# Edge Ideals: Algebraic and Combinatorial Properties

Susan Morey and Rafael H. Villarreal

**Abstract.** Let $\mathcal{C}$ be a clutter and let $I(\mathcal{C}) \subset R$ be its edge ideal. This is a survey paper on the algebraic and combinatorial properties of $R/I(\mathcal{C})$ and $\mathcal{C}$, respectively. We give a criterion to estimate the regularity of $R/I(\mathcal{C})$ and apply this criterion to give new proofs of some formulas for the regularity. If $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay, we present a formula for the regularity of the ideal of vertex covers of $\mathcal{C}$ and give a formula for the projective dimension of $R/I(\mathcal{C})$. We also examine the associated primes of powers of edge ideals, and show that for a graph with a leaf, these sets form an ascending chain.

**Keywords.** Edge Ideal, Regularity, Associated Prime, Sequentially Cohen–Macaulay.

**2010 Mathematics Subject Classification.** Primary: 13-02, 13F55; Secondary: 05C10, 05C25, 05C65, 05E40.

## 1 Introduction

A *clutter* $\mathcal{C}$ is a finite ground set $X$ together with a family $E$ of subsets of $X$ such that if $f_1, f_2 \in E$, then $f_1 \not\subset f_2$. The ground set $X$ is called the *vertex set* of $\mathcal{C}$ and $E$ is called the *edge set* of $\mathcal{C}$, denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. Clutters are simple hypergraphs and are sometimes called *Sperner families* in the literature. We can also think of a clutter as the maximal faces of a simplicial complex over a ground set. One example of a clutter is a graph with the vertices and edges defined in the usual way.

Let $\mathcal{C}$ be a clutter with vertex set $X = \{x_1, \ldots, x_n\}$ and with edge set $E(\mathcal{C})$. Permitting an abuse of notation, we will also denote by $x_i$ the $i^{\text{th}}$ variable in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. The *edge ideal* of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of $R$ generated by all monomials $x_e = \prod_{x_i \in e} x_i$ such that $e \in E(\mathcal{C})$. Edge ideals of graphs and clutters were introduced in [109] and [39, 47, 53], respectively. The assignment $\mathcal{C} \mapsto I(\mathcal{C})$ establishes a natural one-to-one correspondence between the family of clutters and the family of square-free monomial ideals. Edge ideals of clutters are also called *facet ideals* [39].

This is a survey paper on edge ideals, which includes some new proofs of known results and some new results. The study of algebraic and combinatorial properties of edge ideals and clutters (e.g., Cohen–Macaulayness, unmixedness, normality, normally torsion-freeness, shellability, vertex decomposability, stability of associated primes) is of current interest, see [22, 24, 25, 32, 39, 41, 42, 45, 51, 61, 62, 89, 116] and

The second author was partially supported by CONACyT grant 49251-F and SNI.

the references there. In this paper we will focus on the following algebraic properties: the sequentially Cohen–Macaulay property, the stability of associated primes, and the connection between torsion-freeness and combinatorial problems.

The numerical invariants of edge ideals have attracted a great deal of interest [1,46, 53,77,87,90,91,106,110,117,118]. In this paper we focus on the following invariants: projective dimension, regularity, depth and Krull dimension.

We present a few new results on edge ideals. We give a criterion to estimate the regularity of edge ideals (see Theorem 3.14). We apply this criterion to give new proofs of some formulas for the regularity of edge ideals (see Corollary 3.15). If $\mathcal{C}$ is a clutter and $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay, we present a formula for the regularity of the ideal of vertex covers of $\mathcal{C}$ (see Theorem 3.31) and give a formula for the projective dimension of $R/I(\mathcal{C})$ (see Corollary 3.33). We also give a new class of monomial ideals for which the sets of associate primes of powers are known to form ascending chains (Proposition 4.23).

For undefined terminology on commutative algebra, edge ideals, graph theory, and the theory of clutters and hypergraphs we refer to [10, 28, 102], [38, 110], [7, 54], [17, 97], respectively.

## 2 Algebraic and Combinatorial Properties of Edge Ideals

Let $\mathcal{C}$ be a clutter with vertex set $X = \{x_1, \ldots, x_n\}$ and let $I = I(\mathcal{C}) \subset R$ be its edge ideal. A subset $F$ of $X$ is called *independent* or *stable* if $e \not\subset F$ for any $e \in E(\mathcal{C})$. The dual concept of a stable vertex set is a *vertex cover*, i.e., a subset $C$ of $X$ is a vertex cover of $\mathcal{C}$ if and only if $X \setminus C$ is a stable vertex set. A first hint of the rich interaction between the combinatorics of $\mathcal{C}$ and the algebra of $I(\mathcal{C})$ is that the number of vertices in a minimum vertex cover of $\mathcal{C}$ (the *covering number* $\alpha_0(\mathcal{C})$ of $\mathcal{C}$) coincides with the *height* $\operatorname{ht} I(\mathcal{C})$ of the ideal $I(\mathcal{C})$. The number of vertices in a maximum stable set (the *stability number* of $\mathcal{C}$) is denoted by $\beta_0(\mathcal{C})$. Notice that $n = \alpha_0(\mathcal{C}) + \beta_0(\mathcal{C})$.

A less immediate interaction between the two fields comes from passing to a simplicial complex and relating combinatorial properties of the complex to algebraic properties of the ideal. The Stanley–Reisner complex of $I(\mathcal{C})$, denoted by $\Delta_{\mathcal{C}}$, is the simplicial complex whose faces are the independent vertex sets of $\mathcal{C}$. The complex $\Delta_{\mathcal{C}}$ is also called the *independence complex* of $\mathcal{C}$. Recall that $\Delta_{\mathcal{C}}$ is called *pure* if all maximal independent vertex sets of $\mathcal{C}$, with respect to inclusion, have the same number of elements. If $\Delta_{\mathcal{C}}$ is pure (resp. Cohen–Macaulay, shellable, vertex decomposable), we say that $\mathcal{C}$ is *unmixed* (resp. Cohen–Macaulay, shellable, vertex decomposable). Since minor variations of the definition of shellability exist in the literature, we state here the definition used throughout this article.

**Definition 2.1.** A simplicial complex $\Delta$ is *shellable* if the facets (maximal faces) of $\Delta$ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j-1\}$ with $F_j \setminus F_\ell = \{v\}$.

We are interested in determining which families of clutters have the property that $\Delta_{\mathcal{C}}$ is pure, Cohen–Macaulay, or shellable. These properties have been extensively studied, see [10, 89, 92–94, 100, 102, 110, 111] and the references there.

The above definition of shellable is due to Björner and Wachs [6] and is usually referred to as *nonpure shellable*, although here we will drop the adjective "nonpure". Originally, the definition of shellable also required that the simplicial complex be pure, that is, all facets have the same dimension. We will say $\Delta$ is *pure shellable* if it also satisfies this hypothesis. These properties are related to other important properties [10, 102, 110]:

$$\text{pure shellable} \implies \text{constructible} \implies \text{Cohen–Macaulay} \impliedby \text{Gorenstein}.$$

If a shellable complex is not pure, an implication similar to that above holds when Cohen–Macaulay is replaced by sequentially Cohen–Macaulay.

**Definition 2.2.** Let $R = K[x_1, \ldots, x_n]$. A graded $R$-module $M$ is called *sequentially Cohen–Macaulay* (over $K$) if there exists a finite filtration of graded $R$-modules

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each $M_i/M_{i-1}$ is Cohen–Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

We call a clutter $\mathcal{C}$ *sequentially Cohen–Macaulay* if $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay. As first shown by Stanley [102], shellable implies sequentially Cohen–Macaulay.

A related notion for a simplicial complex is that of *vertex decomposability* [5]. If $\Delta$ is a simplicial complex and $v$ is a vertex of $\Delta$, then the subcomplex formed by deleting $v$ is the simplicial complex consisting of the faces of $\Delta$ that do not contain $v$, and the *link* of $v$ is

$$lk(v) = \{F \in \Delta | v \notin F \text{ and } F \cup \{v\} \in \Delta\}.$$

Suppose $\Delta$ is a (not necessarily pure) simplicial complex. We say that $\Delta$ is vertex-decomposable if either $\Delta$ is a simplex, or $\Delta$ contains a vertex $v$ such that both the link of $v$ and the subcomplex formed by deleting $v$ are vertex-decomposable, and such that every facet of the deletion is a facet of $\Delta$. If $\mathcal{C}$ is vertex decomposable, i.e., $\Delta_{\mathcal{C}}$ is vertex decomposable, then $\mathcal{C}$ is shellable and sequentially Cohen–Macaulay [5, 116]. Thus, we have:

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{sequentially Cohen–Macaulay}.$$

Two additional properties related to the properties above are also of interest in this area. One is the unmixed property, which is implied by the Cohen–Macaulay property.

The other is balanced. To define balanced, it is useful to have a matrix that encodes the edges of a graph or clutter.

**Definition 2.3.** Let $f_1, \ldots, f_q$ be the edges of a clutter $\mathcal{C}$. The *incidence matrix* or *clutter matrix* of $\mathcal{C}$ is the $n \times q$ matrix $A = (a_{ij})$ given by $a_{ij} = 1$ if $x_i \in f_j$ and $a_{ij} = 0$ otherwise. We say that $\mathcal{C}$ is a *totally balanced* clutter (resp. *balanced* clutter) if $A$ has no square submatrix of order at least 3 (resp. of odd order) with exactly two 1's in each row and column.

If $G$ is a graph, then $G$ is balanced if and only if $G$ is bipartite and $G$ is totally balanced if and only if $G$ is a forest [96, 97].

While the implications between the properties mentioned above are interesting in their own right, it is useful to identify classes of ideals that satisfy the various properties. We begin with the Cohen–Macaulay and unmixed properties. There are classifications of the following families in terms of combinatorial properties of the graph or clutter:

($c_1$) [94, 111] unmixed bipartite graphs,

($c_2$) [36, 60] Cohen–Macaulay bipartite graphs,

($c_3$) [109] Cohen–Macaulay trees,

($c_4$) [38] totally balanced unmixed clutters,

($c_5$) [89] unmixed clutters with the König property without cycles of length 3 or 4,

($c_6$) [89] unmixed balanced clutters.

We now focus on the sequentially Cohen–Macaulay property.

**Proposition 2.4** ([43]). *The only sequentially Cohen–Macaulay cycles are $C_3$ and $C_5$.*

The next theorem generalizes a result of [36] (see ($c_2$) above) which shows that a bipartite graph $G$ is Cohen–Macaulay if and only if $\Delta_G$ has a pure shelling.

**Theorem 2.5** ([107]). *Let $G$ be a bipartite graph. Then $G$ is shellable if and only if $G$ is sequentially Cohen–Macaulay.*

Recently Van Tuyl [106] has shown that Theorem 2.5 remains valid if we replace shellable by vertex decomposable.

Additional examples of sequentially Cohen–Macaulay ideals depend on the chordal structure of the graph. A graph $G$ is said to be *chordal* if every cycle of $G$ of length at least 4 has a chord. A *chord* of a cycle is an edge joining two non-adjacent vertices of the cycle. Chordal graphs have been extensively studied, and they can be constructed according to a result of G. A. Dirac, see [21, 63, 104]. A chordal graph is called *strongly chordal* if every cycle $C$ of even length at least six has a chord that

divides $C$ into two odd length paths. A *clique* of a graph is a set of mutually adjacent vertices. Totally balanced clutters are precisely the clutters of maximal cliques of strongly chordal graphs by a result of Farber [37]. Faridi [39] introduced the notion of a simplicial forest. In [62, Theorem 3.2] it is shown that $\mathcal{C}$ is the clutter of the facets of a simplicial forest if and only if $\mathcal{C}$ is a totally balanced clutter. Additionally, a clutter $\mathcal{C}$ is called *$d$-uniform* if all its edges have size $d$.

**Theorem 2.6.** *Any of the following clutters is sequentially Cohen–Macaulay*:

(i) [116] *graphs with no chordless cycles of length other than* 3 *or* 5,

(ii) [43] *chordal graphs,*

(iii) [63] *clutters whose ideal of covers has linear quotients (see Definitions* 2.7 *and* 3.1*),*

(iv) [55] *clutters of paths of length $t$ of directed rooted trees,*

(v) [39] *simplicial forests, i.e., totally balanced clutters,*

(vi) [52] *uniform admissible clutters whose covering number is* 3.

The clutters of parts (i)–(v) are in fact shellable, and the clutters of parts (i)–(ii) are in fact vertex decomposable, see [22, 63, 106, 107, 115, 116]. The family of graphs in part (ii) is contained in the family of graphs of part (i) because the only induced cycles of a chordal graph are 3-cycles.

A useful tool in examining invariants related to resolutions comes from a carefully chosen ordering of the generators.

**Definition 2.7.** A monomial ideal $I$ has *linear quotients* if the monomials that generate $I$ can be ordered $g_1, \ldots, g_q$ such that for all $1 \leq i \leq q-1$, $((g_1, \ldots, g_i) : g_{i+1})$ is generated by linear forms.

If an edge ideal $I$ is generated in a single degree and $I$ has linear quotients, then $I$ has a linear resolution (cf. [39, Lemma 5.2]). If $I$ is the edge ideal of a graph, then $I$ has linear quotients if and only if $I$ has a linear resolution if and only if each power of $I$ has a linear resolution [64].

Let $G$ be a graph. Given a subset $A \subset V(G)$, by $G \setminus A$, we mean the graph formed from $G$ by deleting all the vertices in $A$, and all edges incident to a vertex in $A$. A graph $G$ is called *vertex-critical* if $\alpha_0(G \setminus \{v\}) < \alpha_0(G)$ for all $v \in V(G)$. An *edge critical* graph is defined similarly. The final property introduced in this section is a combinatorial decomposition of the vertex set of a graph.

**Definition 2.8** ([2]). A graph $G$ without isolated vertices is called a *$B$-graph* if there is a family $\mathcal{G}$ consisting of independent sets of $G$ such that $V(G) = \bigcup_{C \in \mathcal{G}} C$ and $|C| = \beta_0(G)$ for all $C \in \mathcal{G}$.

The notion of a *B-graph* is at the center of several interesting families of graphs. One has the following implications for any graph $G$ without isolated vertices [2, 110]:

$$\begin{array}{ccccccc} & & \text{edge-critical} & \Longrightarrow & & & \\ & & & & B\text{-graph} & \Longrightarrow & \text{vertex-critical} \\ \text{Cohen–Macaulay} & \Longrightarrow & \text{unmixed} & \Longrightarrow & & & \end{array}$$

In [2] the integer $\alpha_0(G)$ is called the *transversal number* of $G$.

**Theorem 2.9** ([34, 46]). *If $G$ is a $B$-graph, then $\beta_0(G) \leq \alpha_0(G)$.*

# 3 Invariants of Edge Ideals: Regularity, Projective Dimension, Depth

Let $\mathcal{C}$ be a clutter and let $I = I(\mathcal{C})$ be its edge ideal. In this section we study the regularity, depth, projective dimension, and Krull dimension of $R/I(\mathcal{C})$. There are several well-known results relating these invariants that will prove useful. We collect some of them here for ease of reference.

The first result is a basic relation between the dimension and the depth (see for example [28, Proposition 18.2]):

$$\operatorname{depth} R/I(\mathcal{C}) \leq \dim R/I(\mathcal{C}). \tag{3.1}$$

The deviation from equality in the above relationship can be quantified using the projective dimension, as is seen in a formula discovered by Auslander and Buchsbaum (see [28, Theorem 19.9]):

$$\operatorname{pd}_R(R/I(\mathcal{C})) + \operatorname{depth} R/I(\mathcal{C}) = \operatorname{depth}(R). \tag{3.2}$$

Notice that since in the setting of this article $R$ is a polynomial ring in $n$ variables, $\operatorname{depth}(R) = n$.

Another invariant of interest also follows from a closer inspection of a minimal projective resolution of $R/I$. Consider the minimal graded free resolution of $M = R/I$ as an $R$-module:

$$\mathbb{F}_\star : \quad 0 \to \bigoplus_j R(-j)^{b_{gj}} \to \cdots \to \bigoplus_j R(-j)^{b_{1j}} \to R \to R/I \to 0.$$

The *Castelnuovo–Mumford regularity* or simply the *regularity* of $M$ is defined as

$$\operatorname{reg}(M) = \max\{j - i \,|\, b_{ij} \neq 0\}.$$

An excellent reference for the regularity is the book of Eisenbud [29]. There are methods to compute the regularity of $R/I$ avoiding the construction of a minimal

graded free resolution, see [3] and [50, page 614]. These methods work for any homogeneous ideal over an arbitrary field.

We are interested in finding good bounds for the regularity. Of particular interest is to be able to express $\operatorname{reg}(R/I(\mathcal{C}))$ in terms of the combinatorics of $\mathcal{C}$, at least for some special families of clutters. Several authors have studied the regularity of edge ideals of graphs and clutters [18, 20, 53, 74, 75, 77, 81, 90, 103, 106, 114]. The main results are general bounds for the regularity and combinatorial formulas for the regularity of special families of clutters. The estimates for the regularity are in terms of matching numbers and the number of cliques needed to cover the vertex set. Covers will play a particularly important role since they form the basis for a duality.

**Definition 3.1.** The *ideal of covers* of $I(\mathcal{C})$, denoted by $I_c(\mathcal{C})$, is the ideal of $R$ generated by all the monomials $x_{i_1}\cdots x_{i_k}$ such that $\{x_{i_1},\ldots,x_{i_k}\}$ is a vertex cover of $\mathcal{C}$. The ideal $I_c(\mathcal{C})$ is also called the *Alexander dual* of $I(\mathcal{C})$ and is also denoted by $I(\mathcal{C})^\vee$. The clutter of minimal vertex covers of $\mathcal{C}$, denoted by $\mathcal{C}^\vee$, is called the *Alexander dual clutter* or *blocker* of $\mathcal{C}$.

To better understand the Alexander dual, let $e \in E(\mathcal{C})$ and consider the monomial prime ideal $(e) = (\{x_i | x_i \in e\})$. Then the duality is given by:

$$\begin{array}{rcccl} I(\mathcal{C}) = & (x_{e_1}, x_{e_2}, \ldots, x_{e_q}) & = & \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s \\ & \updownarrow & & \updownarrow & \\ I_c(\mathcal{C}) = (e_1) \cap (e_2) \cap \cdots \cap (e_q) & = & (x_{\mathfrak{p}_1}, x_{\mathfrak{p}_2}, \ldots, x_{\mathfrak{p}_s}), & & \end{array} \tag{3.3}$$

where $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ are the associated primes of $I(\mathcal{C})$ and $x_{\mathfrak{p}_k} = \prod_{x_i\in\mathfrak{p}_k} x_i$ for $1 \le k \le s$. Notice the equality $I_c(\mathcal{C}) = I(\mathcal{C}^\vee)$. Since $(\mathcal{C}^\vee)^\vee = \mathcal{C}$, we have $I_c(\mathcal{C}^\vee) = I(\mathcal{C})$. In many cases $I(\mathcal{C})$ reflects properties of $I_c(\mathcal{C})$ and viceversa [27, 58, 86]. The following result illustrates this interaction.

**Theorem 3.2** ([103]). *Let $\mathcal{C}$ be a clutter. If* $\operatorname{ht}(I(\mathcal{C})) \ge 2$*, then*

$$\operatorname{reg} I(\mathcal{C}) = 1 + \operatorname{reg} R/I(\mathcal{C}) = \operatorname{pd} R/I_c(\mathcal{C}),$$

*where $I_c(\mathcal{C})$ is the ideal of minimal vertex covers of $\mathcal{C}$.*

If $|e| \ge 2$ for all $e \in E(\mathcal{C})$, then this formula says that the regularity of $R/I(\mathcal{C})$ equals 1 if and only if $I_c(\mathcal{C})$ is a Cohen–Macaulay ideal of height 2. This formula will be used to show that regularity behaves well when working with edge ideals with disjoint sets of variables (see Proposition 3.4). This formula also holds for edge ideals of height one [61, Proposition 8.1.10].

**Corollary 3.3.** *If* $\operatorname{ht}(I(\mathcal{C})) = 1$*, then* $\operatorname{reg} R/I(\mathcal{C}) = \operatorname{pd} R/I_c(\mathcal{C}) - 1$.

*Proof.* We set $I = I(\mathcal{C})$. The formula clearly holds if $I = (x_1 \cdots x_r)$ is a principal ideal. Assume that $I$ is not principal. Consider the primary decomposition of $I$

$$I = (x_1) \cap \cdots \cap (x_r) \cap \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m,$$

where $L = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ is an edge ideal of height at least 2. Notice that $I = fL$, where $f = x_1 \cdots x_r$. Then the Alexander dual of $I$ is

$$I^\vee = (x_1, \ldots, x_r, x_{\mathfrak{p}_1}, x_{\mathfrak{p}_2}, \ldots, x_{\mathfrak{p}_m}) = (x_1, \ldots, x_r) + L^\vee.$$

The multiplication map $L[-r] \xrightarrow{f} fL$ induces an isomorphism of graded $R$-modules. Thus $\operatorname{reg}(L[-r]) = r + \operatorname{reg}(L) = \operatorname{reg}(I)$. By the Auslander–Buchsbaum formula, one has the equality $\operatorname{pd}(R/I^\vee) = r + \operatorname{pd}(R/L^\vee)$. Therefore, using Theorem 3.2, we get

$$\operatorname{reg}(R/I) = \operatorname{reg}(R/L) + r \overset{3.2}{=} (\operatorname{pd}(R/L^\vee) - 1) + r = \operatorname{pd}(R/I^\vee) - 1.$$

Thus $\operatorname{reg}(R/I) = \operatorname{pd}(R/I^\vee) - 1$, as required. □

Next we show some basic properties of regularity. The first such property is that regularity behaves well when working with the edge ideal of a graph with multiple disjoint components or with isolated vertices, as can be seen by the following proposition.

**Proposition 3.4** ([117, Lemma 7]). *Let $R_1 = K[\mathbf{x}]$ and $R_2 = K[\mathbf{y}]$ be two polynomial rings over a field $K$ and let $R = K[\mathbf{x}, \mathbf{y}]$. If $I_1$ and $I_2$ are edge ideals of $R_1$ and $R_2$ respectively, then*

$$\operatorname{reg} R/(I_1 R + I_2 R) = \operatorname{reg}(R_1/I_1) + \operatorname{reg}(R_2/I_2).$$

*Proof.* By abuse of notation, we will write $I_i$ in place of $I_i R$ for $i = 1, 2$ when it is clear from context that we are using the generators of $I_i$ but extending to an ideal of the larger ring. Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_m\}$ be two disjoint sets of variables. Notice that $(I_1 + I_2)^\vee = I_1^\vee I_2^\vee = I_1^\vee \cap I_2^\vee$ where $I_i^\vee$ is the Alexander dual of $I_i$ (see Definition 3.1). Hence by Theorem 3.2 and using the Auslander–Buchsbaum formula, we get

$$\operatorname{reg}(R/(I_1 + I_2)) = n + m - \operatorname{depth}(R/(I_1^\vee \cap I_2^\vee)) - 1,$$
$$\operatorname{reg}(R_1/I_1) + \operatorname{reg}(R_2/I_2) = n - \operatorname{depth}(R_1/I_1^\vee) - 1 + m - \operatorname{depth}(R_2/I_2^\vee) - 1.$$

Therefore we need only show the equality

$$\operatorname{depth}(R/(I_1^\vee \cap I_2^\vee)) = \operatorname{depth}(R_1/I_1^\vee) + \operatorname{depth}(R_2/I_2^\vee) + 1.$$

Since $\operatorname{depth}(R/(I_1^\vee + I_2^\vee)) = \operatorname{depth}(R_1/I_1^\vee) + \operatorname{depth}(R_2/I_2^\vee)$, the proof reduces to showing the equality

$$\operatorname{depth}(R/(I_1^\vee \cap I_2^\vee)) = \operatorname{depth}(R/(I_1^\vee + I_2^\vee)) + 1. \tag{3.4}$$

We may assume that $\operatorname{depth}(R/I_1^\vee) \geq \operatorname{depth}(R/I_2^\vee)$. There is an exact sequence of graded $R$-modules:

$$0 \longrightarrow R/(I_1^\vee \cap I_2^\vee) \xrightarrow{\varphi} R/I_1^\vee \oplus R/I_2^\vee \xrightarrow{\phi} R/(I_1^\vee + I_2^\vee) \longrightarrow 0, \tag{3.5}$$

where $\varphi(\overline{r}) = (\overline{r}, -\overline{r})$ and $\phi(\overline{r}_1, \overline{r}_2) = \overline{r_1 + r_2}$. From the inequality

$$\begin{aligned}\operatorname{depth}(R/I_1^\vee \oplus R/I_2^\vee) &= \max\{\operatorname{depth}(R/I_i^\vee)\}_{i=1}^2 = \operatorname{depth}(R/I_1^\vee)\\ &= \operatorname{depth}(R_1/I_1^\vee) + m\\ &> \operatorname{depth}(R_1/I_1^\vee) + \operatorname{depth}(R_2/I_2^\vee) = \operatorname{depth}(R/(I_1^\vee + I_2^\vee))\end{aligned}$$

and applying the depth lemma (see [10, Proposition 1.2.9] for example) to (3.5), we obtain (3.4). □

Another useful property of regularity is that one can delete isolated vertices of a graph without changing the regularity of the edge ideal. The following lemma shows that this can be done without significant changes to the projective dimension as well.

**Lemma 3.5.** *Let $R = K[x_1, \ldots, x_n]$ and $I$ be an ideal of $R$. If $I \subset (x_1, \ldots, x_{n-1})$, and $R' = R/(x_n) \cong K[x_1, \ldots, x_{n-1}]$, then $\operatorname{reg}(R/I) = \operatorname{reg}(R'/I)$ and $\operatorname{pd}_R(R/I) = \operatorname{pd}_{R'}(R'/I)$. Similarly, if $x_n \in I$ and $I' = I/(x_n)$, then $\operatorname{reg}(R/I) = \operatorname{reg}(R'/I')$ and $\operatorname{pd}_R(R/I) = \operatorname{pd}_{R'}(R'/I') + 1$.*

*Proof.* The first projective dimension result follows from the Auslander–Buchsbaum formula since $\operatorname{depth}(R/I) = \operatorname{depth}(R'/I) + 1$ and $\operatorname{depth}(R) = \operatorname{depth}(R') + 1$. Since $\operatorname{depth}(R/I) = \operatorname{depth}(R'/I')$ the second result for projective dimension holds as well. The results for regularity follow from Proposition 3.4 by noting that the regularity of a polynomial ring $K[x]$ is 0, as is the regularity of the field $K = K[x]/(x)$. □

While adding variables to the ring will preserve the regularity, other changes to the base ring, such as changing the characteristic, will affect this invariant. The following example shows that, even for graphs, a purely combinatorial description of the regularity might not be possible. Results regarding the role the characteristic of the field plays in the resolution of the ideal appear in [19, 74].

**Example 3.6.** Consider the edge ideal $I \subset R = K[x_1, \ldots, x_{10}]$ generated by the monomials

$$\begin{array}{llllllllll} x_1x_3, & x_1x_4, & x_1x_7, & x_1x_{10}, & x_1x_{11}, & x_2x_4, & x_2x_5, & x_2x_8, & x_2x_{10}, \\ x_2x_{11}, & x_3x_5, & x_3x_6, & x_3x_8, & x_3x_{11}, & x_4x_6, & x_4x_9, & x_4x_{11}, & x_5x_7, \\ x_5x_9, & x_5x_{11}, & x_6x_8, & x_6x_9, & x_7x_9, & x_7x_{10}, & x_8x_{10}. & & \end{array}$$

Using *Macaulay*2 [49] we get that $\operatorname{reg}(R/I) = 3$ if $\operatorname{char}(K) = 2$, and $\operatorname{reg}(R/I) = 2$ if $\operatorname{char}(K) = 3$.

As mentioned in Theorem 2.6 (ii), chordal graphs provide a key example of a class of clutters whose edge ideals are sequentially Cohen–Macaulay. Much work has been done toward finding hypergraph generalizations of chordal graphs, typically by looking at cycles of edges or at tree hypergraphs [32, 53, 113]. The papers [53, 113, 115] contribute to the algebraic approach that is largely motivated by finding hypergraph generalizations that have edge ideals with linear resolutions.

It is useful to consider the homogeneous components of the ideals when using linear resolutions. Let $(I_d)$ denote the ideal generated by all degree $d$ elements of a homogeneous ideal $I$. Then $I$ is called *componentwise linear* if $(I_d)$ has a linear resolution for all $d$. If $I$ is the edge ideal of a clutter, we write $I_{[d]}$ for the ideal generated by all the squarefree monomials of degree $d$ in $I$.

**Theorem 3.7.** *Let $K$ be a field and $\mathcal{C}$ be a clutter. Then*

(i) [27] *$R/I(\mathcal{C})$ is Cohen–Macaulay if and only if $I_c(\mathcal{C})$ has a linear resolution.*

(ii) [58] *$R/I(\mathcal{C})$ is sequentially Cohen–Macaulay if and only if $I_c(\mathcal{C})$ is componentwise linear.*

(iii) [58] *$I(\mathcal{C})$ is componentwise linear if and only if $I(\mathcal{C})_{[d]}$ has a linear resolution for $d \geq 0$.*

(iv) [44] *If $G$ is a graph, then $I(G)$ has a linear resolution if and only if $G^c$ is chordal.*

(v) [103] *If $G$ is a graph, then* $\operatorname{reg}(R/I(G)) = 1$ *if and only if $I_c(G)$ is Cohen–Macaulay.*

A graph whose complement is chordal is called *co-chordal*. A consequence of this result and Theorem 3.2 is that an edge ideal $I(\mathcal{C})$ has regularity 2 if and only if $\Delta_{\mathcal{C}}$ is the independence complex of a co-chordal graph. In this case the complex $\Delta_{\mathcal{C}}$ turns out to be a quasi-forest in the sense of Zheng [118]. In [61, Theorem 9.2.12] it is shown that a complex $\Delta$ is a quasi-forest if and only if $\Delta$ is the clique complex of a chordal graph.

Information about the regularity of a clutter can also be found by examining smaller, closely related clutters. Let $S$ be a set of vertices of a clutter $\mathcal{C}$. The *induced subclutter* on $S$, denoted by $\mathcal{C}[S]$, is the maximal subclutter of $\mathcal{C}$ with vertex set $S$. Thus the vertex set of $\mathcal{C}[S]$ is $S$ and the edges of $\mathcal{C}[S]$ are exactly the edges of $\mathcal{C}$ contained in $S$. Notice that $\mathcal{C}[S]$ may have isolated vertices, i.e., vertices that do not belong to any edge of $\mathcal{C}[S]$. If $\mathcal{C}$ is a discrete clutter, i.e., all the vertices of $\mathcal{C}$ are isolated, we set $I(\mathcal{C}) = 0$ and $\alpha_0(\mathcal{C}) = 0$. A clutter of the form $\mathcal{C}[S]$ for some $S \subset V(\mathcal{C})$ is called an *induced subclutter* of $\mathcal{C}$.

**Proposition 3.8.** *Suppose $\mathcal{D}$ is an induced subclutter of $\mathcal{C}$. Then* $\operatorname{reg}(R/I(\mathcal{D})) \leq \operatorname{reg}(R/I(\mathcal{C}))$.

*Proof.* There is $S \subset V(\mathcal{C})$ such that $\mathcal{D} = \mathcal{C}[S]$. Let $\mathfrak{p}$ be the prime ideal of $R$ generated by the variables in $S$. By duality, we have

$$I_c(\mathcal{C}) = \bigcap_{e\in E(\mathcal{C})} (e) \Longrightarrow I_c(\mathcal{C})_{\mathfrak{p}} = \bigcap_{e\in E(\mathcal{C})} (e)_{\mathfrak{p}} = \bigcap_{e\in E(\mathcal{D})} (e)_{\mathfrak{p}} = I_c(\mathcal{D})_{\mathfrak{p}}.$$

Therefore, using Theorem 3.2 and Lemma 3.5, we get

$$\begin{aligned}\operatorname{reg}(R/I(\mathcal{C})) &= \operatorname{pd}(R/I_c(\mathcal{C})) - 1\\ &\geq \operatorname{pd}(R_{\mathfrak{p}}/I_c(\mathcal{C})_{\mathfrak{p}}) - 1 = \operatorname{pd}(R_{\mathfrak{p}}/I_c(\mathcal{D})_{\mathfrak{p}}) - 1\\ &= \operatorname{pd}(R'/I_c(\mathcal{D})) - 1 = \operatorname{pd}(R/I_c(\mathcal{D})) - 1 = \operatorname{reg}(R/I(\mathcal{D})),\end{aligned}$$

where $R'$ is the polynomial ring $K[S]$. Thus, $\operatorname{reg}(R/I(\mathcal{C})) \geq \operatorname{reg}(R/I(\mathcal{D}))$. □

Several combinatorially defined invariants that bound the regularity or other invariants of a clutter are given in terms of subsets of the edge set of the clutter. An *induced matching* in a clutter $\mathcal{C}$ is a set of pairwise disjoint edges $f_1, \ldots, f_r$ such that the only edges of $\mathcal{C}$ contained in $\bigcup_{i=1}^r f_i$ are $f_1, \ldots, f_r$. We let $\operatorname{im}(\mathcal{C})$ be the number of edges in the largest induced matching.

The next result was shown in [53, Theorem 6.5] for the family of uniform properly-connected hypergraphs.

**Corollary 3.9.** *Let $\mathcal{C}$ be a clutter and let $f_1, \ldots, f_r$ be an induced matching of $\mathcal{C}$ with $d_i = |f_i|$ for $i = 1, \ldots, r$. Then*

(i) $(\sum_{i=1}^r d_i) - r \leq \operatorname{reg}(R/I(\mathcal{C}))$.

(ii) [74, *Lemma* 2.2] $\operatorname{im}(G) \leq \operatorname{reg}(R/I(G))$ *for any graph $G$.*

*Proof.* Let $\mathcal{D} = \mathcal{C}[\cup_{i=1}^r f_i]$. Notice that $I(\mathcal{D}) = (x_{f_1}, \ldots, x_{f_r})$. Thus $I(\mathcal{D})$ is a complete intersection and the regularity of $R/I(\mathcal{D})$ is the degree of its $h$-polynomial. The Hilbert series of $R/I(\mathcal{D})$ is given by

$$HS_{\mathcal{D}}(t) = \frac{\prod_{i=1}^r (1 + t + \cdots + t^{d_i - 1})}{(1-t)^{n-r}}.$$

Thus, the degree of the $h$-polynomial equals $(\sum_{i=1}^r d_i) - r$. Therefore, part (i) follows from Proposition 3.8. Part (ii) follows from part (i). □

**Corollary 3.10.** *If $\mathcal{C}$ is a clutter and $R/I_c(\mathcal{C})$ is Cohen–Macaulay, then* $\operatorname{im}(\mathcal{C}) = 1$.

*Proof.* Let $r$ be the induced matching number of $\mathcal{C}$ and let $d$ be the cardinality of any edge of $\mathcal{C}$. Using Theorem 3.2 and Corollary 3.9, we obtain $d - 1 \geq r(d-1)$. Thus $r = 1$, as required. □

The following example shows that the inequality obtained in Corollary 3.9 (ii) can be strict.

**Example 3.11.** Let $G$ be the complement of a cycle $C_6 = \{x_1, \ldots, x_6\}$ of length six. The edge ideal of $G$ is

$$I(G) = (x_1x_3, x_1x_5, x_1x_4, x_2x_6, x_2x_4, x_2x_5, x_3x_5, x_3x_6, x_4x_6).$$

Using *Macaulay*2 [49], we get $\operatorname{reg}(R/I(G)) = 2$ and $\operatorname{im}(G) = 1$.

**Lemma 3.12** ([28, Corollary 20.19]). *If $0 \to N \to M \to L \to 0$ is a short exact sequence of graded finitely generated $R$-modules, then*

(i) $\operatorname{reg}(N) \leq \max(\operatorname{reg}(M), \operatorname{reg}(L) + 1)$.

(ii) $\operatorname{reg}(M) \leq \max(\operatorname{reg}(N), \operatorname{reg}(L))$.

(iii) $\operatorname{reg}(L) \leq \max(\operatorname{reg}(N) - 1, \operatorname{reg}(M))$.

**Definition 3.13.** If $x$ is a vertex of a graph $G$, then its *neighbor set*, denoted by $N_G(x)$, is the set of vertices of $G$ adjacent to $x$.

The following theorem gives a precise sense in which passing to induced subgraphs can be used to bound the regularity. Recall that a discrete graph is one in which all the vertices are isolated.

**Theorem 3.14.** *Let $\mathcal{F}$ be a family of graphs containing any discrete graph and let $\beta\colon \mathcal{F} \to \mathbb{N}$ be a function satisfying that $\beta(G) = 0$ for any discrete graph $G$, and such that given $G \in \mathcal{F}$, with $E(G) \neq \emptyset$, there is $x \in V(G)$ such that the following two conditions hold:*

(i) *$G \setminus \{x\}$ and $G \setminus (\{x\} \cup N_G(x))$ are in $\mathcal{F}$.*

(ii) *$\beta(G \setminus (\{x\} \cup N_G(x))) < \beta(G)$ and $\beta(G \setminus \{x\}) \leq \beta(G)$.*

*Then $\operatorname{reg}(R/I(G)) \leq \beta(G)$ for any $G \in \mathcal{F}$.*

*Proof.* The proof is by induction on the number of vertices. Let $G$ be a graph in $\mathcal{F}$. If $G$ is a discrete graph, then $I(G) = (0)$ and $\operatorname{reg}(R) = \beta(G) = 0$. Assume that $G$ has at least one edge. There is a vertex $x \in V(G)$ such that the induced subgraphs $G_1 = G \setminus \{x\}$ and $G_2 = G \setminus (\{x\} \cup N_G(x))$ satisfy (i) and (ii). There is an exact sequence of graded $R$-modules

$$0 \longrightarrow R/(I(G)\colon x)[-1] \xrightarrow{x} R/I(G) \longrightarrow R/(x, I(G)) \longrightarrow 0.$$

Notice that $(I(G)\colon x) = (N_G(x), I(G_2))$ and $(x, I(G)) = (x, I(G_1))$. The graphs $G_1$ and $G_2$ have fewer vertices than $G$. It follows directly from the definition of regularity that $\operatorname{reg}(M[-1]) = 1 + \operatorname{reg}(M)$ for any graded $R$-module $M$. Therefore applying the

induction hypothesis to $G_1$ and $G_2$, and using conditions (i) and (ii) and Lemma 3.5, we get

$$\begin{aligned}\operatorname{reg}(R/(I(G):x)[-1]) &= \operatorname{reg}(R/(I(G):x)) + 1 = \operatorname{reg}(R'/I(G_2)) + 1\\ &\leq \beta(G_2) + 1 \leq \beta(G),\\ \operatorname{reg}(R/(x, I(G))) &\leq \beta(G_1) \leq \beta(G)\end{aligned}$$

where $R'$ is the ring in the variables $V(G_2)$. Therefore from Lemma 3.12, the regularity of $R/I(G)$ is bounded by the maximum of the regularities of $R/(I(G):x)[-1]$ and $R/(x, I(G))$. Thus $\operatorname{reg}(R/I(G)) \leq \beta(G)$, as required. □

As an example of how Theorem 3.14 can be applied to obtain combinatorial bounds for the regularity, we provide new proofs for several previously known results. Let $G$ be a graph. We let $\beta'(G)$ be the cardinality of any smallest maximal matching of $G$. Hà and Van Tuyl proved that the regularity of $R/I(G)$ is bounded from above by the matching number of $G$ and Woodroofe improved this result showing that $\beta'(G)$ is an upper bound for the regularity.

**Corollary 3.15.** *Let $G$ be a graph and let $R = K[V(G)]$. Then*

(i) [53, *Corollary* 6.9] $\operatorname{reg}(R/I(G)) = \operatorname{im}(G)$ *for any chordal graph $G$.*

(ii) [53, *Theorem* 6.7; 117] $\operatorname{reg}(R/I(G)) \leq \beta'(G)$.

(iii) [106, *Theorem* 3.3] $\operatorname{reg}(R/I(G)) = \operatorname{im}(G)$ *if $G$ is bipartite and $R/I(G)$ is sequentially Cohen–Macaulay.*

*Proof.* (i) Let $\mathcal{F}$ be the family of chordal graphs and let $G$ be a chordal graph with $E(G) \neq \emptyset$. By Corollary 3.9 and Theorem 3.14 it suffices to prove that there is $x \in V(G)$ such that $\operatorname{im}(G_1) \leq \operatorname{im}(G)$ and $\operatorname{im}(G_2) < \operatorname{im}(G)$, where $G_1$ and $G_2$ are the subgraphs $G \setminus \{x\}$ and $G \setminus (\{x\} \cup N_G(x))$, respectively. The inequality $\operatorname{im}(G_1) \leq \operatorname{im}(G)$ is clear because any induced matching of $G_1$ is an induced matching of $G$. We now show the other inequality. By [104, Theorem 8.3], there is $y \in V(G)$ such that $G[N_G(y) \cup \{y\}]$ is a complete subgraph. Pick $x \in N_G(y)$ and set $f_0 = \{x, y\}$. Consider an induced matching $f_1, \ldots, f_r$ of $G_2$ with $r = \operatorname{im}(G_2)$. We claim that $f_0, f_1, \ldots, f_r$ is an induced matching of $G$. Let $e$ be an edge of $G$ contained in $\bigcup_{i=0}^{r} f_i$. We may assume that $e \cap f_0 \neq \emptyset$ and $e \cap f_i \neq \emptyset$ for some $i \geq 1$, otherwise $e = f_0$ or $e = f_i$ for some $i \geq 1$. Then $e = \{y, z\}$ or $e = \{x, z\}$ for some $z \in f_i$. If $e = \{y, z\}$, then $z \in N_G(y)$ and $x \in N_G(y)$. Hence $\{z, x\} \in E(G)$ and $z \in N_G(x)$, a contradiction because the vertex set of $G_2$ is disjoint from $N_G(x) \cup \{x\}$. If $e = \{x, z\}$, then $z \in N_G(x)$, a contradiction. This completes the proof of the claim. Hence $\operatorname{im}(G_2) < \operatorname{im}(G)$.

(ii) Let $\mathcal{F}$ be the family of all graphs and let $G$ be a graph with $E(G) \neq \emptyset$. By Theorem 3.14 it suffices to prove that there is $x \in V(G)$ such that $\beta'(G_1) \leq \beta'(G)$ and

$\beta'(G_2) < \beta'(G)$, where $G_1$ and $G_2$ are the subgraphs $G \setminus \{x\}$ and $G \setminus (\{x\} \cup N_G(x))$, respectively.

Let $f_1, \ldots, f_r$ be a maximal matching of $G$ with $r = \beta'(G)$ and let $x, y$ be the vertices of $f_1$. Clearly $f_2, \ldots, f_r$ is a matching of $G_1$. Thus we can extend it to a maximal matching $f_2, \ldots, f_r, h_1, \ldots, h_s$ of $G_1$. Notice that $s \leq 1$. Indeed if $s \geq 2$, then $h_i \cap f_1 = \emptyset$ for some $i \in \{1, 2\}$ (otherwise $y \in h_1 \cap h_2$, which is impossible). Hence $f_1, \ldots, f_r, h_i$ is a matching of $G$, a contradiction because $f_1, \ldots, f_r$ is maximal. Therefore $\beta'(G_1) \leq r - 1 + s \leq \beta'(G)$.

The set $f_2, \ldots, f_r$ contains a matching of $G_2$, namely those edges $f_i$ that do not degenerate. Reorder the edges so that $f_2, \ldots, f_m$ are the edges that do not degenerate. Then this set can be extended to a maximal matching $f_2, \ldots, f_m, f'_{m+1}, \ldots, f'_k$ of $G_2$. Now consider $f'_{m+1}$. Since $f_1, \ldots, f_r$ is a maximal matching of $G$, $f'_{m+1}$ has a nontrivial intersection with $f_i$ for some $i$. Note that $i \neq 1$ since $f'_{m+1}$ is an edge of $G_2$, and $i \geq m + 1$ since $f_2, \ldots, f_m$ and $f'_{m+1}$ are all part of a matching of $G_2$. Reorder so that $i = m + 1$. Repeat the process with $f'_{m+2}$. As before, $f'_{m+2}$ has a nontrivial intersection with $f_i$ for some $i \geq m + 1$. If $i = m + 1$, then since $f'_{m+1}$ and $f'_{m+2}$ are disjoint, each must share a different vertex with $f_{m+1}$. But $f_{m+1}$ degenerated when passing to $G_2$, and $G_2$ is induced on the remaining vertices, so this is a contradiction. Thus we may reorder so that $f'_{m+2}$ nontrivially intersects $f_{m+2}$. Repeating this process, we see that $f'_j$ nontrivially intersects $f_j$ for all $m+1 \leq j \leq k$. Thus $k \leq r$. Now $\beta'(G_2) \leq k - 1 \leq r - 1 < \beta'(G)$.

(iii) Let $\mathcal{F}$ be the family of all bipartite graphs $G$ such that $R/I(G)$ is sequentially Cohen–Macaulay, and let $\beta: \mathcal{F} \to \mathbb{N}$ be the function $\beta(G) = \operatorname{im}(G)$. Let $G$ be a graph in $\mathcal{F}$ with $E(G) \neq \emptyset$. By Corollary 3.9 and Theorem 3.14 it suffices to observe that, according to [107, Corollary 2.10], there are adjacent vertices $x$ and $y$ with $\deg(y) = 1$ such that the bipartite graphs $G \setminus (\{x\} \cup N_G(x))$ and $G \setminus (\{y\} \cup N_G(y))$ are sequentially Cohen–Macaulay. Thus conditions (i) and (ii) of Theorem 3.14 are satisfied. □

Corollary 3.15 shows that the regularity of $R/I(G)$ equals $\operatorname{im}(G)$ for any forest $G$, which was first proved by Zheng [118]. If $G$ is an unmixed graph, Kummini [77] showed that $\operatorname{reg}(R/I(G))$ equals the induced matching number of $G$. If $G$ is claw-free and its complement has no induced 4-cycles, then $\operatorname{reg}(R/I(G)) \leq 2$ with equality if its complement is not chordal [90] (note that in this case $\operatorname{reg}(R/I(G)) = \operatorname{im}(G) + 1$). Formulas for the regularity of ideals of mixed products are given in [71]. The regularity and depth of lex segment edge ideals are computed in [33]. The regularity and other algebraic properties of the edge ideal $I(G)$ associated to a Ferrers graph $G$ are studied in detail in [18]. If $R/I(G)$ is Cohen–Macaulay, then a formula for $\operatorname{reg}(R/I(G))$ follows from [95, Corollary 4.2].

The following result about regularity was shown by Kalai and Meshulam for square-free monomial ideals and by Herzog for arbitrary monomial ideals. Similar inequalities hold for the projective dimension.

**Proposition 3.16** ([57,73]). *Let $I_1$ and $I_2$ be monomial ideals of $R$. Then*

(i) $\operatorname{reg} R/(I_1 + I_2) \leq \operatorname{reg}(R/I_1) + \operatorname{reg}(R/I_2)$,

(ii) $\operatorname{reg} R/(I_1 \cap I_2) \leq \operatorname{reg}(R/I_1) + \operatorname{reg}(R/I_2) + 1$.

**Corollary 3.17.** *If $\mathcal{C}_1, \ldots, \mathcal{C}_s$ are clutters on the vertex set $X$, then*

$$\operatorname{reg}\Big(R/I\Big(\bigcup_{i=1}^{s} \mathcal{C}_i\Big)\Big) \leq \operatorname{reg}(R/I(\mathcal{C}_1)) + \cdots + \operatorname{reg}(R/I(\mathcal{C}_s)).$$

*Proof.* The set of edges of $\mathcal{C} = \bigcup_{i=1}^{s} \mathcal{C}_i$ equals $\bigcup_{i=1}^{s} E(\mathcal{C}_i)$. By Proposition 3.16, it suffices to notice the equality $I(\bigcup_{i=1}^{s} \mathcal{C}_i) = \sum_{i=1}^{s} I(\mathcal{C}_i)$. □

A clutter $\mathcal{C}$ is called *co-CM* if $I_c(\mathcal{C})$ is Cohen–Macaulay. A co-CM clutter is uniform because Cohen–Macaulay clutters are unmixed.

**Corollary 3.18.** *If $\mathcal{C}_1, \ldots, \mathcal{C}_s$ are co-CM clutters on the vertex set $X$, then*

$$\operatorname{reg}(R/I(\cup_{i=1}^{s} \mathcal{C}_i)) \leq (d_1 - 1) + \cdots + (d_s - 1),$$

*where $d_i$ is the number of elements in any edge of $\mathcal{C}_i$.*

*Proof.* By Theorem 3.2, we get that $\operatorname{reg} R/I(\mathcal{C}_i) = d_i - 1$ for all $i$. Thus the result follows from Corollary 3.17. □

This result is especially useful for graphs. A graph $G$ is *weakly chordal* if every induced cycle in both $G$ and $G^c$ has length at most 4. It was pointed out in [117] that a weakly chordal graph $G$ can be covered by $\operatorname{im}(G)$ co-CM graphs (this fact was shown in [13]). Thus we have:

**Theorem 3.19** ([117]). *If $G$ is a weakly chordal graph, then* $\operatorname{reg}(R/I(G)) = \operatorname{im}(G)$.

There are bounds for the regularity of $R/I$ in terms of some other algebraic invariants of $R/I$. Recall that the $a$-invariant of $R/I$, denoted by $a(R/I)$, is the degree (as a rational function) of the Hilbert series of $R/I$. Also recall that the independence complex of $I(\mathcal{C})$, denoted by $\Delta_{\mathcal{C}}$, is the simplicial complex whose faces are the independent vertex sets of $\mathcal{C}$. The *arithmetic degree* of $I = I(\mathcal{C})$, denoted by $\operatorname{arith-deg}(I)$, is the number of *facets* (maximal faces with respect to inclusion) of $\Delta_{\mathcal{C}}$. The *arithmetical rank* of $I$, denoted by $\operatorname{ara}(I)$, is the least number of elements of $R$ which generate the ideal $I$ up to radical.

**Theorem 3.20** ([108, Corollary B.4.1]). $a(R/I) \leq \operatorname{reg}(R/I) - \operatorname{depth}(R/I)$, *with equality if $R/I$ is Cohen–Macaulay.*

**Theorem 3.21** ([79; 80, Proposition 3]). $\operatorname{reg}(R/I^{\vee}) = \operatorname{pd}(R/I) - 1 \leq \operatorname{ara}(I) - 1$.

The equality $\operatorname{reg}(R/I^{\vee}) = \operatorname{pd}(R/I) - 1$ was pointed out earlier in Theorem 3.2. There are many instances where the equality $\operatorname{pd}(R/I) = \operatorname{ara}(I)$ holds, see [1, 33, 76] and the references there. For example, for paths, one has $\operatorname{pd}(R/I) = \operatorname{ara}(I)$ [1]. Barile [1] has conjectured that the equality holds for edge ideals of forests. We also have that $n - \min_i\{\operatorname{depth}(R/I^{(i)})\}$ is an upper bound for $\operatorname{ara}(I)$, see [80]. This upper bound tends to be very loose. If $I$ is the edge ideal of a tree, then $I$ is normally torsion free (see Section 4 together with Theorems 4.34 and 4.8). Then $\min_i\{\operatorname{depth}(R/I^{(i)})\} = 1$ by [87, Lemma 2.6]. But when $I$ is the edge ideal of a path with 8 vertices, then the actual value of $\operatorname{ara}(I)$ is 5.

**Theorem 3.22** ([103, Theorem 3.1]). *If* $\operatorname{ht}(I) \geq 2$*, then* $\operatorname{reg}(I) \leq \operatorname{arith-deg}(I)$.

The next open problem is known as the Eisenbud–Goto regularity conjecture [30].

**Conjecture 3.23.** If $\mathfrak{p} \subset (x_1, \ldots, x_n)^2$ is a prime graded ideal, then

$$\operatorname{reg}(R/\mathfrak{p}) \leq \deg(R/\mathfrak{p}) - \operatorname{codim}(R/\mathfrak{p}).$$

A pure $d$-dimensional complex $\Delta$ is called *connected in codimension* 1 if each pair of facets $F, G$ can be connected by a sequence of facets $F = F_0, F_1, \ldots, F_s = G$, such that $\dim(F_{i-1} \cap F_i) = d - 1$ for $1 \leq i \leq s$. According to [5, Proposition 11.7], every Cohen–Macaulay complex is connected in codimension 1.

The following gives a partial answer to the monomial version of the Eisenbud–Goto regularity conjecture.

**Theorem 3.24** ([103]). *Let* $I = I(\mathcal{C})$ *be an edge ideal. If* $\Delta_{\mathcal{C}}$ *is connected in codimension* 1*, then*

$$\operatorname{reg}(R/I) \leq \deg(R/I) - \operatorname{codim}(R/I).$$

The dual notion to the independence complex of $I(\mathcal{C})$ is to start with a complex $\Delta$ and associate to it an ideal whose independence complex is $\Delta$.

**Definition 3.25.** Given a simplicial complex $\Delta$ with vertex set $X = \{x_1, \ldots, x_n\}$, the *Stanley–Reisner ideal* of $\Delta$ is defined as

$$I_{\Delta} = (\{x_{i_1} \cdots x_{i_r} \,|\, i_1 < \cdots < i_r, \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta\}),$$

and its *Stanley–Reisner ring* $K[\Delta]$ is defined as the quotient ring $R/I_{\Delta}$.

A simple proof the next result is given in [44].

**Theorem 3.26** ([101]). *Let* $\mathcal{C}$ *be a clutter and let* $\Delta = \Delta_{\mathcal{C}}$ *be its independence complex. Then*

$$\operatorname{depth} R/I(\mathcal{C}) = 1 + \max\{i \mid K[\Delta^i] \text{ is Cohen–Macaulay}\},$$

*where* $\Delta^i = \{F \in \Delta \mid \dim(F) \leq i\}$ *is the* $i$*-skeleton of* $\Delta$ *and* $-1 \leq i \leq \dim(\Delta)$.

A variation on the concept of the $i$-skeleton will facilitate an extension of the result above to the sequentially Cohen–Macaulay case.

**Definition 3.27.** Let $\Delta$ be a simplicial complex. The *pure $i$-skeleton* of $\Delta$ is defined as:

$$\Delta^{[i]} = \langle \{F \in \Delta | \ \dim(F) = i\} \rangle; \ \ -1 \leq i \leq \dim(\Delta),$$

where $\langle \mathcal{F} \rangle$ denotes the subcomplex generated by $\mathcal{F}$.

Note that $\Delta^{[i]}$ is always pure of dimension $i$. We say that a simplicial complex $\Delta$ is *sequentially Cohen–Macaulay* if its Stanley–Reisner ring has this property. The following results link the sequentially Cohen–Macaulay property to the Cohen–Macaulay property and to the regularity and projective dimension. The first is an interesting result of Duval.

**Theorem 3.28** ([26, Theorem 3.3]). *Let $\Delta$ be a simplicial complex. Then $\Delta$ is sequentially Cohen–Macaulay if and only if the pure $i$-skeleton $\Delta^{[i]}$ is Cohen–Macaulay for $-1 \leq i \leq \dim(\Delta)$.*

**Corollary 3.29.** *$R/I(\mathcal{C})$ is Cohen–Macaulay if and only if $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay and $\mathcal{C}$ is unmixed.*

**Lemma 3.30.** *Let $\mathcal{C}$ be a clutter and let $\Delta = \Delta_{\mathcal{C}}$ be its independence complex. If $\beta_0'(\mathcal{C})$ is the cardinality of a smallest maximal independent set of $\mathcal{C}$, then $\Delta^{[i]} = \Delta^i$ for $i \leq \beta_0'(\mathcal{C}) - 1$.*

*Proof.* First we prove the inclusion $\Delta^{[i]} \subset \Delta^i$. Let $F$ be a face of $\Delta^{[i]}$. Then $F$ is contained in a face of $\Delta$ of dimension $i$, and so $F$ is in $\Delta^i$. Conversely, let $F$ be a face of $\Delta^i$. Then

$$\dim(F) \leq i \leq \beta_0'(\mathcal{C}) - 1 \Longrightarrow |F| \leq i + 1 \leq \beta_0'(\mathcal{C}).$$

Since $\beta_0'(\mathcal{C})$ is the cardinality of any smallest maximal independent set of $\mathcal{C}$, we can extend $F$ to an independent set of $\mathcal{C}$ with $i + 1$ vertices. Thus $F$ is in $\Delta^{[i]}$. □

While $\beta_0'$ regulates the equality of the $i$-skeleton and the pure $i$-skeleton of the independence complex, its complement provides a lower bound for the regularity of the ideal of covers.

**Theorem 3.31.** *Let $\mathcal{C}$ be a clutter, let $I_c(\mathcal{C})$ be its ideal of vertex covers, and let $\alpha_0'(\mathcal{C})$ be the cardinality of a largest minimal vertex cover of $\mathcal{C}$. Then*

$$\operatorname{reg} R/I_c(\mathcal{C}) \geq \alpha_0'(\mathcal{C}) - 1,$$

*with equality if $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay.*

*Proof.* We set $\beta_0'(\mathcal{C}) = n - \alpha_0'(\mathcal{C})$. Using Theorem 3.2 and the Auslander–Buchsbaum formula (see (3.2)), the proof reduces to showing: $\operatorname{depth} R/I(\mathcal{C}) \leq \beta_0'(\mathcal{C})$, with equality if $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay.

First we show that $\operatorname{depth} R/I(\mathcal{C}) \leq \beta_0'(\mathcal{C})$. Assume $\Delta^i$ is Cohen–Macaulay for some $-1 \leq i \leq \dim(\Delta)$, where $\Delta$ is the independence complex of $\mathcal{C}$. According to Theorem 3.26, it suffices to prove that $1 + i \leq \beta_0'(\mathcal{C})$. Notice that $\beta_0'(\mathcal{C})$ is the cardinality of any smallest maximal independent set of $\mathcal{C}$. Thus, we can pick a maximal independent set $F$ of $\mathcal{C}$ with $\beta_0'(\mathcal{C})$ vertices. Since $\Delta^i$ is Cohen–Macaulay, the complex $\Delta^i$ is pure, that is, all maximal faces of $\Delta$ have dimension $i$. If $1 + i > \beta_0'(\mathcal{C})$, then $F$ is a maximal face of $\Delta^i$ of dimension $\beta_0'(\mathcal{C}) - 1$, a contradiction to the purity of $\Delta^i$.

Assume that $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay. By Lemma 3.30 $\Delta^{[i]} = \Delta^i$ for $i \leq \beta_0'(\mathcal{C}) - 1$. Then by Theorem 3.28, the ring $K[\Delta^i]$ is Cohen–Macaulay for $i \leq \beta_0'(\mathcal{C}) - 1$. Therefore, applying Theorem 3.26, we get that the depth of $R/I(\mathcal{C})$ is at least $\beta_0'(\mathcal{C})$. Consequently, in this case one has the equality $\operatorname{depth} R/I(\mathcal{C}) = \beta_0'(\mathcal{C})$. □

The inequality in Theorem 3.31 also follows directly from the definition of regularity because $\operatorname{reg}(I_c(\mathcal{C}))$ is an upper bound for the largest degree of a minimal generator of $I_c(\mathcal{C})$.

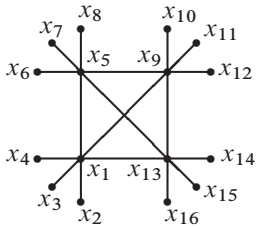
**Remark 3.32.** $\alpha_0'(\mathcal{C})$ is $\max\{|e|\colon e \in E(\mathcal{C}^\vee)\}$ and $\alpha_0'(\mathcal{C}^\vee)$ is $\max\{|e|\colon e \in E(\mathcal{C})\}$. This follows by Alexander duality, see (3.3).

**Corollary 3.33.** *If $I(\mathcal{C})$ is an edge ideal, then $\operatorname{pd}_R(R/I(\mathcal{C})) \geq \alpha_0'(\mathcal{C})$, with equality if $R/I(\mathcal{C})$ is sequentially Cohen–Macaulay.*

*Proof.* It follows from the proof of Theorem 3.31. □

There are many interesting classes of sequentially Cohen–Macaulay clutters where this formula for the projective dimension applies (see Theorem 2.6). The projective dimension of edge ideals of forests was studied in [22, 53], where some recursive formulas are presented. Explicit formulas for the projective dimension for some path ideals of directed rooted trees can be found in [55, Theorem 1.2]. Path ideals of directed graphs were introduced by Conca and De Negri [15]. Fix an integer $t \geq 2$, and suppose that $\mathcal{D}$ is a directed graph, i.e., each edge has been assigned a direction. A sequence of $t$ vertices $x_{i_1}, \ldots, x_{i_t}$ is said to be a *path* of length $t$ if there are $t - 1$ distinct edges $e_1, \ldots, e_{t-1}$ such that $e_j = (x_{i_j}, x_{i_j+1})$ is a directed edge from $x_{i_j}$ to $x_{i_j+1}$. The *path ideal* of $\mathcal{D}$ of length $t$, denoted by $I_t(\mathcal{D})$, is the ideal generated by all monomials $x_{i_1} \cdots x_{i_t}$ such that $x_{i_1}, \ldots, x_{i_t}$ is a path of length $t$ in $\mathcal{D}$. Note that when $t = 2$, then $I_2(\mathcal{D})$ is simply the edge ideal of $\mathcal{D}$.

**Example 3.34.** Let $K$ be any field and let $G$ be the following chordal graph




Then, by Theorem 2.6 and Corollary 3.31, $\operatorname{pd}_R(R/I(G)) = 6$ and $\operatorname{depth} R/I(G) = 10$.

**Corollary 3.35.** *Let $\mathcal{C}$ be a clutter. If $I(\mathcal{C})$ has linear quotients, then*

$$\operatorname{reg} R/I(\mathcal{C}) = \max\{|e|\colon e \in E(\mathcal{C})\} - 1.$$

*Proof.* The ideal of covers $I_c(\mathcal{C})$ is sequentially Cohen–Macaulay by Theorem 2.6 (iv). Hence, using Theorem 3.31, we get $\operatorname{reg}(R/I(\mathcal{C})) = \alpha_0'(\mathcal{C}^\vee) - 1$. To complete the proof notice that $\alpha_0'(\mathcal{C}^\vee) = \max\{|e|\colon e \in E(\mathcal{C})\}$ (see Remark 3.32). □

The converse of Theorem 3.33 is not true.

**Example 3.36.** Let $C_6$ be a cycle of length 6. Then $R/I(C_6)$ is not sequentially Cohen–Macaulay by Proposition 2.4. Using *Macaulay*2, we get $\operatorname{pd}(R/I(C_6)) = \alpha_0'(C_6) = 4$.

When $R/I$ is not known to be Cohen–Macaulay, it can prove useful to have effective bounds on the depth of $R/I$.

**Theorem 3.37.** *Let $G$ be a bipartite graph without isolated vertices. If $G$ has $n$ vertices, then*

$$\operatorname{depth} R/I(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

*Proof.* Let $(V_1, V_2)$ be a bipartition of $G$ with $|V_1| \leq |V_2|$. Note $2|V_1| \leq n$ because $|V_1| + |V_2| = n$. Since $V_1$ is a maximal independent set of vertices one has $\beta_0'(G) \leq |V_1| \leq n/2$. Therefore, using Corollary 3.33 and the Auslander–Buchsbaum formula, we get $\operatorname{depth} R/I(G) \leq n/2$. □

**Corollary 3.38.** *If $G$ is a $B$-graph with $n$ vertices, then*

$$\operatorname{depth} R/I(G) \leq \dim R/I(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

*Proof.* Recall that $n = \alpha_0(G) + \beta_0(G)$. By Theorem 2.9, $\beta_0(G) \leq \alpha_0(G)$, and so $\beta_0(G) \leq \lfloor \frac{n}{2} \rfloor$. The result now follows because $\beta_0(G) = \dim R/I(\mathcal{C})$. □

Lower bounds are given in [87] for the depths of $R/I(G)^t$ for $t \geq 1$ when $I(G)$ is the edge ideal of a tree or forest. Upper bounds for the depth of $R/I(G)$ are given in [46, Corollary 4.15] when $G$ is any graph without isolated vertices. The depth and the Cohen–Macaulay property of ideals of mixed products is studied in [71].

We close this section with an upper bound for the multiplicity of edge rings. Let $\mathcal{C}$ be a clutter. The *multiplicity* of the edge-ring $R/I(\mathcal{C})$, denoted by $e(R/I(\mathcal{C}))$, equals the number of faces of maximum dimension of the independence complex $\Delta_{\mathcal{C}}$, i.e., the multiplicity of $R/I(\mathcal{C})$ equals the number of independent sets of $\mathcal{C}$ with $\beta_0(\mathcal{C})$ vertices. A related invariant that was considered earlier is arith-deg$(I(\mathcal{C}))$, the number of maximal independent sets of $\mathcal{C}$.

**Proposition 3.39** ([46]). *If $\mathcal{C}$ is a $d$-uniform clutter and $I = I(\mathcal{C})$, then $e(R/I) \leq d^{\alpha_0(\mathcal{C})}$.*

## 4 Stability of Associated Primes

One method of gathering information about an ideal is through its associated primes. Let $I$ be an ideal of a ring $R$. In this section, we will examine the sets of associated primes of powers of $I$, that is, the sets

$$\operatorname{Ass}(R/I^t) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is prime and } \mathfrak{p} = (I^t : c) \text{ for some } c \in R\}.$$

When $I$ is a monomial ideal of a polynomial ring $R = K[x_1, \ldots, x_n]$, the associated primes will be monomial ideals, that is, prime ideals which are generated by a subset of the variables. When $I$ is a square-free monomial ideal, the minimal primes of $I$, $\operatorname{Min}(R/I)$, correspond to minimal vertex covers of the clutter $\mathcal{C}$ associated to $I$. In general $\operatorname{Min}(R/I) \subset \operatorname{Ass}(R/I^t)$ for all positive integers $t$. For a square-free monomial ideal, in the case where equality holds for all $t$, the ideal $I$ is said to be *normally torsion-free*. More generally, an ideal $I \subset R$ is called *normally torsion-free* if $\operatorname{Ass}(R/I^i)$ is contained in $\operatorname{Ass}(R/I)$ for all $i \geq 1$ and $I \neq R$.

In [9], Brodmann showed that when $R$ is a Noetherian ring and $I$ is an ideal of $R$, the sets $\operatorname{Ass}(R/I^t)$ stabilize for large $t$. That is, there exists a positive integer $N$ such that $\operatorname{Ass}(R/I^t) = \operatorname{Ass}(R/I^N)$ for all $t \geq N$. We will refer to a minimal such $N$ as the *index of stability* of $I$. There are two natural questions following from this result. In this article, we will focus on the monomial versions of the questions.

**Question 4.1.** Given a monomial ideal $I$, what is an effective upper bound on the index of stability for a given class of monomial ideals?

**Question 4.2.** Given a monomial ideal $I$, which primes are in $\operatorname{Ass}(R/I^t)$ for all sufficiently large $t$?

An interesting variation on Questions 4.1 and 4.2 was posed in [99].

**Question 4.3.** Suppose that $N$ is the index of stability of an ideal $I$. Given a prime $\mathfrak{p} \in \operatorname{Ass}(R/I^N)$, can you find an integer $N_{\mathfrak{p}}$ for which $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ for $t \geq N_{\mathfrak{p}}$?

Brodmann also showed that the sets $\operatorname{Ass}(I^{t-1}/I^t)$ stabilize. Thus in the general setting, one could ask similar questions about these sets. However, for monomial ideals the following lemma shows that in order to find information about $\operatorname{Ass}(R/I^t)$, one may instead study $\operatorname{Ass}(I^{t-1}/I^t)$.

**Lemma 4.4.** *Let $I$ be a monomial ideal. Then* $\operatorname{Ass}(I^{t-1}/I^t) = \operatorname{Ass}(R/I^t)$.

*Proof.* Suppose that $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$. Then $\mathfrak{p} = (I^t : c)$ for some monomial $c \in R$. But since $\mathfrak{p}$ is necessarily a monomial prime, generated by a subset of the variables, then if $xc \in I^t$ for a variable $x \in \mathfrak{p}$, then $c \in I^{t-1}$ and so $\mathfrak{p} \in \operatorname{Ass}(I^{t-1}/I^t)$. The other inclusion is automatic. □

Note that this method was used in [105] to show that the corresponding equality also holds for the integral closures of the powers of $I$.

For special classes of ideals, there have been some results that use properties of the ideals to find bounds on $N$. For example, if $I$ is generated by a regular sequence, then by [67], $I$ is normally torsion-free, or $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I)$ for all $t$, and thus $N = 1$. If instead $I$ is generated by a $d$-sequence and is strongly Cohen–Macaulay, then it was shown in [88] that $N$ is bounded above by the dimension of the ring. In particular, $N \leq n-g+1$ where $n$ is the dimension of the ring and $g$ is the height of the ideal. We are particularly interested in finding similar bounds for classes of monomial ideals.

In [65], Hoa used integer programming techniques to give an upper bound on $N$ for general monomial ideals. Let $n$ be the number of variables, $s$ the number of generators of $I$, and $d$ the maximal degree of a generator.

**Theorem 4.5** ([65, Theorem 2.12]). *If $I$ is a monomial ideal, then the index of stability is bounded above by*

$$\max\left\{d(ns+s+d)(\sqrt{n})^{n+1}(\sqrt{2}d)^{(n+1)(s-1)}, s(s+n)^4 s^{n+2} d^2 (2d^2)^{s^2-s+1}\right\}.$$

Notice that this bound can be extremely large. For general monomial ideals, examples are given in [65] to show that the bound should depend on $d$ and $n$. However, if we restrict to special classes of monomial ideals, much smaller bounds can be found. For example, an alternate bound has been shown to hold for integral closures of powers of monomial ideals.

**Theorem 4.6** ([105, Theorem 16]). *If $I$ is a monomial ideal, and $N_0 = n2^{n-1}d^{n-2}$, then* $\operatorname{Ass}(R/\overline{I^t}) = \operatorname{Ass}(R/\overline{I^{N_0}})$ *for $t \geq N_0$ when $n \geq 2$.*

Here again $n$ is the number of variables and $d$ is the maximal degree of a generator. For the class of normal monomial ideals, this bound on the index of stability can be significantly lower than the general bound given above. When $n = 2$, the index of stability of the integral closures is lower still.

**Lemma 4.7** ([85]). *If $n \leq 2$, then $\operatorname{Ass}(R/\overline{I^t}) = \operatorname{Ass}(R/\overline{I})$ for all $t \geq 1$.*

Note that this result is of interest for general monomial ideals; however, when $n = 2$ a square-free monomial ideal will be a complete intersection. Of particular interest for this article are results that use combinatorial and graph-theoretic properties to yield insights into the associated primes and index of stability of monomial ideals. One pivotal result in this area establishes a classification of all graphs for which $N = 1$.

**Theorem 4.8** ([100, Theorem 5.9]). *Let $G$ be a graph and $I$ its edge ideal. Then $G$ is bipartite if and only if $I$ is normally torsion-free.*

The result above shows that $N = 1$ for the edge ideal of a graph if and only if the graph is bipartite. Since minimal primes correspond to minimal vertex covers, this completely answers Questions 4.1 and 4.2 for bipartite graphs. In addition if $I$ is the edge ideal of a balanced clutter, then $N = 1$ [45].

Suppose now that $G$ is a graph that is not bipartite. Then $G$ contains at least one odd cycle. For such graphs, a method of describing embedded associated primes, and a bound on where the stability occurs, were given in [14]. The method of building embedded primes centered around the odd cycles, so we first give an alternate proof of the description of the associated primes for this base case.

**Lemma 4.9.** *Suppose $G$ is a cycle of length $n = 2k + 1$ and $I$ is the edge ideal of $G$. Then $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I)$ if $t \leq k$ and $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I) \cup \{\mathfrak{m}\}$ if $t \geq k + 1$. Moreover, when $t \geq k + 1$, $\mathfrak{m} = (I^t : c)$ for a monomial $c$ of degree $2t - 1$.*

*Proof.* If $\mathfrak{p} \neq \mathfrak{m}$ is a prime ideal, then $I_{\mathfrak{p}}$ is the edge ideal of a bipartite graph, and thus by Theorem 4.8 $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ (for any $t \geq 1$) if and only if $\mathfrak{p}$ is a minimal prime of $I$. Notice also that the deletion of any vertex $x_i$ (which corresponds to passing to the quotient ring $R/(x_i)$) results in a bipartite graph as well. Thus by [51, Corollary 3.6], $\mathfrak{m} \notin \operatorname{Ass}(R/I^t)$ for $t \leq k$ since a maximal matching has $k$ edges. For $t \geq k + 1$, define $b = \left(\prod_{i=1}^{n} x_i\right)$ and $c = b(x_1x_2)^{t-k-1}$ where $x_1x_2$ is any edge of $G$. Then since $c$ has degree $2t - 1$, $c \notin I^t$, but $G$ is a cycle, $x_ib \in I^{k+1}$ and $x_ic \in I^t$. Thus $\mathfrak{m} = (I^t : c)$ and so $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$ for $t \geq k + 1$. □

**Corollary 4.10.** *Suppose $G$ is a connected graph containing an odd cycle of length $2k + 1$ and suppose that every vertex of $G$ that is not in the cycle is a leaf. Then $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I) \cup \{\mathfrak{m}\}$ if $t \geq k+1$. Moreover, when $t \geq k+1$, $\mathfrak{m} = (I^t : c)$ for a monomial $c$ of degree $2t - 1$.*

*Proof.* Let $b$ and $c$ be defined as in the proof of Lemma 4.9. Notice that if $x$ is a leaf, then $x$ is connected to a unique vertex in the cycle and that $xb \in I^{k+1}$. The remainder of the proof follows as in Lemma 4.9. □

If $G$ is a more general graph, the embedded associated primes of $I = I(G)$ are formed by working outward from the odd cycles. This was done in [14], including a detailed explanation of how to work outward from multiple odd cycles. Before providing more concise proofs of the process, we first give an informal, but illustrative, description. Suppose $C$ is a cycle with $2k+1$ vertices $x_1, \ldots, x_{2k+1}$. Color the vertices of $C$ red and color any noncolored vertex that is adjacent to a red vertex blue. The set of colored vertices, together with a minimal vertex cover of the set of edges neither of whose vertices is colored, will be an embedded associated prime of $I^t$ for all $t \geq k+1$. To find additional embedded primes of higher powers, select any blue vertex to turn red and turn any uncolored neighbors of this vertex blue. The set of colored vertices, together with a minimal vertex cover of the noncolored edges, will be an embedded associated prime of $I^t$ for all $t \geq k+2$. This process continues until all vertices are colored red or blue.

The method of building new associated primes for a power of $I$ from primes associated to lower powers relies on localization. Since localization will generally cause the graph (or clutter) to become disconnected, we first need the following lemma.

**Lemma 4.11** ([51, Lemma 3.4], see also [14, Lemma 2.1]). *Suppose $I$ is a square-free monomial ideal in $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ such that $I = I_1 S + I_2 S$, where $I_1 \subset S_1 = K[x_1, \ldots, x_r]$ and $I_2 \subset S_2 = K[y_1, \ldots, y_s]$. Then $\mathfrak{p} \in \operatorname{Ass}(S/I^t)$ if and only if $\mathfrak{p} = \mathfrak{p}_1 S + \mathfrak{p}_2 S$, where $\mathfrak{p}_1 \in \operatorname{Ass}(S_1/I_1^{t_1})$ and $\mathfrak{p}_2 \in \operatorname{Ass}(S_2/I_2^{t_2})$ with $(t_1 - 1) + (t_2 - 1) = t - 1$.*

Note that this lemma easily generalizes to an ideal $I = (I_1, I_2, \ldots, I_s)$ where the $I_i$ are edge ideals of disjoint clutters. Then $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ if and only if $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ with $\mathfrak{p}_i \in \operatorname{Ass}(R/I_i^{t_i})$ where $(t_1-1)+(t_2-1)+\cdots+(t_s-1) = (t-1)$.

We now fix a notation to show how to build embedded associated primes. Consider $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ for $I$ the edge ideal of a graph $G$. Without loss of generality, Lemma 4.11 allows us to assume $G$ does not have isolated vertices. If $\mathfrak{p} \neq \mathfrak{m}$, then since $\mathfrak{p} \in \operatorname{Ass}(R/I^t) \Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}(R_{\mathfrak{p}}/(I_{\mathfrak{p}})^t)$, consider $I_{\mathfrak{p}}$. Write $I_{\mathfrak{p}} = (I_a, I_b)$ where $I_a$ is generated by all generators of $I_{\mathfrak{p}}$ of degree two and $I_b$ is the prime ideal generated by the degree one generators of $I_{\mathfrak{p}}$, which correspond to the isolated vertices of the graph associated to $I_{\mathfrak{p}}$. Note that the graph corresponding to $I_a$ need not be connected. If $I_a = (0)$, then $\mathfrak{p}$ is a minimal prime of $I$, so assume $I_a \neq (0)$. Define $\mathfrak{p}_a$ to be the monomial prime generated by variables of $I_a$. Define $N_1 = \bigcup_{x \in \mathfrak{p}_a} N(x)$, where $N(x)$ is the neighbor set of $x$ in $G$, and let $\mathfrak{p}_1 = \mathfrak{p}_a \cup N_1$. Notice that if $x \in \mathfrak{p}_a$, then $x$ is not isolated in $I_{\mathfrak{p}}$, so $N_1 \subset \mathfrak{p}$ and thus $N_1 \subset \mathfrak{p}_a \cup I_b$. Define $\mathfrak{p}_2 = \mathfrak{p} \backslash \mathfrak{p}_1 = I_b \backslash N_1$, and $N_2 = \bigcup_{x \in \mathfrak{p}_1} N(x) \backslash \mathfrak{p}$. If $G_1$ is the induced subgraph of $G$ on the vertices in $\mathfrak{p}_1 \cup N_2$, $G_2$ is the induced subgraph of $G$ on vertices in $V \backslash \mathfrak{p}_1$, and

$I_i = I(G_i)$ for $i = 1, 2$, then $I_{\mathfrak{p}} = ((I_1)_{\mathfrak{p}_1}, (I_2)_{\mathfrak{p}_2})$ and $\mathfrak{p}_2$ is a minimal vertex cover of $I_2$. By design, any vertex appearing in both $G_1$ and $G_2$ is not in $\mathfrak{p}$, and thus $(I_1)_{\mathfrak{p}_1}$ and $(I_2)_{\mathfrak{p}_2}$ do not share a vertex. Thus by Lemma 4.11 and the fact that associated primes localize, $\mathfrak{p} \in \operatorname{Ass}(R_{\mathfrak{p}}/(I_{\mathfrak{p}})^t)$ if and only if $\mathfrak{p}_1 \in \operatorname{Ass}(R_1/(I_1)^t)$ and $\mathfrak{p}_1$ is the maximal ideal of $R_1 = K[x \mid x \in \mathfrak{p}_1]$. For convenience, define $R_a = K[x \mid x \in \mathfrak{p}_a]$.

**Proposition 4.12.** *Let $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$. Using the notation from above, assume $\mathfrak{p}_1 = (I_1^t : c)$ for some monomial $c \in R_a$ of degree at most $2t - 1$. Let $x \in \mathfrak{p}_1$. Let $\mathfrak{p}_1' = \mathfrak{p}_1 \cup N(x)$, and let $\mathfrak{p}_2'$ be any minimal vertex cover of the edges of $G_2'$ where $G_2'$ is the induced subgraph of $G$ on the vertices $V \backslash \mathfrak{p}_1'$. Let $N_2' = \bigcup_{x \in \mathfrak{p}_1'} N(x) \backslash (\mathfrak{p}_1' \cup \mathfrak{p}_2')$. Let $G_1'$ be the induced subgraph of $G$ with vertices in $\mathfrak{p}_1' \cup N_2'$. Then $\mathfrak{p}' = (\mathfrak{p}_1', \mathfrak{p}_2') \in \operatorname{Ass}(R/I^{t+1})$.*

*Proof.* If $v$ is an isolated vertex of $G_2'$ then $N(v) \subset \mathfrak{p}_1'$ and thus every edge of $G$ containing $v$ is covered by $\mathfrak{p}_1'$. Hence $\mathfrak{p}'$ is a vertex cover of $G$. Since $x \in \mathfrak{p}_1$, there is an edge $xy \in G_1$ with $y \in \mathfrak{p}_a$. Consider $c' = cxy$. Then the degree of $c'$ is at most $2t + 1$, so $c' \notin (I_1)^{t+1}$. If $I_1' = I(G_1')$, then $c' \notin (I_1')^{t+1}$ as well. If $z \in \mathfrak{p}_1$, then $z(cxy) = (zc)(xy) \in (I_1')^{t+1}$. If $z \in N(x)$, then $z(cxy) = (cy)(zx) \in (I_1')^{t+1}$ since $y \in \mathfrak{p}_1$. Thus $\mathfrak{p}_1' \subset ((I_1')^{t+1} : c')$. Suppose $z \notin \mathfrak{p}_1'$ is a vertex of $G_1'$. Then $z \in N_2'$. Then $z \notin N(x)$ and $z \notin N(y)$ since $y \in \mathfrak{p}_a$, so $zx$ and $zy$ are not edges of $G_1'$. Also $z \notin \mathfrak{p}_1$ and $c \in R_a$, so $zc \notin (I_1')^{t+1}$. Thus the inclusion must be an equality.

Since $\mathfrak{p}_2'$ is a minimal vertex cover of the edges of $G_2'$, then $\mathfrak{p}_2' \in \operatorname{Ass}(R/I_2')$ where $I_2'$ is the edge ideal of $G_2'$ (where isolated vertices of $G_2'$ are not included in $I_2'$). Note that $I_{\mathfrak{p}'} = ((I_1')_{\mathfrak{p}_1'}, (I_2')_{\mathfrak{p}_2'})$ and so the result follows from Lemma 4.11. □

Note that if $G$ contains an odd cycle of length $2k + 1$, then embedded associated primes satisfying the hypotheses of Proposition 4.12 exist for $t \geq k + 1$ by Lemma 4.9 and Corollary 4.10. Starting with an induced odd cycle $C$ one can now recover all the primes described in [14, Theorem 3.3]. In addition, combining Corollary 4.10 with Lemma 4.11 as a starting place for Proposition 4.12 recovers the result from [14, Theorem 3.7] as well. Define $\operatorname{Ass}(R/I^t)^*$ to be the set of embedded associated primes of $I^t$ produced in Proposition 4.12 by starting from any odd cycle, or collection of odd cycles, of the graph. Then $\operatorname{Ass}(R/I^t)^* \subset \operatorname{Ass}(R/I^s)$ for all $s \geq t$. To see this, recall that if $\mathfrak{p}$ is not a minimal prime, then there is a vertex $x$ such that $x \cup N(x) \subset \mathfrak{p}$. Choosing such an $x$ results in $\mathfrak{p}_1 = \mathfrak{p}_1'$ and the process shows that $\mathfrak{p} \in \operatorname{Ass}(R/I^{t+1})$. Notice also that the sets $\operatorname{Ass}(R/I^t)^*$ stabilize. In particular, $\operatorname{Ass}(R/I^t)^* = \operatorname{Ass}(R/I^n)^*$ for all $t \geq n$ where $n$ is the number of variables. Notice that choosing $x \in N_1$ each time will eventually result in $\mathfrak{m} \in \operatorname{Ass}(R/I^t)^*$ for some $t$. Counting the maximal number of steps this could take provides a bound on the index of stability. Following the process above for a particular graph can often yield a significantly lower power $M$ for which $\operatorname{Ass}(R/I^t)^*$ stabilize. These results are collected below.

**Theorem 4.13.** *Let $I$ be the edge ideal of a connected graph $G$ that is not bipartite. Suppose $G$ has $n$ vertices and $s$ leaves, and $N$ is the index of stability of $I$.*

(i) [14, *Theorem* 4.1] *The process used in Proposition* 4.12 *produces all embedded associated primes in the stable set. That is,* $\operatorname{Ass}(R/I^N) = \operatorname{Min}(R/I) \cup \operatorname{Ass}(R/I^N)^*$.

(ii) [14, *Corollary* 4.3], *(Proposition* 4.12*) If the smallest odd cycle of $G$ has length $2k+1$, then $N \leq n-k-s$.*

(iii) [14, *Theorem* 5.6, *Corollary* 5.7] *If $G$ has a unique odd cycle, then* $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I) \cup \operatorname{Ass}(R/I^t)^*$ *for all $t$. Moreover, the sets* $\operatorname{Ass}(R/I^t)$ *form an ascending chain.*

(iv) *Suppose $\mathfrak{p} \in \operatorname{Ass}(R/I^N)$, and $N_0$ is the smallest positive integer for which $\mathfrak{p} \in \operatorname{Ass}(R/I^{N_0})^*$. Then $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ for all $t \geq N_0$.*

To interpret Theorem 4.13 in light of our earlier questions, notice that (i) answers Question 4.2, (ii) answers Question 4.1, and (iv) provides a good upper bound for $N_{\mathfrak{p}}$ in Question 4.3. The significance of (iii) is to answer a fourth question of interest. Before presenting that question, we first discuss some extensions of the above results to graphs containing loops.

**Corollary 4.14.** *Let $I$ be a monomial ideal, not necessarily square-free, such that the generators of $I$ have degree at most two. Define* $\operatorname{Ass}(R/I^t)^*$ *to be the set of embedded associate primes of $I^t$ produced in Proposition* 4.12 *by starting from any odd cycle, or collection of odd cycles where generators of $I$ that are not square-free are considered to be cycles of length one. Then the results of Theorem* 4.13 *hold for $I$.*

*Proof.* If $I$ has generators of degree one, then write $I = (I_1, I_2)$ where $I_2$ is generated in degree one. Then $I_2$ is a complete intersection, so by using Lemma 4.11 we may replace $I$ by $I_1$ and assume $I$ is generated in degree two. If $I$ is not square-free, consider a generator $x^2 \in I$. This generator can be represented as a loop (cycle of length one) in the graph. Define $\mathfrak{p}_a = (x)$ and $N_1 = N(x)$. Note that $\mathfrak{p}_1 = \mathfrak{p}_a \cup N_1 = (I_1 : c)$ where $I_1$ is the induced graph on $x \cup N(x)$ and $c = x$. Then $\mathfrak{p}_1$ satisfies the hypotheses of Proposition 4.12. The results now follow from the proof of Proposition 4.12. □

Notice that ideals that are not square-free will generally have embedded primes starting with $t = 1$ since the smallest odd cycle has length $1 = 2(0)+1$, so $k+1 = 1$. The above corollary can be extended to allow for any pure powers of variables to be generators of the ideal $I$.

**Corollary 4.15.** *Let $I = (I_1, I_2)$ where $I_2$ is the edge ideal of a graph $G$ and $I_1 = (x_{i_1}^{s_1}, \ldots, x_{i_r}^{s_r})$ for any powers $s_j \geq 1$. Then the results of Theorem* 4.13 *hold for $I$ with* $\operatorname{Ass}(R/I^t)^*$ *defined as in Corollary* 4.14.

*Proof.* As before, we may assume $x_j \geq 2$ for all $j$. Let $K = (x_{i_1}^2, \ldots, x_{i_r}^2)$ and let $J = (K, I_2)$. Then $J$ satisfies the hypotheses of Corollary 4.14. Let $\mathfrak{p} \in \operatorname{Ass}(R/J^t)^*$ be formed by starting with $\mathfrak{p}_a = (x_{i_1})$ and let $\mathfrak{p}_1 = (J_1^t : c)$ where $J_1$ and $c$ are defined as in Corollary 4.14. Suppose $x_{i_1}, \ldots, x_{i_v} \in \mathfrak{p}_1$. Let $q_j \geq 0$ be the least integers such that $c' = x_{i_1}^{q_1} \cdots x_{i_v}^{q_v} \cdot c \notin I_1^t$. Then it is straightforward to check that $\mathfrak{p}_1 = (I_1^t : c')$ and so $\mathfrak{p} \in \operatorname{Ass}(R/I^t)^*$. Thus higher powers of variables can also be treated as loops and the results of Theorem 4.13 hold. □

We now return to Theorem 4.13 (iii). In general, the sets $\operatorname{Ass}(R/I^t)^*$ form an ascending chain. Theorem 4.13 (iii) gives a class of graphs for which $\operatorname{Ass}(R/I^t)^*$ describe every embedded prime of a power of $I^t$ optimally. Thus $\operatorname{Ass}(R/I^t)$ will form a chain. This happens for many classes of monomial ideals, and leads to the fourth question.

**Question 4.16.** If $I$ is a square-free monomial ideal, is $\operatorname{Ass}(R/I^t) \subset \operatorname{Ass}(R/I^{t+1})$ for all $t$?

For monomial ideals, Question 4.16 is of interest for low powers of $I$. For sufficiently large powers, the sets of associated primes are known to form an ascending chain, and a bound beyond which the sets $\operatorname{Ass}(I^t/I^{t+1})$ form a chain has been shown by multiple authors (see [85, 99]). This bound depends on two graded algebras which encode information on the powers of $I$, and which will prove useful in other results. The first is the *Rees algebra* $R[It]$ of $I$, which is defined by

$$R[It] = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots$$

and the second is the *associated graded ring* of $I$,

$$\operatorname{gr}_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots .$$

Notice that while the result is for $\operatorname{Ass}(I^t/I^{t+1})$, for monomial ideals $\operatorname{Ass}(R/I^{t+1})$ will also form a chain.

**Theorem 4.17** ([85, 99]). $\operatorname{Ass}(I^t/I^{t+1})$ *is increasing for* $t > a^0_{R[It]_+}(\operatorname{gr}_I(R))$.

Note that square-free is essential in Question 4.16. Examples of monomial ideals for which the associated primes do not form an ascending chain have been given in [59, 65]. Those examples were designed for other purposes and so are more complex than what is needed here. A simple example can be found by taking the product of consecutive edges of an odd cycle.

**Example 4.18.** Let $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$, and let $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5)$. Then $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$ for $t = 1, 4$, but $\mathfrak{m} \notin \operatorname{Ass}(R/I^t)$ for $t = 2, 3$.

The ideal in Example 4.18 can be viewed as multiplying adjacent edges in a 5-cycle to form generators of $I$, and so has a simple combinatorial realization. A similar result holds for longer odd cycles, where the maximal ideal is not the only associate prime to appear and disappear. However, if instead

$$I = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$$

is the path ideal of the pentagon, then $\operatorname{Ass}(R/I^t) = \operatorname{Min}(R/I) \cup \{\mathfrak{m}\}$ for $t \geq 2$ and thus $\operatorname{Ass}(R/I^t)$ form an ascending chain (see [51, Example 3.14]).

There are some interesting cases where associated primes are known to form ascending chains. The first listed is quite general, but has applications to square-free monomial ideals.

**Theorem 4.19** ([84, Proposition 3.9], see also [56, Proposition 16.3]). *If $R$ is a Noetherian ring, then* $\operatorname{Ass}(R/\overline{I^t})$ *form an ascending chain.*

In order to present the next class of ideals for which the associated primes are known to form ascending chains, we first need some some background definitions.

**Definition 4.20.** Let $G$ be a graph. A *colouring* of the vertices of $G$ is an assignment of colours to the vertices of $G$ such that adjacent vertices have distinct colours. The *chromatic number* of $G$ is the minimal number of colours in a colouring of $G$. A graph is called *perfect* if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$.

An excellent reference for the theory of perfect graphs is the book of Golumbic [48]. Using perfect graphs, we now give an example to show how Theorem 4.19 can be applied to classes of square-free monomial ideals. An alternate proof appears in [42, Corollary 5.11].

**Example 4.21.** If $I$ is the ideal of minimal vertex covers of a perfect graph, then $\operatorname{Ass}(R/I^t)$ form an ascending chain.

*Proof.* By [112, Theorem 2.10], $R[It]$ is normal. Thus $I^t = \overline{I^t}$ for all $t$, so by Theorem 4.19, $\operatorname{Ass}(R/I^t)$ form an ascending chain. □

Similar results hold for other classes of monomial ideals for which $R[It]$ is known to be normal. For example, in [15, Corollary 4.2] it is shown that a path ideal of a rooted tree has a normal Rees algebra. Thus by Theorem 4.19, $\operatorname{Ass}(R/I^t)$ form an ascending chain for such ideals. Note that a path ideal can be viewed as the edge ideal of a carefully chosen uniform clutter.

It is interesting to compare the result of Example 4.21 to [42, Theorem 5.9], where it is shown that if $I$ is the ideal of minimal vertex covers of a perfect graph, then the set of primes associated to any fixed power has the saturated chain property. Here $\operatorname{Ass}(R/I^t)$ has the saturated chain property if for every $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$, either $\mathfrak{p}$ is minimal or there is a $Q \subsetneq \mathfrak{p}$ with $Q \in \operatorname{Ass}(R/I^t)$ and height $Q =$ height $\mathfrak{p} - 1$.

Notice that Theorem 4.13 shows that in the case of a graph with a unique odd cycle, Question 4.16 has an affirmative answer. This result can be generalized to any graph containing a leaf. First we need a slight variation of a previously known result.

**Lemma 4.22** ([88, Lemma 2.3]). *Suppose $I = I(G)$ is the edge ideal of a graph and $a \in I/I^2$ is a regular element of the associated graded ring $\operatorname{gr}_I(R)$. Then the sets $\operatorname{Ass}(R/I^t)$ form an ascending chain. Moreover, $\operatorname{Ass}(R/I^t) = \operatorname{Ass}(I^{t-1}/I^t)$ for all $t \geq 1$.*

*Proof.* Let $a \in I/I^2$ be a regular element of $\operatorname{gr}_I(R)$. Assume $\mathfrak{p} \in \operatorname{Ass}(I^t/I^{t+1})$. Then there is a $c \in I^t/I^{t+1}$ with $\mathfrak{p} = (0 :_{R/I} c)$. But then $\mathfrak{p} = (0 :_{R/I} ac)$, and $a$ lives in degree one, so $\mathfrak{p} \in \operatorname{Ass}(I^{t+1}/I^{t+2})$. So these sets form an ascending chain. Now the standard short exact sequence

$$0 \to I^t/I^{t+1} \to R/I^{t+1} \to R/I^t \to 0$$

gives

$$\operatorname{Ass}(I^t/I^{t+1}) \subset \operatorname{Ass}(R/I^{t+1}) \subset \operatorname{Ass}(R/I^t) \cup \operatorname{Ass}(I^t/I^{t+1})$$

and the result follows by induction. □

**Proposition 4.23.** *Let $G$ be a graph containing a leaf $x$ and let $I = I(G)$ be its edge ideal. Then $\operatorname{Ass}(R/I^t) \subset \operatorname{Ass}(R/I^{t+1})$ for all $t$. That is, the sets of associated primes of the powers of $I$ form an ascending chain.*

*Proof.* Since $x$ is a leaf of $G$, there is a unique generator $e = xy \in I$ divisible by $x$. Let $a$ denote the image of $e$ in $I/I^2$. We claim $a$ is a regular element of $\operatorname{gr}_I(R)$. To see this, it suffices to show that if $fe \in I^{t+1}$ for some $t$, then $f \in I^t$. Since $I$ is a monomial ideal and $e$ is a monomial, $fe \in I^{t+1}$ if and only if every term of $fe$ is in $I^{t+1}$. Thus we may assume $f$ is a monomial and $fxy = e_1 e_2 \cdots e_{t+1} h$ for some edges $e_i$ of $G$ and some monomial $h$. Suppose $f \notin I^t$. Then $x$ divides $e_i$ for some $i$, say $i = t + 1$. Since $x$ is a leaf, $e_i = xy$ and by cancellation $f = e_1 \cdots e_t h \in I^t$. Thus $a$ is a regular element of $\operatorname{gr}_I(R)$ and by Lemma 4.22 the result follows. □

When extending the above results to more general square-free monomial ideals, one needs to pass from graphs to clutters. An obstruction to extending the results is the

lack of an analog to Theorem 4.8 [100, Theorem 5.9]. One possible analog appears as a conjecture of Conforti and Cornuéjols, see [17, Conjecture 1.6], which we discuss later in this section. This conjecture is stated in the language of combinatorial optimization. It says that a clutter $\mathcal{C}$ has the *max-flow min-cut* (MFMC, see Definition 4.30) property if and only if $\mathcal{C}$ has the *packing* property. These criterion for clutters have been shown in recent years to have algebraic translations [45] which will be discussed in greater detail later in the section. An ideal satisfies the packing property if the monomial grade of $I$ (see Definition 4.29) is equal to the height of $I$ and this same equality holds for every minor of $I$ [17, 45]. Here a minor is formed by either localizing at a collection of variables, passing to the image of $I$ in a quotient ring $R/(x_{i_1}, x_{i_2}, \ldots, x_{i_s})$, or a combination of the two. In [47, Corollary 3.14] and [62, Corollary 1.6], it was shown that $\mathcal{C}$ satisfies MFMC if and only if the corresponding edge ideal $I(\mathcal{C})$ is normally torsion-free. This allows the conjecture to be restated (cf. [45, Conjecture 4.18]) as: if $\mathcal{C}$ has the packing property, then $I(\mathcal{C})$ is normally torsion-free.

Since a proof of this conjecture does not yet exist, the techniques used to describe the embedded associated primes, the stable set of associated primes, and the index of stability for graphs are difficult to extend. However some partial results are known. The first gives some conditions under which it is known that the maximal ideal is, or is not, an associated prime. In special cases, this can provide a seed for additional embedded associated primes using techniques such as those in Proposition 4.12.

**Theorem 4.24.** *If $I$ is a square-free monomial ideal, every proper minor of $I$ is normally torsion-free, and $\beta_1$ is the monomial grade of $I$, then*

(i) [51, *Corollary* 3.6] $\mathfrak{m} \notin \operatorname{Ass}(R/I^t)$ *for* $t \leq \beta_1$.

(ii) [51, *Theorem* 4.6] *If $I$ fails the packing property, then* $\mathfrak{m} \in \operatorname{Ass}(R/I^{\beta_1+1})$.

(iii) [51, *Proposition* 3.9] *If $I$ is unmixed and satisfies the packing property, then $I$ is normally torsion-free.*

Other recent results have taken a different approach. Instead of working directly with the edge ideal of a clutter $\mathcal{C}$, one can work with its Alexander dual, which is again the edge ideal of a clutter. Using this approach, the embedded associated primes of the Alexander dual have been linked to colorings of a clutter. Recall that $\chi(\mathcal{C})$ is the minimal number $d$ for which there is a partition $X_1, \ldots, X_d$ of the vertices of $\mathcal{C}$ for which for all edges $f$ of $\mathcal{C}$, $f \not\subset X_i$ for every $i$. A clutter is *critically $d$-chromatic* if $\chi(\mathcal{C}) = d$ but $\chi(\mathcal{C}\backslash\{x\}) < d$ for every vertex $x$.

**Theorem 4.25.** (i) [42, Corollary 4.6] *If $I$ is the ideal of covers of a clutter $\mathcal{C}$, and if the induced subclutter $\mathcal{C}_\mathfrak{p}$ on the vertices in $\mathfrak{p}$ is critically $(d+1)$-chromatic, then $\mathfrak{p} \in \operatorname{Ass}(R/I^d)$ but $\mathfrak{p} \notin \operatorname{Ass}(R/I^t)$ for any $t \leq d-1$.*

(ii) [42, *Theorem* 5.9] *If $I$ is the ideal of covers of a perfect graph $G$, then $\mathfrak{p} \in \operatorname{Ass}(R/I^t)$ if and only if the induced graph on the vertices in $\mathfrak{p}$ is a clique of size at most $t+1$.*

If one restricts to a particular power, then additional results on embedded associate primes are known. For example, in [41, Corollary 3.4] it is shown that if $I$ is the edge ideal of the Alexander dual of a graph $G$, then embedded primes of $R/I^2$ are in one-to-one correspondence with induced odd cycles of $G$. More precisely, $\mathfrak{p} \in \operatorname{Ass}(R/I^2)$ is an embedded prime if and only if the induced subgraph of $G$ on the vertices in $\mathfrak{p}$ is an induced odd cycle of $G$.

An interesting class of ideals, which is in a sense dual to the edge ideals of graphs, is unmixed square-free monomial ideals of height two. These are the Alexander duals of edge ideals of graphs, which can be viewed as edge ideals of clutters where, instead of requiring that each edge has two vertices, it is instead required that each minimal vertex cover has two vertices. For such ideals it has been shown in [40, Theorem 1.2] that an affirmative answer to a conjecture on graph colorings, [40, Conjecture 1.1], would imply an affirmative answer to Question 4.16. In [40, Corollary 3.11] it is shown that this conjecture holds for cliques, odd holes, and odd antiholes. Thus the Alexander duals of these special classes of graphs provide additional examples where Question 4.16 has an affirmative answer.

We now provide a more detailed discussion of the Conforti–Cornuéjols conjecture, followed by a collection of results which provide families of clutters where the conjecture is known to be true (such a family was already given in Theorem 4.24 (iii)). We also discuss some algebraic versions of this conjecture and how it relates to the depth of powers of edge ideals and to normality and torsion-freeness.

Having defined the notion of a *minor* for edge ideals, using the correpondence between clutters and square-free monomias ideals, we also have the notion of a *minor* of a clutter. We say that $\mathcal{C}$ has the *packing property* if $I(\mathcal{C})$ has this property.

**Definition 4.26.** Let $A$ be the incidence matrix of a clutter $\mathcal{C}$. The *set covering polyhedron* is the rational polyhedron:

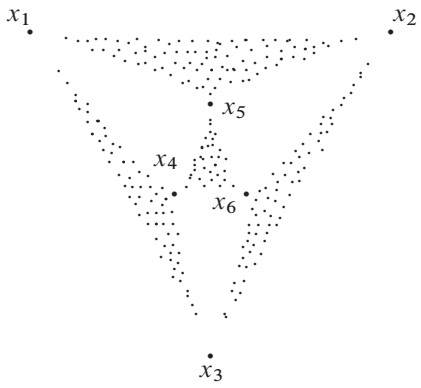
$$Q(A) = \{x \in \mathbb{R}^n \mid x \geq \mathbf{0},\ xA \geq \mathbf{1}\},$$

where $\mathbf{0}$ and $\mathbf{1}$ are vectors whose entries are equal to 0 and 1 respectively. Often we denote the vectors $\mathbf{0}$, $\mathbf{1}$ simply by 0, 1. We say that $Q(A)$ is *integral* if it has only integral vertices.

**Theorem 4.27** (A. Lehman [78; 17, Theorem 1.8]). *If a clutter $\mathcal{C}$ has the packing property, then $Q(A)$ is integral.*

The converse is not true. A famous example is the clutter $\mathcal{Q}_6$, given below. It does not pack and $Q(A)$ is integral.

**Example 4.28.** Let $I = (x_1x_2x_5, x_1x_3x_4, x_2x_3x_6, x_4x_5x_6)$. The figure:




corresponds to the clutter associated to $I$. This clutter will be denoted by $\mathcal{Q}_6$. Using *Normaliz* [11] we obtain that $\overline{R[It]} = R[It][x_1 \cdots x_6 t^2]$. Thus $R[It]$ is not normal. An interesting property of this example is that $\operatorname{Ass}(R/\overline{I^i}) = \operatorname{Ass}(R/I)$ for all $i$ (see [45]).

**Definition 4.29.** A set of edges of a clutter $\mathcal{C}$ is *independent* if no two of them have a common vertex. We denote the maximum number of independent edges of $\mathcal{C}$ by $\beta_1(\mathcal{C})$. We call $\beta_1(\mathcal{C})$ the *edge independence number* of $\mathcal{C}$ or the *monomial grade* of $I$.

Let $A$ be the incidence matrix of $\mathcal{C}$. The edge independence number and the covering number are related to min-max problems because they satisfy:

$$\begin{aligned}\alpha_0(\mathcal{C}) &\geq \min\{\langle 1, x\rangle |\, x \geq 0; xA \geq 1\}\\ &= \max\{\langle y, 1\rangle |\, y \geq 0; Ay \leq 1\} \geq \beta_1(\mathcal{C}).\end{aligned}$$

Notice that $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ if and only if both sides of the equality have integral optimum solutions.

**Definition 4.30.** A clutter $\mathcal{C}$, with incidence matrix $A$, satisfies the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle \alpha, x\rangle |\, x \geq 0; xA \geq 1\} = \max\{\langle y, 1\rangle |\, y \geq 0; Ay \leq \alpha\} \tag{4.1}$$

have integral optimum solutions $x$ and $y$ for each non-negative integral vector $\alpha$. The system $x \geq 0; xA \geq 1$ is called *totally dual integral* (TDI) if the maximum in (4.1) has an integral optimum solution $y$ for each integral vector $\alpha$ with finite maximum.

**Definition 4.31.** If $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ we say that the clutter $\mathcal{C}$ (or the ideal $I$) has the *König property*.

Note that $\mathcal{C}$ has the packing property if and only if every minor of $\mathcal{C}$ satisfies the König property. This leads to the following well-known result.

**Corollary 4.32** ([17]). *If a clutter $\mathcal{C}$ has the max-flow min-cut property, then $\mathcal{C}$ has the packing property.*

*Proof.* Assume that the clutter $\mathcal{C}$ has the max-flow min-cut property. This property is closed under taking minors. Thus it suffices to prove that $\mathcal{C}$ has the König property. We denote the incidence matrix of $\mathcal{C}$ by $A$. By hypothesis the LP-duality equation

$$\min\{\langle 1, x\rangle |\, x \geq 0; xA \geq 1\} = \max\{\langle y, 1\rangle |\, y \geq 0; Ay \leq 1\}$$

has optimum integral solutions $x$, $y$. To complete the proof notice that the left hand side of this equality is $\alpha_0(\mathcal{C})$ and the right-hand side is $\beta_1(\mathcal{C})$. □

Conforti and Cornuéjols [16] conjecture that the converse is also true.

**Conjecture 4.33** (Conforti–Cornuéjols). If a clutter $\mathcal{C}$ has the packing property, then $\mathcal{C}$ has the max-flow min-cut property.

An algebraic description of the packing property has already been given. In order to use algebraic techniques to attack this combinatorial conjecture, an algebraic translation is needed for the max-flow min-cut property. There are several equivalent algebraic descriptions of the max-flow min-cut property, as seen in the following result.

**Theorem 4.34** ([35,47,69]). *Let $\mathcal{C}$ be a clutter and let $I$ be its edge ideal. The following conditions are equivalent*:

(i) $\operatorname{gr}_I(R)$ *is reduced.*

(ii) $R[It]$ *is normal and $Q(A)$ is an integral polyhedron.*

(iii) $x \geq 0;\ xA \geq 1$ *is a* TDI system.

(iv) $\mathcal{C}$ *has the max-flow min-cut property.*

(v) $I^i = I^{(i)}$ *for $i \geq 1$, where $I^{(i)}$ is the $i$th symbolic power.*

(vi) $I$ *is normally torsion-free.*

By Theorems 4.34 and 4.27, Conjecture 4.33 reduces to:

**Conjecture 4.35** ([45]). If $I$ has the packing property, then $R[It]$ is normal.

Several variations of condition (ii) of Theorem 4.34 are possible. In particular, there are combinatorial conditions on the clutter that can be used to replace the normality of the Rees algebra. One such condition is defined below.

**Definition 4.36.** Let $\mathcal{C}^\vee$ be the clutter of minimal vertex covers of $\mathcal{C}$. The clutter $\mathcal{C}$ is called *diadic* if $|e \cap e'| \leq 2$ for $e \in E(\mathcal{C})$ and $e' \in E(\mathcal{C}^\vee)$

**Proposition 4.37** ([45]). *If $Q(A)$ is integral and $\mathcal{C}$ is diadic, then $I$ is normally torsion-free.*

Theorem 4.34 can be used to exhibit families of normally torsion-free ideals. Recall that a matrix $A$ is called *totally unimodular* if each $i \times i$ subdeterminant of $A$ is 0 or $\pm 1$ for all $i \geq 1$.

**Corollary 4.38.** *If $A$ is totally unimodular, then $I$ and $I^\vee$ are normally torsion-free.*

*Proof.* By [97] the linear system $x \geq 0$; $xA \geq 1$ is TDI. Hence $I$ is normally torsion-free by Theorem 4.34. Let $\mathcal{C}^\vee$ be the blocker (or Alexander dual) of $\mathcal{C}$. By [97, Corollary 83.1a (v), page 1441], we get that $\mathcal{C}^\vee$ satisfies the max-flow min-cut property. Hence $I(\mathcal{C}^\vee)$ is normally torsion-free by Theorem 4.34. Thus $I^\vee$ is normally torsion-free because $I(\mathcal{C}^\vee) = I^\vee$. □

In particular if $I$ is the edge ideal of a bipartite graph, then $I$ and $I^\vee$ are normally torsion-free.

Theorem 4.34 shows that the Rees algebra and the associated graded ring play an important role in the study of the max-flow min-cut property. An invariant related to the blowup algebras will also be useful. The *analytic spread* of an edge ideal $I$ is given by $\ell(I) = \dim R[It]/\mathfrak{m}R[It]$. If $\mathcal{C}$ is uniform, the analytic spread of $I$ is the rank of the incidence matrix of $\mathcal{C}$. The analytic spread of a monomial ideal can be computed in terms of the Newton polyhedron of $I$, see [4]. The next result follows directly from [83, Theorem 3].

**Proposition 4.39.** *If $Q(A)$ is integral, then $\ell(I) < n = \dim(R)$.*

To relate this result on $\ell(I)$ to Conjecture 4.33 (or equivalently to Conjecture 4.35) we first need to recall the following bound on the depths of the powers of an ideal $I$.

**Theorem 4.40.** $\inf_i\{\operatorname{depth}(R/I^i)\} \leq \dim(R) - \ell(I)$. *If $\operatorname{gr}_I(R)$ is Cohen–Macaulay, then the equality holds.*

This inequality is due to Burch [12] (cf. [70, Theorem 5.4.7]), while the equality comes from [31]. By a result of Brodmann [8], depth $R/I^k$ is constant for $k \gg 0$. Broadmann improved Burch's inequality by showing that the constant value is bounded by $\dim(R) - \ell(I)$. For a study of the initial and limit behaviour of the numerical function $f(k) = \operatorname{depth} R/I^k$ see [59].

**Theorem 4.41** ([68]). *Let $R$ be a Cohen–Macaulay ring and let $I$ be an ideal of $R$ containing regular elements. If $R[It]$ is Cohen–Macaulay, then $\operatorname{gr}_I(R)$ is Cohen–Macaulay.*

**Proposition 4.42.** *Let $\mathcal{C}$ be a clutter and let $I$ be its edge ideal. Let $J_i$ be the ideal obtained from $I$ by making $x_i = 1$. If $Q(A)$ is integral, then $I$ is normal if and only if $J_i$ is normal for all $i$ and* $\operatorname{depth}(R/I^k) \geq 1$ *for all $k \geq 1$.*

*Proof.* Assume that $I$ is normal. The normality of an edge ideal is closed under taking minors [35], hence $J_i$ is normal for all $i$. By hypothesis the Rees algebra $R[It]$ is normal. Then $R[It]$ is Cohen–Macaulay by a theorem of Hochster [66]. Then the ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay by Theorem 4.41. Hence using Theorem 4.40 and Proposition 4.39 we get that $\operatorname{depth}(R/I^i) \geq 1$ for all $i$. The converse follows readily adapting the arguments given in the proof of the normality criterion presented in [35]. □

By Proposition 4.42 and Theorem 4.27, we get that Conjecture 4.33 also reduces to:

**Conjecture 4.43.** If $I$ has the packing property, then $\operatorname{depth}(R/I^i) \geq 1$ for all $i \geq 1$.

We conclude this section with a collection of results giving conditions under which Conjecture 4.33, or its equivalent statements mentioned above, is known to hold. For uniform clutters it suffices to prove Conjecture 4.33 for Cohen–Macaulay clutters [23].

**Proposition 4.44** ([45]). *Let $\mathcal{C}$ be the collection of bases of a matroid. If $\mathcal{C}$ satisfies the packing property, then $\mathcal{C}$ satisfies the max-flow min-cut property.*

When $G$ is a graph, integrality of $Q(A)$ is sufficient in condition (ii) of Theorem 4.34, and the packing property is sufficient to imply the max-flow min-cut property, thus providing another class of examples for which Conjecture 4.33 holds.

**Proposition 4.45** ([17, 45]). *If $G$ is a graph and $I = I(G)$, then the following are equivalent:*

(i) $\operatorname{gr}_I(R)$ *is reduced.*

(ii) $G$ *is bipartite.*

(iii) $Q(A)$ *is integral.*

(iv) $G$ *has the packing property.*

(v) $G$ *has the max-flow min-cut property.*

(vi) $\overline{I^i} = I^{(i)}$ *for $i \geq 1$.*

**Definition 4.46.** A clutter is *binary* if its edges and its minimal vertex covers intersect in an odd number of vertices.

**Theorem 4.47** ([98]). *A binary clutter $\mathcal{C}$ has the max-flow min-cut property if and only if $\mathcal{Q}_6$ is not a minor of $\mathcal{C}$.*

**Corollary 4.48.** *If $\mathcal{C}$ is a binary clutter with the packing property, then $\mathcal{C}$ has the max-flow min-cut property.*

**Proposition 4.49** ([112]). *Let $\mathcal{C}$ be a uniform clutter and let $A$ be its incidence matrix. If the polyhedra*

$$P(A) = \{x \,|\, x \geq 0;\ xA \leq 1\} \quad \textit{and} \quad Q(A) = \{x \,|\, x \geq 0;\ xA \geq 1\}$$

*are integral, then $\mathcal{C}$ has the max-flow min-cut property.*

In light of Theorem 4.27, this result implies that if $P(A)$ is integral and $\mathcal{C}$ has the packing property, then $\mathcal{C}$ has the max-flow min-cut property. An open problem is to show that this result holds for non-uniform clutters (see [82, Conjecture 1.1]).

A *Meyniel graph* is a simple graph in which every odd cycle of length at least five has at least two chords. The following gives some support to [82, Conjecture 1.1] because Meyniel graphs are perfect [97, Theorem 66.6].

**Theorem 4.50** ([82]). *Let $\mathcal{C}$ be the clutter of maximal cliques of a Meyniel graph. If $\mathcal{C}$ has the packing property, then $\mathcal{C}$ has the max-flow min-cut property.*

Let $P = (X, \prec)$ be a *partially ordered set* (*poset* for short) on the finite vertex set $X$ and let $G$ be its *comparability graph*. Recall that the vertex set of $G$ is $X$ and the edge set of $G$ is the set of all unordered pairs $\{x_i, x_j\}$ such that $x_i$ and $x_j$ are comparable.

**Theorem 4.51** ([25]). *If $G$ is a comparability graph and $\mathcal{C}$ is the clutter of maximal cliques of $G$, then the edge ideal $I(\mathcal{C})$ is normally torsion free.*

**Theorem 4.52** ([24]). *Let $\mathcal{C}$ be a uniform clutter with a perfect matching such that $\mathcal{C}$ has the packing property and $\alpha_0(\mathcal{C}) = 2$. If the columns of the incidence matrix of $\mathcal{C}$ are linearly independent, then $\mathcal{C}$ has the max-flow min-cut property.*

**Acknowledgments.** The software package *Macaulay2* [49] was used to compute many of the examples in this paper, including Example 4.18, which was originally discovered by three undegraduate students (Mike Alwill, Katherine Pavelek, and Jennifer von Reis) in an unpublished project in 2002. The authors would like to thank an anonymous referee for providing us with useful comments and suggestions.

## Bibliography

[1] Barile, M., On the arithmetical rank of the edge ideals of forests. Comm Algebra. 2008; 36(12):4678–4703.

[2] Berge, C., Some common properties for regularizable graphs, edge-critical graphs and $B$-graphs. In: Rosa, A., Sabidussi, G., Turgeon, J., editors. Theory and practice of combinatorics. vol. 60 of North-Holland mathematics studies. Amsterdam: North-Holland; 1982. p. 31–44.

[3] Bermejo, I., Gimenez, P., Saturation and Castelnuovo–Mumford regularity. J. Algebra. 2006;303(2):592–617.

[4] Bivià-Ausina, C., The analytic spread of monomial ideals. Comm Algebra. 2003;31(7): 3487–3496.

[5] Björner, A., Topological methods. In: Handbook of combinatorics, Vol. 1, 2. Amsterdam: Elsevier; 1819–1872, 1995.

[6] Björner, A., Wachs, M., Shellable nonpure complexes and posets I, Trans Amer Math Soc. 1996;348:1299–1327.

[7] Bollobás, B., Modern graph theory. vol. 184 ofGraduate texts in mathematics 184. New York: Springer-Verlag; 1998.

[8] Brodmann, M., The asymptotic nature of the analytic spread. Math Proc Cambridge Philos Soc. 1979;86:35–39.

[9] Brodmann, M., Asymptotic stability of Ass($M/I^nM$). Proc Amer Math Soc. 1979;74: 16–18.

[10] Bruns, W., Herzog, J., Cohen–Macaulay rings. Revised ed. Cambridge: Cambridge University Press; 1997.

[11] Bruns, W., Ichim, B., Normaliz 2.0, Computing normalizations of affine semigroups. 2008. Available from: http://www.math.uos.de/normaliz.

[12] Burch, L., Codimension and analytic spread. Proc Cambridge Philos Soc. 1972;72:369–373.

[13] Busch, A. H., Dragan, F. F., Sritharan, R., New min-max theorems for weakly chordal and dually chordal graphs. vol. 6509 of Lecture notes in computer science. Springer-Verlag; 2010. p. 207–218.

[14] Chen, J., Morey, S., Sung, A., The stable set of associated primes of the ideal of a graph. Rocky Mountain J Math. 2002;32:71–89.

[15] Conca, A., De Negri, E., $M$-sequences, graph ideals, and ladder ideals of linear type. J Algebra. 1999;211:599–624.

[16] Conforti, M., Cornuéjols, G., Clutters that pack and the Max-Flow Min-Cut property: a conjecture. In: Pulleyblank, W. R., Shepherd, F. B., editors. The Fourth Bellairs Workshop on Combinatorial Optimization. 1993.

[17] Cornuéjols, G., Combinatorial optimization: packing and covering. vol. 74 of CBMS-NSF regional conference series in applied mathematics. SIAM; 2001.

[18] Corso, A., Nagel, U., Monomial and toric ideals associated to ferrers graphs. Trans Amer Math Soc. 2009;361(3):1371–1395.

[19] Dalili, K., Kummini, M., Dependence of betti numbers on characteristic. Preprint. arXiv:1009.4243v1. 2010.

[20] Dalili, K., Faridi, S., Traves, W., The reconstruction conjecture and edge ideals. Discrete Math. 2008;308:2002–2010.

[21] Dirac, G. A., On rigid circuit graphs. Abh Math Semin Univ Hambg. 1961;38:71–76.

[22] Dochtermann, A., Engstrom, A., Algebraic properties of edge ideals via combinatorial topology. Electron J Combin. 2009;16(2):R2.

[23] Dupont, L. A., Reyes, E., Villarreal, R. H., Cohen–Macaulay clutters with combinatorial optimization properties and parallelizations of normal edge ideals. São Paulo J Math Sci. 2009;3(1):61–75.

[24] Dupont, L. A., Villarreal, R. H., Algebraic and combinatorial properties of ideals and algebras of uniform clutters of TDI systems. J Comb Optim. 2011;21(3):269–292.

[25] Dupont, L. A., Villarreal, R. H., Edge ideals of clique clutters of comparability graphs and the normality of monomial ideals. Math Scand. 2010;106(1):88–98.

[26] Duval, A. M., Algebraic shifting and sequentially Cohen–Macaulay simplicial complexes. Electron J Combin. 1996;3(1):Research paper 21.

[27] Eagon, J. A., Reiner, V., Resolutions of Stanley–Reisner rings and Alexander duality. J Pure Appl Algebra. 1998;130:265–275.

[28] Eisenbud, D., Commutative algebra with a view toward algebraic geometry. vol. 150 of Graduate texts in mathematics. Springer-Verlag; 1995.

[29] Eisenbud, D., The geometry of syzygies: a second course in commutative algebra and algebraic geometry. vol. 229 of Graduate texts in mathematics. New York: Springer; 2005.

[30] Eisenbud, D., Goto, S., Linear free resolutions and minimal multiplicity. J Algebra. 1984;88(1):189–133.

[31] Eisenbud, D., Huneke, C., Cohen–Macaulay Rees algebras and their specializations. J Algebra. 1983;81:202–224.

[32] Emtander, E., A class of hypergraphs that generalizes chordal graphs. Math Scand. 2010;106(1):50–66.

[33] Ene, V., Olteanu, O., Terai, N., Arithmetical rank of lexsegment edge ideals. Bull Math Soc Sci Math Roumanie (N S). 2010;53(4):315–327.

[34] Erdős, P., Gallai, T., On the minimal number of vertices representing the edges of a graph. Magyar Tud Akad Mat Kutató Int Közl. 1961;6:181–203.

[35] Escobar, C., Villarreal, R. H., Yoshino, Y., Torsion freeness and normality of blowup rings of monomial ideals. In: Commutative algebra. vol. 244 of Lecture notes in pure and applied mathematics. Boca Raton, FL: Chapman & Hall/CRC; 2006. p. 69–84.

[36] Estrada, M., Villarreal, R. H., Cohen–Macaulay bipartite graphs. Arch Math. 1997;68: 124–128.

[37] Farber, M., Characterizations of strongly chordal graphs. Discrete Math. 1983;43(2–3): 173–189.

[38] Faridi, S., Cohen–Macaulay properties of square-free monomial ideals. J Combin Theory Ser A. 2005;109(2):299–329.

[39] Faridi, S., Simplicial trees are sequentially Cohen–Macaulay. J Pure Appl Algebra. 2004;190:121–136.

[40] Francisco, C., Hà, H. T., Van Tuyl, A., A conjecture on critical graphs and connections to the persistence of associated primes. Discrete Math. 2010;310:2176–2182.

[41] Francisco, C., Hà, H. T., Van Tuyl, A., Associated primes of monomial ideals and odd holes in graphs. J Algebraic Combin. 2010;32:287–301.

[42] Francisco, C., Hà, H. T., Van Tuyl, A., Colorings of hypergraphs, perfect graphs, and associated primes of powers of monomial ideals. J Algebra. 2011;331(1):224–242.

[43] Francisco, C. A., Van Tuyl, A., Sequentially Cohen–Macaulay edge ideals. Proc Amer Math Soc. 2007;135:2327–2337.

[44] Fröberg, R., On Stanley–Reisner rings. In: Balcerzyk, S., Józefiak, T., Krempa, J., Simson, D., Vogel, W., editors. Topics in algebra, Part 2. Polish Scientific Publishers; 1990. p. 57–70.

[45] Gitler, I., Reyes, E., Villarreal, R. H., Blowup algebras of square-free monomial ideals and some links to combinatorial optimization problems. Rocky Mountain J Math. 2009; 39(1):71–102.

[46] Gitler, I., Valencia, C., Bounds for invariants of edge-rings. Comm Algebra. 2005;33: 1603–1616.

[47] Gitler, I., Valencia, C., Villarreal, R. H., A note on Rees algebras and the MFMC property. Beiträge Algebra Geom. 2007;48(1):141–150.

[48] Golumbic, M. C., Algorithmic graph theory and perfect graphs, second edition. vol. 57 of Annals of discrete mathematics. Amsterdam: Elsevier Science B.V.; 2004.

[49] Grayson, D. R., Stillman, M. E., Macaulay2, a software system for research in algebraic geometry. 1996. Available from: http://www.math.uiuc.edu/Macaulay2/.

[50] Greuel, G. M., Pfister, G., A singular introduction to commutative algebra. 2nd extended ed. Berlin: Springer; 2008.

[51] Hà, H. T., Morey, S., Embedded associated primes of powers of square-free monomial ideals. J Pure Appl Algebra. 2010;214(4):301–308.

[52] Hà, H. T., Morey, S., Villarreal, R. H., Cohen–Macaulay admissible clutters. J Commut Algebra. 2009;1(3):463–480.

[53] Hà, H. T., Van Tuyl, A., Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers. J Algebraic Combin. 2008;27:215–245.

[54] Harary, F., Graph theory. Reading, MA: Addison-Wesley; 1972.

[55] He, J., Van Tuyl, A., Algebraic properties of the path ideal of a tree. Comm Algebra. 2010;38(5):1725–1742.

[56] Herrmann, M., Ikeda, S., Orbanz, U., Equimultiplicity and blowing up. An algebraic study. Berlin: Springer-Verlag; 1988.

[57] Herzog, J., A Generalization of the Taylor complex construction. Comm Algebra. 2007; 35(5):1747–1756.

[58] Herzog, J., Hibi, T., Componentwise linear ideals. Nagoya Math J. 1999;153:141–153.

[59] Herzog, J., Hibi, T., The depth of powers of an ideal. J Algebra. 2005;291:534–550.

[60] Herzog, J., Hibi, T., Distributive lattices, bipartite graphs and Alexander duality. J Algebraic Combin. 2005;22(3):289–302.

[61] Herzog, J., Hibi, T., Monomial ideals. vol. 260 of Graduate texts in mathematics. Springer-Verlag; 2011.

[62] Herzog, J., Hibi, T., Trung, N. V., Zheng, X., Standard graded vertex cover algebras, cycles and leaves. Trans Amer Math Soc. 2008;360:6231–6249.

[63] Herzog, J., Hibi, T., Zheng, X., Dirac's theorem on chordal graphs and Alexander duality. European J Combin. 2004;25(7):949–960.

[64] Herzog, J., Hibi, T., Zheng, X., Monomial ideals whose powers have a linear resolution. Math Scand. 2004;95:23–32.

[65] Hoa, L. T., Stability of associated primes of monomial ideals. Vietnam J Math. 2006; 34(4):473–487.

[66] Hochster, M., Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes. Ann of Math. 1972;96:318–337.

[67] Hochster, M., Criteria for the equality of ordinary and symbolic powers of primes. Math Z. 1973;133:53–65.

[68] Huneke, C., On the associated graded ring of an ideal. Illinois J Math. 1982;26:121–137.

[69] Huneke, C., Simis, A., Vasconcelos, W. V., Reduced normal cones are domains. Contemp Math. 1989;88:95–101.

[70] Huneke, C., Swanson, I., Integral closure of ideals rings, and modules. vol. 336 of London Mathematical Society lecture note series. Cambridge: Cambridge University Press; 2006.

[71] Ionescu, C., Rinaldo, G., Some algebraic invariants related to mixed product ideals. Arch Math. 2008;91:20–30.

[72] Jacques, S., Betti numbers of graph ideals [PhD thesis]. University of Sheffield; 2004. arXiv:math/0410107v1.

[73] Kalai, G., Meshulam, R., Unions and intersections of Leray complexes. J Combin Theory Ser A. 2006;113:1586–1592.

[74] Katzman, M., Characteristic-independence of Betti numbers of graph ideals. J Combin Theory Ser A. 2006;113(3):435–454.

[75] Kiani, D., Moradi, S., Bounds for the regularity of edge ideal of vertex decomposable and shellable graphs. Preprint. arXiv:1007.4056v1. 2010.

[76] Kimura, K., Terai, N., Yoshida, K., Arithmetical rank of squarefree monomial ideals of small arithmetic degree. J Algebraic Combin. 2009;29:389–404.

[77] Kummini, M., Regularity, depth and arithmetic rank of bipartite edge ideals. J Algebraic Combin. 2009;30(4):429–445.

[78] Lehman, A., On the width-length inequality and degenerate projective planes. In: Cook, W., Seymour, P., editors. Polyhedral Combinatorics. vol. 1 of DIMACS series in discrete mathematics and theoretical computer science. American Mathematical Society; 1990, p. 101–105.

[79] Lyubeznik, G., On the local cohomology modules $H^i_{\mathcal{A}}(R)$ for ideals $\mathcal{A}$ generated by monomials in an $R$-sequence. In: Greco, S., Strano, R., editors. Complete intersections. Lectures given at the 1st 1983 Session of the Centro Internazionale Matematico Estivo (C.I.M.E.); 1983, Jun 13–21; Acireale, Italy. Berlin-Heidelberg: Springer. p. 214–220.

[80] Lyubeznik, G., On the arithmetical rank of monomial ideals. J Algebra. 1988;112:86–89.

[81] Mahmoudi, M., Mousivand, A., Crupi, M., Rinaldo, G., Terai, N., Yassemi, S., Vertex decomposability and regularity of very well-covered graphs. J Pure Appl Algebra. 2011; 215(10):2473–2480.

[82] Martínez-Bernal, J., O'Shea, E., Villarreal, R. H., Ehrhart clutters: regularity and max-flow min-cut. Electron J Combin. 2010;17(1):R52.

[83] McAdam, S., Asymptotic prime divisors and analytic spreads. Proc Amer Math Soc. 1980;80:555–559.

[84] McAdam, S., Asymptotic prime divisors. vol. 103 of Lecture notes in mathematics. New York: Springer-Verlag; 1983.

[85] McAdam, S., Eakin, P., The asymptotic Ass. J Algebra. 1979;61(1):71–81.

[86] Miller, E., Sturmfels, B., Combinatorial commutative algebra. vol. 227 of Graduate texts in mathematics. Springer; 2004.

[87] Morey, S., Depths of powers of the edge ideal of a tree. Comm Algebra. 2010;38(11): 4042–4055.

[88] Morey, S., Stability of associated primes and equality of ordinary and symbolic powers of ideals. Comm Algebra. 1999;27:3221–3231.

[89] Morey, S., Reyes, E., Villarreal, R. H., Cohen–Macaulay, shellable and unmixed clutters with a perfect matching of König type. J Pure Appl Algebra. 2008;212(7):1770–1786.

[90] Nevo, E., Regularity of edge ideals of $C_4$-free graphs via the topology of the lcm-lattice. J Combin Theory Ser A. 2011;118(2):491–501.

[91] Peeva, I., Stillman, M., Open problems on syzygies and Hilbert functions. J Commut Algebra. 2009;1(1):159–195.

[92] Plummer, M. D., Some covering concepts in graphs. J Combin Theory. 1970;8:91–98.

[93] Plummer, M. D., Well-covered graphs: a survey. Quaest Math. 1993;16(3):253–287.

[94] Ravindra, G., Well-covered graphs. J Comb Inf Syst Sci. 1977;2(1):20–21.

[95] Renteln, P., The Hilbert series of the face ring of a flag complex. Graphs Combin. 2002; 18(3):605–619.

[96] Schrijver, A., Theory of linear and integer programming. New York: John Wiley & Sons; 1986.

[97] Schrijver, A., Combinatorial optimization. vol. 24 of Algorithms and combinatorics. Berlin: Springer-Verlag; 2003.

[98] Seymour, P. D., The matroids with the max-flow min-cut property. J Combin Theory Ser B. 1977;23:189–222.

[99] Sharp, R., Convergence of sequences of sets of associated primes. Proc Amer Math Soc. 2003;131:3009–3017.

[100] Simis, A., Vasconcelos, W. V., Villarreal, R. H., On the ideal theory of graphs. J Algebra. 1994;167:389–416.

[101] Smith, D. E., On the Cohen–Macaulay property in commutative algebra and simplicial topology. Pacific J Math. 1990;141:165–196.

[102] Stanley, R. P., Combinatorics and commutative algebra. 2nd ed. vol. 41 of Progress in mathematics. Boston, MA: Birkhäuser Boston, Inc., 1996.

[103] Terai, N., Alexander duality theorem and Stanley–Reisner rings. Sūrikaisekikenkyūsho Kōkyūroku 1999;1078(Free resolutions of coordinate rings of projective varieties and related topics; 1998; Kyoto):174–184.

[104] Toft, B., Colouring, stable sets and perfect graphs. In: Graham, R. L., Grötschel, M., Lovász, L., editors. Handbook of combinatorics I. Elsevier; 1995. p. 233–288.

[105] Trung, T. N., Stability of associated primes of integral closures of monomial ideals. J Combin Theory Ser A 2009;116:44–54.

[106] Van Tuyl, A., Sequentially Cohen–Macaulay bipartite graphs: vertex decomposability and regularity. Arch Math. 2009;93(5):451–459.

[107] Van Tuyl, A., Villarreal, R. H., Shellable graphs and sequentially Cohen–Macaulay bipartite graphs. J Combin Theory Ser A. 2008;115(5):799–814.

[108] Vasconcelos, W. V., Computational methods in commutative algebra and algebraic geometry. Springer-Verlag: 1998.

[109] Villarreal, R. H., Cohen–Macaulay graphs. Manuscripta Math. 1990;66:277–293.

[110] Villarreal, R. H., Monomial algebras. vol. 238 of Monographs and textbooks in pure and applied mathematics. New York: Marcel Dekker; 2001.

[111] Villarreal, R. H., Unmixed bipartite graphs. Rev Colombiana Mat. 2007;41(2):393–395.

[112] Villarreal, R. H., Rees algebras and polyhedral cones of ideals of vertex covers of perfect graphs. J Algebraic Combin. 2008;27:293–305.

[113] Voloshin, V. I., Coloring mixed hypergraphs: theory, algorithms and applications. vol. 17 of Fields institute monographs. Providence, RI: American Mathematical Society; 2002.

[114] Whieldon, G., Jump sequences of edge ideals. Preprint. arXiv:1012.0108v1. 2010.

[115] Woodroofe, R., Chordal and sequentially Cohen–Macaulay clutters. Electron J Combin. 2011;18(1):R208.

[116] Woodroofe, R., Vertex decomposable graphs and obstructions to shellability. Proc Amer Math Soc. 2009;137:3235–3246.

[117] Woodroofe, R., Matchings, coverings, and Castelnuovo–Mumford regularity. Preprint. arXiv:1009.2756. 2010.

[118] Zheng, X., Resolutions of facet ideals. Comm Algebra. 2004;32:2301–2324.

## Author Information

Susan Morey, Department of Mathematics, Texas State University, San Marcos, TX, USA.
E-mail: morey@txstate.edu

Rafael H. Villarreal, Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Mexico City, D.F., Mexico.
E-mail: vila@math.cinvestav.mx