii), so assume that A and T are coherent and B is finitely presented over A. Lemma 2.1 (v) implies that T is finitely generated over R, thus C is finitely generated over R and over T by Lemma 2.1 (iv). Since T is coherent and C is finitely generated over T, it follows that C is finitely presented over T, and we conclude that T is finitely presented over R by Lemma 2.2 (ii).

(c) Assume that the conductor square ($\square$) is regular and C is finitely generated over R or over T. We need to prove that (i) $\Rightarrow$ (ii), so we assume that R is coherent and the extension $R \hookrightarrow T$ is finite. Lemma 2.1 (iv) implies that C is finitely generated over R and over T. By Proposition 2.5 (ii), the R-module T is isomorphic to an ideal of R. Since T is a finitely generated over the coherent ring R, we conclude that T is finitely presented over R. Theorem 5.2 implies that T is coherent, and that $A = R/C$ is coherent. $\square$

6 The n-generator Property in Pullbacks

This section is devoted to the behavior of the (strong) n-generator property in the conductor square ($\square$). We recall the following definitions.

Definition 6.1. Let R be any commutative ring.

(i) An ideal I of R is *n-generated* if there exist $a_1, \ldots, a_n \in I$ such that $I = (a_1, \ldots, a_n)$.

(ii) An ideal I of R is *strongly n-generated* if for every nonzero $a \in I$, there exist $a_1, \ldots, a_{n-1} \in I$ such that $I = (a, a_1, \ldots, a_{n-1})$. It is also common to say that I is "$(n-1)\frac{1}{2}$-generated."

(iii) The ring R is said to have the *(strong) n-generator property* if every finitely generated ideal is (strongly) n-generated.

We record some familiar examples here.

Example 6.2. (i) A domain has the 1-generator property if and only if it is Bézout, by definition. In particular principal ideal domains have the 1-generator property.

(ii) Every Prüfer domain of finite character has the strong 2-generator property. In particular, Dedekind domains have the strong 2-generator property. See [23, Theorem 2.2 (a)].

(iii) Every integrally closed domain with the 2-generator property is a Prüfer domain by [23, Proposition 1.11].

(iv) It is routine to show that if a commutative ring S has the strong n-generator property ($n \geq 2$), then any proper homomorphic image of S must have the $(n-1)$-generator property.

As best we know, there are no comprehensive theorems in the literature regarding the transference of the n-generator property in the most general setting of ($\square$). For example, the proof of Theorem 1.1 (iv) relies heavily on the fact that T contains a retract field. In the theorem that follows, the retract condition in T is dropped.

Theorem 6.3 ([28, Theorem]). *Suppose that C is a maximal ideal in the conductor square* ($\square$) *and let $I \not\subseteq C$ be an ideal of R. If IA is an n-generated ideal of A and if IT is an m-generated ideal of T, then I is* $\max\{2, n, m\}$*-generated.*

In order to study the n-generator property in a conductor square of the type ($\square$) where C is not a prime ideal, we put a strong condition on the ring T making it a PID. In doing so, we are able to give some partial results regarding the transference of the (strong) n-generator property in a conductor square ($\square$) where C is a finite intersection of maximal ideals. The set up for these results is next.

Definition 6.4. Let D be an integral domain that is not a field. Let K be its field of fractions. In the diagram ($\square$) above, we set $T = K[X]$ and $C = F_1 \cdots F_r K[X]$ where $F_1, \ldots, F_r$ are irreducible polynomials over the field K that are pairwise coprime in $K[X]$. Now we have that $B = \prod_{i=1}^{r} K[\theta_i]$ where, for each index $i \leq r$, the element θ_i is a root of F_i in some extension field of K. If D_i is any subring of $K[\theta_i]$ that contains $D[\theta_i]$, then a conductor square ($\square$) with $A = \prod_{i=1}^{r} D_i$ yields a ring R between $D[X]$ and $K[X]$ with a non-zero conductor from $K[X]$ into R.

$$\begin{array}{ccc} R & \hookrightarrow & K[X] \\ \downarrow & & \downarrow \\ \prod_{i=1}^{r} D_i & \hookrightarrow & \prod_{i=1}^{r} K[\theta_i] \end{array} \qquad (\boxtimes)$$

In this case we will say that R *is defined by a conductor square of the type* ($\boxtimes$). It is worth noting that one can assume without loss of generality in this construction that each F_i is monic with coefficients in D.

Example 6.5. Let D be an integral domain with field of fractions K and let $E = \{e_1, \ldots, e_r\}$ be any finite subset of D. As noted in the introduction, setting $C = (X - e_1) \cdots (X - e_r)K[X]$ and $A = \prod_{i=1}^{r} D$, we find that $R = \mathrm{Int}(E, D) = \{g \in K[X] \mid g(E) \subset D\}$, the ring of integer-valued polynomials on D determined by the subset E, is defined by a conductor square of the type (⊠).

More generally, it was observed by Elliot [19, Proposition 6.1] that $\mathrm{Int}(S, D)$ is defined by the conductor square (□) where S is any subset of D, $T = K[X]$, $B = K^S$, the map $T \twoheadrightarrow B$ is evaluation at S, and $A = D^S$.

Next, we generalize some definitions made for the ring $\mathrm{Int}(E, D)$ where E is a finite subset of D. For more information on the various Skolem properties, see [11].

Definition 6.6. Suppose that R is a domain defined by a conductor square of type (⊠).

(i) We call an ideal $U \subseteq R$ *unitary* if $U \cap D \neq 0$. It is straightforward to show that an ideal U is unitary if and only if $UK[X] = K[X]$ if and only if $U \cap K \neq 0$.

(ii) For each subset $I \subseteq K[X]$ and each element $\theta \in F$ where F is an extension field of K, set $I(\theta) = \{g(\theta) \mid g \in I\} \subseteq F$. Note that in a conductor square of type (⊠), the set $I(\theta_i)$ is an ideal of D_i, moreover it is the ideal ID_i.

(iii) We say that the domain R has the *almost strong super Skolem property* if, for every pair of unitary ideals $U, V \subseteq R$ and every index $k \leq r$, one has $U = V$ if and only if $U(\theta_k) = V(\theta_k)$.[2]

The point of the next few results is to remove the Prüfer assumption from several results of [4]. We begin with [4, Theorem 5.4].

Theorem 6.7 ([4, Theorem 5.4]). *Suppose that R is a domain defined by a conductor square of the type* (⊠). *Then R has the almost strong super Skolem property.*

Proof. Let U and V be unitary ideals of R such that $U(\theta_i) = V(\theta_i)$ for $i = 1, \ldots, r$.

Claim 1. $C \subseteq U \cap V$. (Compare to [4, Lemma 5.2].) We show that $C \subseteq U$; the containment $C \subseteq V$ then follows by symmetry. Let $g \in C$. Since U is unitary, there is a non-zero element $d \in U \cap D$. Since d is a non-zero constant in $K[X]$, we have $g/d \in K[X]$. Furthermore, we have $(g/d)(\theta_i) = g(\theta_i)/d = 0$ since d is constant and F_i divides g. By definition, this means that $g/d \in R$, so the condition $d \in U$ implies that $g = d(g/d) \in dR \subseteq U$, as desired.

Claim 2. For any ideal $I \subseteq R$, we have $IA = (I + C)/C = \bigoplus_{i=1}^{r} I(\theta_i)$. (Compare to [4, Lemma 5.3].) By definition, we have $A = \prod_{i=1}^{r} D_i \cong R/C$, hence the equality $IA = (I + C)/C$. Since the map $R \to A$ is given by $f \mapsto (f(\theta_1), \ldots, f(\theta_r))$, the containment $IA \subseteq \bigoplus_{i=1}^{r} I(\theta_i)$ is routine. For the reverse containment, let $x = (f_1(\theta_1), \ldots, f_r(\theta_r)) \in \bigoplus_{i=1}^{r} I(\theta_i)$ with each $f_i \in I$. Let $e_1, \ldots, e_r \in \prod_i D_i$ denote

[2] Note that this differs from the terminology used in [4], but is consistent with [11].

the primitive idempotents, and fix liftings $t_1, \dots, t_r \in R$. By assumption, this implies that $t_i(\theta_j) = \delta_{ij}$, the Kroenecker delta. The element $f = \sum_i t_i f_i$ is in I, since each f_i is in I, and the image of f in $\prod_i D_i$ is $\sum_i e_i f_i(\theta_i) = x$, so we have $x \in IA$, as desired.

To complete the proof of the theorem, note that the assumption $U(\theta_i) = V(\theta_i)$ for $i = 1, \dots, r$ explains the second equality in the following display

$$U/C = UA = \bigoplus_{i=1}^{r} U(\theta_i) = \bigoplus_{i=1}^{r} V(\theta_i) = VA = V/C.$$

The other equalities are from Claims 1 and 2. It follows that $U = V$, as desired. □

The almost strong super Skolem property guaranteed by the previous result is key for the proof of the next theorem, which in turn yields the two subsequent results. Compare to [4, Theorems 6.2 and 6.3, and Corollary 6.4].

Theorem 6.8. *Let R be a domain defined by a conductor square of the type* (⊠)*, and let $U \subseteq R$ be a unitary ideal.*

(i) *U is principal if and only if there is a non-zero $v \in U \cap K$ such that $U = Rv$ if and only if there is a non-zero $v \in U \cap K$ such that $U(\theta_k) = D_k v$ for each $k \leq r$.*

(ii) *U is strongly 2-generated if and only if $U(\theta_k)$ is principal for each $k \leq r$.*

(iii) *For $n \geq 2$, the ideal U is strongly $(n+1)$-generated if and only if U is n-generated if and only if $U(\theta_k)$ is n-generated for all $k \leq r$.*

Proof. (i) If there is a non-zero $v \in U \cap K$ such that $U = Rv$, then U is principal and $U(\theta_k) = D_k v$ for each $k \leq r$. If U is principal and unitary, write $U = Rv$ for some $v \in U$. The unitary condition implies that $K[X] = UK[X] = vK[X]$, and it follows that v is a constant in $K[X]$, so $v \in K \cap U$.

Assume that there is a non-zero $v \in K \cap U$ such that $U(\theta_k) = D_k v$ for $k = 1, \dots, r$. Set $V = Rv$, and observe that V is necessarily unitary since $v \in V \cap K$. For each index $k \leq r$, we have $V(\theta_k) = D_k v = U(\theta_k)$. Theorem 6.7 implies that $U = V = Rv$.

(ii) ($\Rightarrow$) If U is strongly 2-generated, then its homomorphic image $U(\theta_k)$ is principal in the proper quotient D_k by Example 6.2 (iv).

($\Leftarrow$) Choose any non-zero $f \in U$. Since each $U(\theta_k)$ is principal, we can write $\overline{U} = \prod_{k=1}^{r} D_k d_k$, where each $d_k \in D_k$; moreover, there exist polynomials $r_k \in R$ such that $r_k(\theta_k) = d_k$. Now, form the polynomial $g = \sum_{k=1}^{r} r_k e_k$, where each $\overline{e_k} \in \prod_i D_i$ is the primitive idempotent corresponding to D_k, and set $V' = Rf + Rg$.

If V' is unitary then $V'(\theta_k) = D_k f(\theta_k) + D_k d_k = D_k d_k = U(\theta_k)$, and we are done by Theorem 6.7. If not, then we show (as in [13, Theorem 4]) how to find a polynomial $h \in K[X]$ such that $g' = g + hF_1 \cdots F_r$ is relatively prime to f in $K[X]$;

once this is shown, then $g'(\theta_k) = g(\theta_k) = d_k \in D_k$ for each index k, which implies that $g' \in R$ and $V = Rf + Rg'$ is unitary, so again Theorem 6.7 ensures that $U = V$.

Note that the fact that U is unitary implies that $U(\theta_k) \neq 0$ for each k, so $d_k \neq 0$ and it follows that g is relatively prime to $F_1 \cdots F_r$. Write $f = f_1 h$ where f_1 and h are relatively prime, each irreducible factor of f_1 divides g, and each irreducible factor of h does not divide g. To show that f and $g' = g + hF_1 \cdots F_r$ are relatively prime in $K[X]$, we let $p \in K[X]$ be an irreducible factor of f and show that p does not divide g'. Since p divides $f = f_1 h$, there are two cases.

Case 1: $p \mid f_1$. In this case, we have $p \mid g$ and $p \nmid h$ by construction of f_1. Since g is relatively prime to $F_1 \cdots F_r$ and $p \mid g$, we have $p \nmid F_1 \cdots F_r$, so $p \nmid hF_1 \cdots F_r$ and $p \nmid g + hF_1 \cdots F_r = g'$.

Case 2: $p \mid h$. In this case, we have $p \nmid g$ by construction of h. Since $p \mid h$, we have $p \mid hF_1 \cdots F_r$, so $p \nmid g + hF_1 \cdots F_r = g'$.

(iii) Certainly if U is n-generated, then it is strongly $(n+1)$-generated, and if U is strongly $(n+1)$-generated, then each of the proper homomorphic images $U(\theta_k)$ is n-generated by Example 6.2 (iv). Thus, we assume that $n \geq 2$ and that each $U(\theta_k)$ is n-generated. It suffices to prove that U is also n-generated. Write $U(\theta_k) = D_k d_{1,k} + \cdots + D_k d_{n,k}$; then for each index $k \leq r$, there exist polynomials $a_{i,k} \in R$ such that $a_{i,k}(\theta_k) = d_{i,k}$. Since U is unitary, we have $U(\theta_k) \neq 0$ for each k, so we have $d_{j,k} \neq 0$ for some j. Reorder the $d_{i,k}$ if necessary to assume that $d_{1,k} \neq 0$ for each k. As in [9, Theorem 3], for each $i \leq n$ put $g_i = a_{i,1}e_1 + \cdots + a_{i,r}e_r$, where $\overline{e_k} \in \prod_i D_i$ is the primitive idempotent corresponding to D_k for all $k \leq r$, and set $V = Rg_1 + \cdots + Rg_n$. Note that $g_1(\theta_k) = d_{i,k} \neq 0$ for all k.

If V is unitary, then the condition $U(\theta_k) = V(\theta_k)$ for each $k \leq r$ implies that $U = V$, by Theorem 6.7. If V is not unitary, then, as above, we can find a polynomial $h \in K[X]$ such that $g_1' = g_1 + hF_1 \cdots F_r$ is relatively prime to g_2 in $K[X]$. It follows that $V' = Rg_1' + Rg_2 + \cdots + Rg_n$ is unitary and that $U(\theta_k) = V'(\theta_k)$ for each index $k \leq r$. Again by Theorem 6.7, we get that $U = V$. □

As a consequence of the preceding result, we obtain the following theorem and its corollary describing the behavior of the (strong) n-generator property in a conductor square (⊠).

Theorem 6.9. *Let R be a domain defined by a conductor square of type* (⊠).

(i) *If $n \geq 2$ and R has the strong n-generator property, then D_k has the $(n-1)$-generator property for each index k.*

(ii) *If $n \geq 2$ and D_k has the n-generator property for each index k, then R has the n-generator property.*

(iii) *The ring R has the strong 2-generator property if and only if D_k is Bézout for each index k.*

Proof. Note that every finitely generated ideal in R is isomorphic to a unitary ideal; argue as in [33] or see [7]. Thus, the desired result follows from Example 6.2 (iv) and Theorem 6.8. □

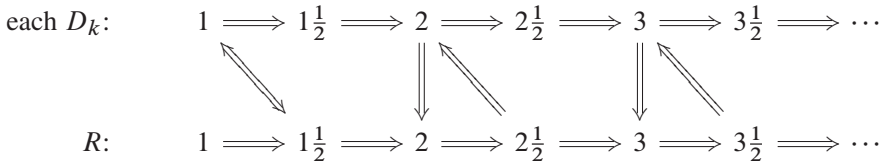
Corollary 6.10. *If R is a domain defined by a conductor square of type* (⊠)*, then the following conditions are equivalent for $n \geq 2$:*

(i) *For each index k, the ring D_k has the n-generator property.*

(ii) *R has the n-generator property.*

(iii) *R has the strong $(n+1)$-generator property.*

We summarize the implications from the preceding results in the next diagram.



In contrast to the previous results, note that the Prüfer hypothesis in the next corollary is crucial; see Example 6.12.

Corollary 6.11 ([4, Theorem 6.6]). *If R is any Prüfer domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that the conductor with respect to $\mathbb{Q}[X]$ is non-zero, then R has the 2-generator property.*

Proof. By [4, Proposition 4.5], the ring R is a Prüfer domain defined by a conductor square of the type (⊠). Since R is integrally closed in $Q(R) = K(X)$, it is integrally closed in $K[X]$. It follows from [21, Lemma 1.1.4 (8)] that $\prod_k D_k$ is integrally closed in $\prod_k K[\theta_k]$, that is, that each D_k is integrally closed in its quotient field $K[\theta_k]$. The containment $\mathbb{Z}[\theta_k] \subset D_k$, implies that each D_k contains the integral closure $\mathbb{Z}_k$ of $\mathbb{Z}$ in $\mathbb{Q}[\theta_k]$. The Krull-Akizuki Theorem says that $\mathbb{Z}_k$ is a Dedekind domain. Since D_k is an overring of $\mathbb{Z}_k$, it too is a Dedekind domain by [31, Theorem 6.21] and therefore has the (strong) 2-generator property. Now apply Corollary 6.10. □

Next, we show that, if $r = 1$ and D_1 is Bézout in the conductor square (⊠), then R need not be Bézout, in contrast to the statement of [4, Theorem 6.3 (4)]. In particular, [4, Example 6.8 (1)] incorrectly states that the ring R in the next example is Bézout. Note that Theorem 6.9 (iii) implies that R does have the strong 2-generator property.

Example 6.12. We consider the specific conductor square

$$\begin{array}{ccc} R & \hookrightarrow & \mathbb{Q}[X] \\ \downarrow & & \downarrow \\ \mathbb{Z}[i] & \hookrightarrow & \mathbb{Q}[i] \end{array}$$

which has conductor ideal $C = (X^2 + 1)$. It is straightforward to show that $R = \mathbb{Z} + \mathbb{Z}X + (X^2+1)\mathbb{Q}[X]$; in other words, a polynomial $f \in \mathbb{Q}[X]$ is in R if and only if the remainder after dividing by X^2+1 is in $\mathbb{Z}[X]$. The ring $\mathbb{Z}[i]$ is Bézout. To show that R is not Bézout, we show that the ideal $U = (X + 1, X^2 + 1)R$ is not principal.

By way of contradiction, suppose that U is principal. Since the polynomials $X + 1$ and $X^2 + 1$ are relatively prime in $\mathbb{Q}[X]$, the ideal U is unitary. Theorem 6.8 (i) provides a non-zero element $c \in U \cap \mathbb{Q}$ such that $U = cR$. Since $X + 1 \in U = cR$, we have $(X + 1)/c \in R$. Given the explicit description of R, the condition $c \in R$ implies that $c \in \mathbb{Z}$, and the condition $(X + 1)/c \in R$ implies that $c = \pm 1$. We conclude that $1 = \pm c \in U = (X + 1, X^2 + 1)R$, so there are elements $p, q \in R$ such that $1 = (X + 1)p + (X^2 + 1)q$. Rewriting p and q using the explicit description of R, we conclude that there are elements $\tilde{p} \in \mathbb{Z}[X]$ and $\tilde{q} \in \mathbb{Q}[X]$ such that $1 = (X+1)\tilde{p}+(X^2+1)\tilde{q}$. Evaluating at i, we obtain the equation $1 = (i+1)\tilde{p}(i)$ which implies that $(1 - i)/2 = 1/(1 + i) = \tilde{p}(i) \in \mathbb{Z}[i]$ a contradiction.

7 Factorization in Pullbacks

In this section, we highlight a few examples in the theory of factorization supplied by pullback constructions. First we recall some relevant definitions.

Definition 7.1. Let D be any integral domain.

(i) We denote by $D^\bullet$ the set of all nonzero nonunits of D.

(ii) We denote by $\mathcal{A}(D)$ the set of all atoms (irreducible elements) of D.

(iii) We call D an *atomic domain* if for every $a \in D^\bullet$, one has a factorization $a = p_1 p_2 \cdots p_n$ where each $p_i \in \mathcal{A}(D)$ and $n \geq 1$.

(iv) We say that D is *ACCP* if it satisfies the ascending chain condition on principal ideals.

(v) We call D a *half factorial domain (HFD)* if for every $a \in D^\bullet$, one has a factorization $a = p_1 p_2 \cdots p_n$ where each $p_i \in \mathcal{A}(D)$ and $n \geq 1$. Moreover, if $a = q_1 q_2 \cdots q_n$ is any other such factorization, then $m = n$.

The following implications are straightforward:

$$\begin{array}{ccccccc} \text{UFD} & \Longrightarrow & \text{HFD} & \Longrightarrow & \text{ACCP} & \Longrightarrow & \text{Atomic} \\ & & & & \Uparrow & & \\ & & \text{Noetherian} & \Longrightarrow & \text{Mori} & & \end{array}$$

It is worth noting that these factorization properties are *not* well behaved in a conductor square of the type (□). In fact, one of the most basic constructions $\mathbb{Z} + X\mathbb{Q}[X]$ is not even atomic [17, Exercise 9.3.4] while the rings $\mathbb{Z}$ and $\mathbb{Q}[X]$ are UFDs. More generally, if $A \subseteq B$, then $A + XB[X]$ is a UFD if and only if $A = B$ and A is a UFD. In order to investigate the weaker half factorial condition in the $A + XB[X]$ construction, [15] makes the following definitions.

Definition 7.2. Let D be any integral domain.

(i) Two nonzero elements $x, y \in D$ are called *v-coprime* if $xD \cap yD = xyD$.

(ii) A subset $S \subset D$ is called a *splitting multiplicative set* of D if every $d \in D$ is expressible as $d = st$ where $s \in S$ and t is v-coprime to every element of S.

We can now give a characterization of the HFD property in the $A + XB[X]$ construction.

Theorem 7.3 ([15, Corollary 3.5]). *Let $A \subseteq B$ be any pair of integral domains such that B has a "proper" element $b \in B$ (no unit of B multiplies b into A). In the conductor square (□), set $T = B[X]$, $C = XB[X]$, and $R = A + XB[X]$. The following statements are equivalent:*

(i) $S = \{g \in R \mid g(0) \neq 0\}$ *is a splitting set of R.*

(ii) *R is an HFD and $A - \{0\}$ is a splitting set of R.*

(iii) *B is integrally closed and $A - \{0\}$ is a splitting set of R.*

Example 7.4. We use the conductor square (□) to exhibit some examples in the theory of factorization.

(i) [32, Example 26] Though it is true that a domain D is a UFD if and only if $D[X]$ is a UFD, the same cannot be said about HFDs. Indeed, the ring $R = \mathbb{R} + X\mathbb{C}[X]$ is a Noetherian HFD while the polynomial ring $R[t]$ is not an HFD. For example, $X \cdot X \cdot (1 + t^2) = X^2 + X^2t^2 = (X + iXt)(X - iXt)$ has an irreducible factorization of length 2 and of length 3.

(ii) [32, Example 27] In [14, Theorem 2.2], it is shown that if D is an integral domain such that its polynomial ring $D[X]$ is an HFD, then D must be integrally closed. However, we cannot conclude that D is completely integrally closed. Let A be any UFD and let X, Y be indeterminates. If $R = A + XA[X, Y]$, then R is an HFD and its polynomial ring $R[t]$ is an HFD as well. However, R is neither a

UFD nor completely integrally closed. For example, X, XY, XY^2 are all atoms so that $X \cdot XY^2 = X^2Y^2 = XY \cdot XY$ is not a unique factorization into atoms. Moreover, $Y \notin R$ while $XY^n \in R$ for all $n \geq 1$ so that R is not completely integrally closed.

(iii) [32, Example 25] The integral closure $\overline{D}$ of an atomic integral domain D may not be atomic. Let $\overline{\mathbb{Z}}$ denote the set of all algebraic integers and set $R = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$. Then R satisfies ACCP and is therefore atomic. However, the integral closure $\overline{R} = \overline{\mathbb{Z}}[X]$ of R is not atomic.

We conclude this paper with a result that guarantees that the ring R in the conductor square ($\boxtimes$) is atomic.

Theorem 7.5 ([8]). *For the conductor square* ($\boxtimes$)*, we set* $C = X(X-1)K[X]$*, so that* $B = K \times K$ *and* $A = D_1 \times D_2$*. Also, set* $S = \{d_1d_2 \mid d_1 \in D_1, d_2 \in D_2\}$ *and* $\mathfrak{J}_0(R) = R \cap K$*. If the following conditions hold, then the ring* R *defined by the conductor square* ($\boxtimes$) *is atomic:*

(i) $S = K$.

(ii) *Every nonunit of* $\mathfrak{J}_0(R)$ *is also a nonunit of* D_1 *and* D_2.

(iii) *The* $\mathfrak{J}_0(R)$*-modules* D_1 *and* D_2 *satisfy ACC on their cyclic submodules.*

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Author Information

Jason G. Boynton, Department of Mathematics, North Dakota State University, Fargo, ND, USA.
E-mail: jason.boynton@ndsu.edu

Sean Sather-Wagstaff, Department of Mathematics, North Dakota State University, Fargo, ND, USA.
E-mail: sean.sather-wagstaff@ndsu.edu