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Finiteness and Homological Conditions in Commutative Group Rings

Sarah Glaz and Ryan Schwarz

Abstract. This article surveys the known results for several related families of ring properties in the context of commutative group rings. These properties include finiteness conditions, homological conditions, and conditions that connect these two families. We briefly survey the classical results, highlight the recent progress, and point out open problems and possible future directions of investigation in these areas.

Keywords. Group Rings, Noetherian Rings, Coherent Rings, Finite Conductor Rings, Weak Global Dimension, Von Neumann Regular Rings, Semihereditary Rings, Prüfer Conditions, Zero Divisors, PP Rings, PF Rings.

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In memory of James Brewer, with respect and affection.

1 Introduction

Let R be a commutative ring with identity and let G be an Abelian group written multiplicatively. The group ring RG is the free R module on the elements of G with multiplication induced by G. An element x in RG has a unique expression: $x = \sum_{g \in G} x_g g$, where $x_g \in R$ and all but finitely many x_g are zero. With addition, multiplication, and scalar multiplication by elements of R defined analogously to the standard polynomial operations, RG becomes a commutative R algebra.

Properties of the group ring RG, particularly in conjunction with questions of descent and ascent of these properties between R and RG, have been of interest for at least 70 years. In his book *Commutative Semigroup Rings* [14], Gilmer traces the beginning of a systematic interest in the nature of RG, for general rings R and groups G, to Higman's article [28] published in 1940. The commutative case became of interest in its own right when the general results reached a stage of specialization at the start of the 1960s. Many of the classical ring theoretic results for the commutative case were collected in two books published in 1983–84: Gilmer [14], and Karpilovsky [32].

In this article, we survey the known results for several related families of properties in the context of commutative group rings. These properties include: finiteness properties (such as Noetherianess, coherence, quasi coherence, and finite conductor properties), homological properties (such as weak global dimension behavior, von Neumann regularity, semihereditarity, and regularity properties), and properties which connect

these two families (such as zero divisor controlling conditions, and Prüfer conditions). Most of the work in this area has been done after the publication of [14], and [32], and employs homological algebra techniques – a direction that was not considered in [14], and only marginally touched in [32]. In addition to highlighting the recent progress in these areas, the article points out open problems and possible future directions of investigation.

Section 1 explores finiteness conditions in the commutative group ring setting. Particularly, this section includes the necessary and sufficient conditions for a group ring to be Noetherian or Artinian (Connell [9]); the necessary and sufficient conditions for a group ring to be coherent (Glaz [16]); and a discussion, particular cases, and open questions, regarding the finite-conductor and quasi coherence properties. This discussion brings in a number of factoriality properties that are closely related to the finite conductor condition, such as GCD, G-GCD and UFD conditions.

Section 2 delves into homological conditions in the group ring setting. The section includes the determination of necessary and sufficient conditions for a group ring to be von Neumann regular (Auslander [2], McLaughlin [37], and Villamayor [44]), semihereditary (Glaz [16]), or coherent regular (Glaz [20]). It also exhibits a formula connecting the weak global dimension of RG with the weak global dimension of R and rank G (Douglas [11], Glaz [16]), and ends with a discussion of possible future directions of exploration of properties such as global dimension, Cohen–Macaulay and Gorenstein ring conditions.

Section 3 considers three zero divisor controlling conditions that can be explored by homological techniques: the PP condition, the PF condition, and the condition that $Q(R)$, the total ring of fractions of the ring R, is von Neumann regular. The section includes the determination of conditions for ascent and descent of these properties between R and RG, (Schwarz and Glaz [43]). It also highlights the applications of these results to the exploration of Prüfer conditions in group ring setting (Schwarz and Glaz [43]). The Prüfer conditions under exploration include arithmetical rings, Gaussian rings, locally Prüfer rings, and Prüfer rings.

2 Finiteness Conditions

Let RG be the group ring associated with a commutative ring R and a multiplicative Abelian group G. The first finiteness conditions to be considered were the properties of being a Noetherian or an Artinian ring. The results in this direction are due to Connell [9].

Theorem 2.1 ([9]). *Let R be a commutative ring and let G be an Abelian group. Then RG is a Noetherian ring if and only if R is a Noetherian ring and G is a finitely generated group.*

As a corollary of this result, Connell [9] determined when a commutative group ring is Artinian.

Corollary 2.2 ([9]). *Let R be a commutative ring and let G be an Abelian group. Then RG is an Artinian ring if and only if R is an Artinian ring and G is a finite group.*

The next finiteness property to be considered was coherence. We start by recalling a few definitions.

Definition 2.3. Let R be a commutative ring. R is said to be *stably coherent* if the polynomial rings in finitely many variables over R are all coherent rings.

It is well known that contrary to the situation for Noetherian rings, the coherence of a ring R does not necessarily ascend to $R[x]$, the polynomial ring in one variable over R (see, for example, [17, Chapter 7] for Soublin's example of such a case). But it is still an open question whether the coherence of the polynomial ring in one variable over a ring R implies the coherence of the polynomial rings in any finite number of variables over R. In all cases where the coherence of R ascends to $R[x]$ this is indeed the case, although the proofs do not employ an inductive argument on the number of variables (as is the case for Noetherian rings). The class of rings which are known to be stably coherent is of considerable size. To name a few: Noetherian rings, von Neumann regular rings, semihereditary rings, coherent rings of global dimension two, and others, are all stably coherent rings. For more details see [17, Chapter 7].

Definition 2.4. Let G be a multiplicative Abelian group. The *rank of* G, denoted by $\operatorname{rank} G$ can be defined as follows: A set of non identity elements of G, $\{g_1, \ldots, g_k\}$, is called *independent* if the equation $g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k} = 1$, where $0 < n_i \in \mathbb{Z}$, implies that $g_1^{n_1} = g_2^{n_2} = \cdots = g_k^{n_k} = 1$. An infinite set of elements of G is called independent if every finite subset of it is independent. By Zorn's Lemma, for every group G we can select an independent set of elements that contains only elements of infinite order and is maximal with respect to this property. The cardinality of this set is $\operatorname{rank} G$. Note that if $\operatorname{rank} G > 0$, then G contains a free subgroup of order $\operatorname{rank} G$.

With these definitions we can now describe the conditions under which RG is a coherent ring. The necessary and sufficient conditions for the coherence of RG were found by Glaz [16].

Theorem 2.5 ([16]). *Let R be a commutative ring and let G be an Abelian group.*

(i) *If G is a torsion group, then RG is a coherent ring if and only if R is a coherent ring.*

(ii) *If $0 < \operatorname{rank} G = n < \infty$, then RG is a coherent ring if and only if the polynomial ring in n variables over R is a coherent ring.*

(iii) *If $\operatorname{rank} G = \infty$, then RG is a coherent ring if and only if R is a stably coherent ring.*

Several finiteness properties exist which relax the coherence conditions on the ring without completely eliminating them. Prominent among those are the finite conductor and quasi coherence properties.

Definition 2.6. Let R be a commutative ring. For an element c in R denote by $(0 : c)$ the annihilator of c. R is said to be a *finite conductor ring* if $aR \cap bR$ and $(0 : c)$ are finitely generated ideals of R for all elements a, b, and c in R. A ring R is said to be a *quasi coherent ring* if $a_1R \cap \cdots \cap a_nR$ and $(0 : c)$ are finitely generated ideals of R for all elements $a_1, \ldots, a_n$, and c in R.

The finite conductor property for integral domains first came into prominence in McAdam's work [36]. Quasi coherence for integral domains was defined by Dobbs [10]. The definitions for general rings are due to Glaz [18]. The theory of these rings is developed in [18]; while [19] provides a survey of the results in this direction and a multitude of examples. Among the class of finite conductor (also quasi coherent) rings we count all coherent rings, UFDs, and GCD domains (that is, integral domains where any two non zero elements have a greatest common divisor), and G-GCD domains (that is, integral domains in which the intersection of two invertible ideals is an invertible ideal). Glaz [18] generalized this last class of rings to rings with zero divisors, and called them G-GCD rings (A ring R is a G-GCD ring if principal ideals of R are projective and the intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R). Neither the finite conductor, not the quasi coherence properties have been investigated in the general group ring setting. A few cases, where the finite conductor (or quasi coherent) ring is a particular integral domain, have been solved by Gilmer and Parker [15]. We provide these results in Theorems 2.7 and 2.9 below:

Theorem 2.7 ([15]). *Let R be a commutative ring and let G be an Abelian group. Then RG is a GCD domain if and only if R is a GCD domain and G is a torsion free group.*

Definition 2.8. Let G be an Abelian group. G is said to be *cyclically Noetherian* if G satisfies the ascending chain conditions for cyclic subgroups.

Theorem 2.9 ([15]). *Let R be an integral domain and let G be a torsion free Abelian group. Then RG is a UFD if and only if R is a UFD and G is cyclically Noetherian.*

In the general ring setting, it follows from [19, Proposition 3.2] that both the finite conductor and the quasi coherence properties descend from RG to R. Regarding ascent of these properties from R to RG, [19, Proposition 3.1] reduces the question to the case where G is finitely generated. Beyond this not much is known about the ascent of either property, not even for the simple case, where R is a G-GCD ring and the structure of RG can be made very explicit (for example, when G an infinite cyclic group). We venture to make the following conjecture:

Conjecture 2.10. If R is a G-GCD ring and G is a finitely generated free Abelian group, then the finite conductor and the quasi coherent properties ascend from R to RG.

A further exploration of these conditions in the group ring setting may shed light on a problem that is still open: Are the finite conductor and the quasi coherence properties distinct [18, 19]?

3 Homological Dimensions and Regularity

Let RG be the group ring associated with a commutative ring R and a multiplicative Abelian group G. The first homological condition to be considered in the commutative group ring setting was von Neumann regularity, that is, the case of weak global dimension equal to zero. The determination of conditions under which RG is von Neumann regular, given in Theorem 3.2, was discovered independently, and almost simultaneously, by Auslander in 1957 [2], McLaughlin in 1958 [37], and Villamayor in 1959 [44]. Their work was also the first to mention a condition linking the ring R and the group G that plays an important role in the majority of results involving homological considerations.

Definition 3.1. Let G be an Abelian group and let R be a commutative ring. *R is said to be uniquely divisible by the order of every element of G* if for every g in G of finite order n, n divides every element $r \in R$, and if for $r \in R$, we have $r = ns = nt$ for some $t, s \in R$, then $s = t$.

Since R is a ring with identity, R is uniquely divisible by an integer n if and only if n is a unit in R. It follows that for an Abelian group G the condition of Definition 3.1 is equivalent to asking that for every element g of G, with order of g equal to p, where p is a prime number, p is a unit in R.

Theorem 3.2 ([2,37,44]). *Let R be a commutative ring and let G be an Abelian group. RG is a von Neumann regular ring if and only if the following three conditions hold:*

(i) *R is a von Neumann regular ring.*

(ii) *G is a torsion group.*

(iii) *R is uniquely divisible by the order of every element of G.*

A year after the solution of the von Neumann regular case, Douglas [11] found a general connection between the weak global dimension of R and that of RG. This result was proved independently and by a different method for the case where R is a coherent ring by Glaz in 1987 [16]. Combining the conditions required for the coherence of RG with the formulas for the weak global dimension made it possible to determine when a commutative group ring is semihereditary.

Theorem 3.3 ([11, 16]). *Let R be a commutative ring and let G be an Abelian group. Then* w. gl. dim $RG < \infty$ *if and only if the following three conditions hold:*

(i) w. gl. dim $R < \infty$

(ii) rank $G < \infty$

(iii) *R is uniquely divisible by the order of every element of G*

Moreover, when condition (iii) *holds we have:*

$$\text{w. gl. dim } RG = \text{w. gl. dim } R + \text{rank } G.$$

One corollary of this theorem is another proof of the characterization of von Neumann regular group rings given in Theorem 3.2.

Recall that a ring R is a *semihereditary* ring if every finitely generated ideal of R is projective. The class of semihereditary rings possesses the next level of homological complexity after the class of von Neumann regular rings. Specifically:

Theorem 3.4 ([17]). *Let R be a commutative ring. The following conditions are equivalent:*

(i) *R is a semihereditary ring.*

(ii) *R is a coherent ring and* w. gl. dim $R \leq 1$.

(iii) *$Q(R)$, the total ring of fractions of R, is a von Neumann regular ring and R_m is a valuation domain for every maximal ideal m of R.*

Equipped with this characterization Glaz [16] determined necessary and sufficient conditions for a group ring to be semihereditary.

Theorem 3.5 ([16]). *Let R be a commutative ring and let G be an Abelian group. Then RG is a semihereditary ring if and only if exactly one of the following conditions hold:*

(i) *R is a von Neumann regular ring,* rank $G = 1$, *and R is uniquely divisible by the order of every element of G.*

(ii) *R is a semihereditary ring, G is a torsion group, and R is uniquely divisible by the order of every element of G.*

Since the group ring of an infinite cyclic group over R is isomorphic to $R[x, x^{-1}]$, where x is an indeterminate over R, we obtain as a bonus the following corollary.

Corollary 3.6 ([16]). *Let R be a commutative ring and let x be an indeterminate over R. Then $R[x, x^{-1}]$ is a semihereditary ring if and only if R is a von Neumann regular ring.*

Definition 3.7. A commutative ring R is said to be a *regular ring* if every finitely generated ideal of R has finite projective dimension.

This notion coincides with the usual definition of regularity if the ring R is Noetherian. The notion had been extended to coherent rings with a considerable degree of success. Examples of coherent regular rings include all coherent rings of finite weak global dimension. In particular von Neumann regular rings and semihereditary rings are coherent regular rings. But, contrary to the situation for Noetherian rings, there are local coherent regular rings of infinite weak global dimension. One such ring is, for example, $k[[x_1, x_2, \dots]]$, the power series ring in infinitely many indeterminates over a field k. For a detailed account of the notion of regularity in the context of coherent rings, see [17, Chapter 6].

It was therefore natural that the determination of necessary and sufficient conditions for a group ring to be coherent of finite weak global dimension raised the following question [16]: When is a group ring a coherent regular ring? This was answered by Glaz in the follow up paper [20].

Theorem 3.8 ([20]). *Let R be a commutative ring and let G be an Abelian group such that RG is a coherent ring. Then RG is a regular ring if and only if the following two conditions hold:*

(i) *R is a coherent regular ring.*

(ii) *R is uniquely divisible by the order of every element of G.*

We note that it is not known if this result holds without the coherence assumption. In general, although the notion of regularity of rings makes sense without any finiteness assumption, the theory of regular rings that do not possess some finiteness condition is still to be developed. Not much is known about regular rings that are not, at least, coherent.

There are a number of open homological questions whose solutions will considerably enrich our knowledge of the nature of group rings. A natural occurring question is:

Question 3.9. Under what conditions can one find a formula (perhaps similar to the formula found in Theorem 3.3) that connects the global dimension of RG, the global dimension of R and some invariant of the group G?

Very little progress has been made in this direction. Particularly, the only known result is in the case of global dimension zero, the so called *semisimple rings*. This is a classical result called Maschke's Theorem, stated in Theorem 3.10, which can be found, for example, in [32]. It is not known in general under what conditions a commutative group ring is semisimple or hereditary (that is, of global dimension equal to one).

Theorem 3.10 ([32]). *Let G be a finite group and let K be a field. Then KG is a semisimple ring if and only if the characteristic of K does not divide the order of the group G.*

Another interesting direction to consider is a relatively recent development in coherent ring theory, the extension of the Cohen–Macaulay and Gorenstein ring notions to the non-Noetherian setting. Theories of coherent Cohen–Macaulay and Gorenstein rings have been developed by Hamilton, Marley, and Hummel (see, for example, [25, 26, 29]). [30] provides an in-depth overview of the recent developments in the subject and an extensive bibliography. It will be of much interest to explore the conditions under which coherent group rings acquire the Cohen–Macaulay or Gorenstein properties.

4 Zero Divisor Controlling Conditions

Let RG be the group ring associated with a commutative ring R and a multiplicative Abelian group G. This section focuses on the recent results obtained by Schwarz and Glaz [43] regarding a number of zero divisor controlling conditions that can be explored using homological algebra techniques and some of the applications of these results to Prüfer conditions. The determination of conditions under which RG is a domain goes back Higman's 1940 article [28]:

Theorem 4.1 ([28]). *Let R be a commutative ring and let G be an Abelian group. Then RG is an integral domain if and only if R is an integral domain and G is a torsion free group.*

We note that for a commutative ring the property of being an integral domain may be viewed as a homological condition on principal ideals of the ring. Specifically, a commutative ring R is an integral domain if and only if its principal ideals are free [43]. Therefore Theorem 4.1 states that principal ideals of RG are free if and only if principal ideals of R are free and G is a torsion free group. Related homological conditions on principal ideals yield two other zero divisor controlling conditions.

Definition 4.2. A commutative ring R is said to be a *PP ring* (or *weak Baer ring*) if principal ideals of R are projective. R is said to be a *PF ring* if principal ideals of R are flat.

PP rings were first introduced by Hattori [27] and Endo [12] in 1960. Hattori aimed to develop a torsion theory for modules over general rings. This condition has implications on the nature of the annihilator ideals of elements of the ring, and as such on the nature of the zero divisors. Specifically:

Theorem 4.3 ([4]). *Let R be a commutative ring. The following conditions are equivalent:*

(i) *R is a PP ring.*

(ii) *For every element a in R, the ideal $(0 : a)$ is generated by an idempotent.*

(iii) *Every element of R can be expressed as a product of a non zero divisor and an idempotent.*

Although possessing a weaker condition, PF rings can be more explicitly linked to domains. Specifically:

Theorem 4.4 ([17, 35]). *Let R be a commutative ring. The following conditions are equivalent:*

(i) *R is a PF ring.*

(ii) *R_p is a domain for every prime ideal p of R.*

(iii) *R_m is a domain for every maximal ideal m of R.*

(iv) *R is a reduced ring and every maximal ideal m of R contains a unique minimal prime ideal p. In this case $p = \{r \in R$: there is a $u \in R - m$ such that $ur = 0\}$ and $R_p = Q(R_m)$, the quotient field of R_m.*

The two conditions are related to another zero divisor controlling condition, namely the requirement that $Q(R)$, *the total ring of fractions of R is von Neumann regular*. Denote by Min R the set of all minimal prime ideals of R with the induced Zariski topology. The three zero divisor controlling conditions are closely linked in the theorem below, which is Glaz's [17] correction of a result of Quentel [39, 40]:

Theorem 4.5 ([17,39,40]). *Let R be a commutative ring. The following conditions are equivalent:*

(i) *R is a PP ring.*

(ii) *R is a PF ring and $Q(R)$ is a von Neumann regular ring.*

(iii) *R is a PF ring and* Min R *is compact in the Zariski topology.*

PP and PF rings make frequent appearances in the literature in a great variety of contexts. The condition that $Q(R)$ is von Neumann regular appears classically in the characterization of semihereditary rings, Theorem 3.4, and has also appeared in a wide variety of both classical and current investigations. For a small sample of papers where some or all three of these conditions appear, see [1, 7, 12, 13, 17, 27, 31, 33–35, 39–42]. Recently all three conditions, but particularly the condition requiring the total ring of quotients to be von Neumann regular, played an important role in the development of the theory of Prüfer conditions in rings with zero divisors, see [3–6, 23] and [24] for a comprehensive survey of this area. We further elaborate on this point later in this section. In the context of group rings, the (not necessarily commutative) PP condition was touched in Pelaez and Teply [38] and in Chen and Zan [8]. All three conditions are explored in depth in Schwarz and Glaz [43], which contains further references to other works involving these conditions. Most of the following results are taken from Schwarz and Glaz [43].

We first resolved the case of von Neumann regularity of the total ring of quotients.

Theorem 4.6 ([43]). *Let R be a commutative ring, and let G be a group that is either torsion free or R is uniquely divisible by the order of every element of G. If $Q(R)$ is a von Neumann regular ring, then $Q(RG)$ is a von Neumann regular ring.*

We note that the converse of this theorem is not true, even if the group is torsion free. [43] provides an example of a torsion free group G (in fact infinite cyclic) and a ring R with $Q(RG)$ von Neumann regular, but $Q(R)$ not von Neumann regular.

In contrast, if the group is torsion free the situation for the PP and the PF conditions is more symmetrical.

Theorem 4.7 ([43]). *Let R be a commutative ring, and let G be a torsion free group. Then RG is a PF ring if and only if R is a PF ring.*

Putting there two results together we conclude:

Corollary 4.8 ([43]). *Let R be a commutative ring and let G be a torsion free group. Then RG is a PP ring if and only if R is a PP ring.*

We extended the descent results to the general case:

Theorem 4.9 ([43]). *Let R be a commutative ring and let G be an Abelian group. If RG is a PF ring (respectively, a PP ring), then R is a PF ring (respectively, a PP ring) and G is either a torsion free group or R is uniquely divisible by every element of G.*

The converse of Theorem 4.9 does not hold if G is not a torsion free group for either the PP or the PF case. Chen and Zan [8] provide an example of a ring R which is PP and a group G such that R is uniquely divisible by the order of every element of G, but RG is not a PP ring. Schwarz and Glaz [43] show that in this case RG is not a PF ring either.

We conclude this section with an application of the results we obtained for the three zero divisor controlling conditions in this section to the extension of the Prüfer conditions to rings with zero divisors. Prüfer domains admit many equivalent definitions. Through the years several of these conditions were explored in a general ring setting, and although there are strong connections between them, in general these conditions were not found to be equivalent.

Definition 4.10. A ring R is said to be an *arithmetical ring* if ideals of R_m are totally ordered by inclusion for each maximal ideal m of R. Let R be a commutative ring and let $f \in R[x]$, the polynomial ring in one variable over R. The so-called *content of f*, denoted $c(f)$, is the ideal of R generated by the coefficients of f. R is said to be a *Gaussian ring* if $c(fg) = c(f)c(g)$ for all $f, g \in R[x]$. R is said to be a *Prüfer ring* if every finitely generated regular ideal of R is invertible. R is said to be *a locally Prüfer ring* if R_p is a Prüfer ring for every prime ideal p of R.

In particular, we considered the following extensions of a Prüfer domain notion to rings with zero divisors:

(i) R is a semihereditary ring.

(ii) w. gl. dim $R \leq 1$

(iii) R is an arithmetical ring.

(iv) R is a Gaussian ring.

(v) R is a locally Prüfer ring.

(vi) R is a Prüfer ring.

These six Prüfer conditions had been extensively studied for the last 5 to 7 years. For a comprehensive survey and an extensive list of references on the subject see [4] and [24]. In particular, [3, 6, 22] show that the Prüfer conditions listed above satisfy the following diagram of strict implications.

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)}.$$

Glaz [23] and Bazzoni and Glaz [3, 4] found conditions that allow for reversals of implications for properties (i)–(iv) and (vi), while Boynton [6] covered the same ground for property (v). In particular, the three zero divisor controlling conditions described in this section allow several reversals of implications and if $Q(R)$ is a von Neumann regular ring, then conditions (i)–(vi) are equivalent for the ring R.

Theorem 3.5 gives the exact conditions under which a commutative group ring satisfies the first Prüfer condition, that is, the semihereditary condition. As a corollary of Theorem 3.3, we obtain similar conditions for a commutative group ring to satisfy the second Prüfer condition, that is, w. gl. dim $R \leq 1$.

Theorem 4.11 ([43]). *Let R be a commutative ring and let G be an Abelian group. Then* w. gl. dim $RG = 1$ *if and only if exactly one of the following conditions hold:*

(i) *R is a von Neumann regular ring,* rank $G = 1$*, and R is uniquely divisible by the order of every element of G.*

(ii) w. gl. dim $R = 1$*, G is a torsion group, and R is uniquely divisible by the order of every element of G.*

Using the result of Theorem 4.6 we can prove the following:

Theorem 4.12 ([43]). *Let R be a commutative ring, and let G be a torsion free or a mixed Abelian group. Then the following conditions are equivalent:*

(i) *RG is a semihereditary ring.*

(ii) w. gl. dim $RG = 1$

(iii) *RG is an arithmetical ring.*

(iv) *RG is a Gaussian ring.*

(v) *RG is a locally Prüfer ring.*

(vi) *RG is a Prüfer ring.*

(vii) *R is a von Neumann regular ring and* $\operatorname{rank} G = 1$.

Some of these equivalences (iii), (vi), and (vii) were proved by different methods in [14] and [15]. As a consequence we obtain as a corollary:

Corollary 4.13 ([43]). *Let R be a commutative ring and let x be an indeterminate over R. Then $R[x, x^{-1}]$ satisfies any of the six Prüfer conditions if and only if R is a von Neumann regular ring.*

If G is a torsion group and R is uniquely divisible by the order of every element of G, we obtain a result similar to Theorem 4.12 under the assumption that the total ring of fractions of R is von Neumann regular.

Theorem 4.14 ([43]). *Let R be a commutative ring such that $Q(R)$, the total ring of fractions of R, is von Neumann regular. Let G be a torsion Abelian group and assume that R is uniquely divisible by the order of every element of G. Then the following conditions are equivalent:*

(i) *RG is a semihereditary ring.*

(ii) $\text{w. gl. dim } RG = 1$

(iii) *RG is an arithmetical ring.*

(iv) *RG is a Gaussian ring.*

(v) *RG is a locally Prüfer ring.*

(vi) *RG is a Prüfer ring.*

(vii) *R is a semihereditary ring.*

An example is given in [43], which shows that the conclusions of Theorem 4.14 need not hold without the assumption that $Q(R)$ is von Neumann regular. At this point the conditions on R and G under which RG satisfies any of the individual Prüfer conditions (iii)–(vi) are not clear. We note that there are several scattered results in the literature giving conditions under which a commutative group ring satisfies one of the Prüfer conditions (iii)–(vi), but those seem to be ad hoc conditions that do not generalize. For example, below is a result from [14]:

Theorem 4.15 ([14]). *Let R be a local arithmetical ring with maximal ideal m and let $c = \operatorname{char}(R/m)$. Assume that G is a finite group of prime power order p^k, where $c \neq p$. Then RG is an arithmetical ring.*

Some examples of group rings that satisfy some, but not other of the six Prüfer conditions also muddy the waters (see [43] for more details). The answer may, or may not, lie in the exploration of other zero divisor controlling conditions in RG. In any case, it is worthwhile exploring other zero divisor controlling conditions in the group ring setting. See [21] for a survey of many of these conditions that appear in the literature.

As a concluding remark, we point out that some of the properties described in this article in the setting when R is a commutative ring and G is an Abelian group have been extended to the case where G is an Abelian monoid. But not all the properties described in this article were considered in this case. In particular, the recent work on Prüfer conditions and zero divisor controlling conditions have not yet been considered in the Abelian monoid setting. Given that in the past such extensions yielded rich and interesting results, this is one direction worth pursuing.

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Author Information

Sarah Glaz, Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA.
E-mail: sarah.glaz@uconn.edu

Ryan Schwarz, Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA.
E-mail: ryan.schwarz@uconn.edu