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Recent Progress in Coherent Rings: a Homological Perspective

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Abstract. Theories of coherent Cohen–Macaulay and Gorenstein rings have recently been developed by Hamilton and Marley, and Hummel and Marley, respectively. This work summarizes these theories after introducing the homological framework upon which they are built. We also explore recent developments in the theory of homological dimensions. These developments may provide further insight into the properties of coherent Cohen–Macaulay and Gorenstein rings, in addition to insight into the development of a characterization for coherent complete intersection rings.

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1 Introduction

Homological dimensions have been studied by Auslander and Bridger [4], Gerko [23], Avramov, Gasharov and Peeva [6], as well as many others, to create dimensions characterizing local *Noetherian* regular, complete intersection, Gorenstein, and Cohen–Macaulay rings. Others, including Bennis and Mahdou [9, 10], and Mao and Ding [37, 39], explored (global) Gorenstein dimensions in the coherent and Noetherian contexts. Through these explorations, a homological dimension introduced by Stenström [46] has been connected to flat and Gorenstein dimensions.

Concurrent with these activities is the exploration of the meaning of regular, complete intersection, Gorenstein and Cohen–Macaulay in the coherent context. A coherent ring is *regular* if every finitely generated ideal of the ring has finite projective dimension [11]. Glaz posed the question of whether there existed a theory of coherent Cohen Macaulay rings such that coherent regular rings are Cohen–Macaulay (see [26] and [27]). Hamilton and Marley [32] provided a positive answer to this question through homological methods. Hummel and Marley [34] extended the notion of Gorenstein dimension to lay the foundation for a theory of non-Noetherian, and even non-coherent, Gorenstein rings. This foundation has played a part in creating a rich theory of coherent rings where coherent regular, Gorenstein, and Cohen–Macaulay rings behave (mostly) like their Noetherian counterparts. Complete intersections are

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thus far the missing character in this theory; their foundations likely still lie within the realm of homological dimensions.

The groundwork for the coherent theory comes from both homological dimensions and local cohomology. This work describes how the ever growing homological theory impacts the theory of coherent rings, and explores the homological methods that may lead to the expansion of this theory.

2 Coherent Rings and Grade

We say $(R, \mathfrak{m})$ is a local ring if it has a unique maximal ideal $\mathfrak{m}$. In this work all rings will be commutative, unless otherwise noted.

2.1 Coherent Rings and $(\mathrm{FP})_\infty^R$ Modules

A finitely generated module M of a ring R is *coherent* if every finitely generated submodule of M is finitely presented. A ring R is coherent if it is coherent as an R-module. Additional characterizations of coherent rings can be found in [25].

One important characterization of coherent rings is that any finitely presented module M over a coherent ring has an infinite resolution by finite free modules [25]. Moreover, $M^* = \mathrm{Hom}_R(M, R)$ has the same property. This property of modules in a coherent ring is denoted $(\mathrm{FP})_\infty^R$ by Bieri.

Definition 2.1 ([12]). Let R be a ring and let M be an R-module. M is $(\mathrm{FP})_\infty^R$ if M admits an infinite resolution of finitely generated free modules. If, instead, M admits a finite free resolution of length n, we say M is $(\mathrm{FP})_n^R$.

It follows that for any finitely presented R-module M over a coherent ring R, M and M^* are $(\mathrm{FP})_\infty^R$. Many of the properties of $(\mathrm{FP})_\infty^R$ modules have been explored in [12], [42], and [34]. Most of the interesting properties of $(\mathrm{FP})_\infty^R$ modules usually occur in the case where both the module and its dual are $(\mathrm{FP})_\infty^R$, that is, in the coherent-like case [34]. The assumption of M and M^* being in $(\mathrm{FP})_\infty^R$ carries the full force of coherence, without additional restrictions associated with the coherence assumption (see [34] and Section 4.2).

2.2 Non-Noetherian Grade

In the Noetherian case, the classical Noetherian notion of the *Depth*, or grade, of a module over an ideal I is defined as $\mathrm{Depth}_I\, M = \sup\{n | x_1, \dots, x_n \in I$ is an M-regular sequence$\}$. In the Noetherian case, Depth exhibits the following property.

Proposition 2.2 ([43]). *Let R be a local Noetherian ring, and let M be a finitely generated R-module. Then* $\mathrm{Depth}_I\, M > 0$ *if and only if* $(0 :_M I) = 0$.

However, there are examples of non-Noetherian rings where Proposition 2.2 does not hold (see, for instance [32]). To rectify this incongruity, Hochster extended Depth to non-Noetherian rings.

Definition 2.3 ([33]). Let R be a ring, let M be an R-module, and let I be an ideal of R. The depth of M with respect to I is defined as

$$\operatorname{depth}_I M = \sup\{\operatorname{Depth}_{IS}(M \otimes_R S) | S \text{ faithfully flat extension of R}\}.$$

If $(R, \mathfrak{m})$ is a local ring and $I = \mathfrak{m}$, then denote $\operatorname{depth}_{\mathfrak{m}} M = \operatorname{depth}_R M$, or $\operatorname{depth} M$ when the ring is unambiguous.

In the literature, depth has also been called polynomial, or p-depth (see [32, 42]). This definition of depth has most of the expected properties, which are summarized below.

Proposition 2.4 ([8; 33; 25, Chapter 7; 43, Chapter 5; 13, Section 9]). *Let M be an R-module and let I be an ideal of R such that $IM \neq M$.*

(i) $\operatorname{depth}_I M = \sup\{\operatorname{depth}_J M | J \subset I, J$ *finitely generated ideal*$\}$.

(ii) *Let $I = (x_1, \dots, x_n)$ and $H_j(\mathbf{x}, M)$ denote the jth Koszul homology of $\mathbf{x} = x_1, \dots, x_n$ on M, then* $\operatorname{depth}_I M = \inf\{i \geq 0 | H_{n-i}(\mathbf{x}, M) \neq 0\}$.

(iii) $\operatorname{depth}_I M = \operatorname{depth}_{IS}(M \otimes_R S)$ *for any faithfully flat R-algebra S.*

(iv) *If* $\operatorname{depth}_I M > 0$, *then* $\operatorname{Depth}_{IS}(M \otimes_R S) > 0$ *where $S = R[X]$ is a polynomial ring in one variable over R.*

(v) *If I is generated by n elements, then* $\operatorname{depth}_I M = \operatorname{Depth}_{IS}(M \otimes_R S)$ *where $S = R[X_1, \dots X_n]$.*

(vi) $\operatorname{depth}_I M = \operatorname{depth}_{\sqrt{I}} M$.

(vii) *If $x \in I$ is M-regular, then* $\operatorname{depth}_I M = \operatorname{depth}_I M/xM + 1$.

(viii) *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of R-modules such that $IL \neq L$ and $IN \neq N$. If* $\operatorname{depth}_I M > \operatorname{depth}_I N$, *then* $\operatorname{depth}_I L = \operatorname{depth}_I N + 1$.

If in addition one assumes that R/I is $(\mathrm{FP})_\infty^R$, Hummel and Marley obtain the following homological characterization of depth for coherent rings.

Proposition 2.5 ([34]). *Let R be a ring, let M be an R-module, and let I be an ideal such that $IM \neq M$. If R/I is $(\mathrm{FP})_n^R$, then the following conditions are equivalent.*

(i) $\operatorname{depth}_I M \geq n$.

(ii) $\operatorname{Ext}_R^i(R/I, M) = 0$, *for* $0 \leq i < n$.

In particular, if $(R, \mathfrak{m})$ is a local coherent ring and $M \neq 0$ such that $\mathfrak{m}M \neq M$, then

$$\operatorname{depth} M = \sup\{n \geq 0 | \operatorname{Ext}_R^i(R/I, M) = 0 \text{ for all } i < n \text{ for some f.g. ideal } I \subset \mathfrak{m}\}.$$

This result connects the polynomial depth to the r-depth (as denoted by Barger [8]) based upon the vanishing of Ext$(R/I, -)$-modules.

There are other notions of grade in the non-Noetherian case, based on the vanishing of the homology of the Hom of a Koszul a complex, of the Čech cohomology of a module, of the local cohomology of a module, of the Ext$(R/I^n, -)$-modules, or of chain maps between complexes. Asgharzadeh and Tousi [3] explore the connections between these different grades to create additional characterizations of coherent Cohen–Macaulay modules (see Section 3).

3 Cohen–Macaulay Rings

Non-Noetherian Cohen–Macaulay rings were defined by Hamilton and Marley in [32]. Their homological approach to this question uses Schenzel's [45] notion of parameter sequences. Over non-Noetherian rings, parameter sequences play the role of systems of parameters over Noetherian rings. However parameter sequences are defined homologically rather than by height conditions.

Let $\mathbf{x} = x_1, \ldots, x_n$ be a finite sequence of elements in R. Given an R-module M, let $\check{H}^i_{\mathbf{x}}(M)$ be the ith Čech cohomology of M with respect to $\mathbf{x}$ and $H^i_{\mathbf{x}}(M)$ be the ith local cohomology of M. Schenzel gives the following definitions.

Definition 3.1 ([45]). Let R be a ring.

(i) The sequence $\mathbf{x}$ is weakly proregular if for all $i \geq 0$ and all R-modules M the natural map $H^i_{\mathbf{x}}(M) \longrightarrow \check{H}^i_{\mathbf{x}}(M)$ is an isomorphism.

(ii) The sequence $\mathbf{x} = x_1, \ldots, x_n$ is a parameter sequence if $\mathbf{x}$ is weakly proregular, $(\mathbf{x})R \neq R$, and $H^n_{\mathbf{x}}(R)_p \neq 0$ for all prime ideals p containing $\mathbf{x}$.

(iii) The sequence $\mathbf{x}$ is a strong parameter sequence if $x_1, \ldots, x_i$ is a parameter sequence for all $1 \leq i \leq n$.

Using these definitions, Hamilton and Marley define Cohen–Macaulay.

Definition 3.2 ([32]). A ring R is Cohen–Macaulay if every strong parameter sequence of R is a regular sequence.

Hamilton and Marley show that Cohen–Macaulay rings have the following properties.

Proposition 3.3 ([32]). *The following conditions are equivalent for a ring R.*

(i) *R is Cohen–Macaulay.*

(ii) Depth$(\mathbf{x})R = \ell(x)$ *for every strong parameter sequence* $\mathbf{x}$ *of* R.

(iii) depth$(\mathbf{x})R = \ell(x)$ *for every strong parameter sequence* $\mathbf{x}$ *of* R.

(iv) $H_i(\mathbf{x}; R) = 0$ *for all* $i \geq 1$ *and every strong parameter sequence* $\mathbf{x}$ *of* R.
(v) $H^i_{\mathbf{x}}(R) = 0$ *for all* $i < \ell(x)$ *and every strong parameter sequence* $\mathbf{x}$ *of* R.

The following results of Hamilton and Marley show the extent to which the Cohen–Macaulay property coincides with the Noetherian case.

Proposition 3.4 ([32]). *Let R be a ring.*
(i) *Let* $f : R \longrightarrow S$ *be a faithfully flat ring homomorphism, if S is Cohen–Macaulay then so is R.*
(ii) *If* $R[x]$ *is Cohen–Macaulay, then so is R.*
(iii) *If* $R_{\mathfrak{m}}$ *is Cohen–Macaulay for all maximal ideals* $\mathfrak{m}$ *of R, then so is R.*

It is unknown whether the converse of the statements above are true. In particular, since it is not known that the Cohen–Macaulay property localizes, Hamilton and Marley introduce the following modified definition of Cohen–Macaulay.

Definition 3.5 ([32]). A ring R is locally Cohen–Macaulay if R_p is Cohen–Macaulay for all $p \in \operatorname{Spec} R$.

Thus coherent regular and locally Cohen–Macaulay rings are related in the same way as their Noetherian counterparts.

Theorem 3.6 ([32]). *Coherent regular rings are locally Cohen–Macaulay.*

On the other hand, with the removal of the Noetherian assumption, the Cohen–Macaulay property is not retained after reduction by a non-zerodivisor, as seen in the following example.

Example 3.7 ([32, Example 4.9]). Let $S = \mathbb{C}[[x, y]]$ be the ring of formal power series in x and y over the field of complex numbers. Let $R = \mathbb{C} + x\mathbb{C}[[x, y]] \subseteq S$. R is a local Cohen–Macaulay domain, but R/xyR is not Cohen–Macaulay.

Towards the conclusion of their work, Hamilton and Marley [32] consider additional characterizations of Cohen–Macaulay rings. We begin with a few additional definitions. A prime ideal P is *weakly associated to* M if P is minimal over $(0 :_R x)$ for some $x \in R$. The set of weakly associated primes of M is denoted $\operatorname{wAss}_R M$. An ideal I of a ring is said to be *unmixed* if $\operatorname{wAss}_R R/I = \operatorname{Min}_R R/I$, the minimal primes of R/I over R. Using these definitions, additional properties of Cohen–Macaulay rings include the following.

Theorem 3.8 ([32]). *Let R be a ring.*
(i) *If every ideal of R generated by a strong parameter sequence is unmixed, then R is Cohen–Macaulay.*
(ii) *If R is an excellent Noetherian domain of characteristic* $p > 0$*, then* R^+ *is Cohen–Macaulay.*

(iii) *Let R be a coherent ring with* $\dim R \leq 2$*, and let G be a finite group of automorphisms of R with $|G|$ a unit in R. Let R^G be the subring of invariants of R under the action of G and assume R is a finite R^G-module. Then R^G is a coherent locally Cohen–Macaulay module.*

Asgharzadeh and Tousi [3] also look at other characterization of Cohen–Macaulay rings in this context. However they use an approach used by Hamilton [31] to explore the question of coherent Cohen–Macaulay rings. Their work [3] compares characterizations of Cohen–Macaulay based upon height conditions of prime ideals to the definition of Hamilton and Marley. These other characterizations of Cohen–Macaulay use the notion of Koszul grade introduced by Alfonsi.

Definition 3.9 ([1]). Let R be a ring, let $I = (\underline{x}) = (x_1, \ldots, x_r)$ be an ideal of R, and let M be an R-module. If $\mathbb{K}_\bullet(\underline{x})$ is the Koszul complex of $(\underline{x})$, the Koszul grade is given by $\text{K.grade}_R(I, M) = \inf\{i \in \mathbb{N} \cup \{0\} | H^i(\text{Hom}_R(\mathbb{K}_\bullet(\underline{x}), M)) \neq 0\}$.

In the following definition, let $\mu(I)$ denote the minimal number of elements of a ring R needed to generate an ideal I of R. Denote the support of an R-module M by $\text{Supp}_R(M)$ and let $\text{Max}(R)$ denote the set of maximal ideals of the ring R.

Definition 3.10 ([3]). Let R be a ring and let M be an R-module.

(i) Hamilton–Marley Cohen–Macaulay [32]: R is Hamilton–Marley Cohen–Macaulay if R satisfies Definition 3.2.

(ii) Glaz Cohen–Macaulay [28]: M is Glaz Cohen–Macaulay if

$$\text{height}_R(p) = \text{K.grade}_{R_p}(pR_p, M_p).$$

(iii) WB Cohen–Macaulay [31]: R is WB Cohen–Macaulay if for each ideal I with height $I \geq \mu(I)$, then I is unmixed (also known as weak Bourbaki unmixed).

(iv) Spec Cohen–Macaulay [3]: M is Spec Cohen–Macaulay if

$$\text{height}_M(I) = \text{K.grade}_R(I, M) \text{ for all ideals } I \in \text{Supp}_R(M).$$

(v) Max ideals Cohen–Macaulay [3]: M is Max ideals Cohen–Macaulay if

$$\text{height}_M(I) = \text{K.grade}_R(I, M) \text{ for all ideals } I \in \text{Supp}_R(M) \cap \text{Max}(R).$$

(vi) f.g ideals Cohen–Macaulay [3]: M is f.g. ideals Cohen–Macaulay if

$$\text{height}_M(I) = \text{K.grade}_R(I, M) \text{ for all finitely generated ideals } I \text{ of } R.$$

(vii) ideals Cohen–Macaulay [3]: M is ideals Cohen–Macaulay if

$$\text{height}_M(I) = \text{K.grade}_R(I, M) \text{ for all ideals } I \text{ of } R.$$

Asgharzadeh and Tousi [3] show the following relations between these definitions of Cohen–Macaulay

Max ideals $\Leftarrow$ Spec $\Leftrightarrow$ ideals $\Rightarrow$ Glaz $\Rightarrow$ f.g ideals $\Rightarrow$ Hamilton-Marley $\Leftarrow$ WB,

and provide examples of the non-existence of some of the missing implications above. See [3] for additional details.

4 Gorenstein Dimensions and the Auslander–Bridger Property

4.1 Gorenstein Dimensions

In [23], Gerko lists several properties that any generalized homological dimension should naturally fulfill. These are listed below for later reference.

Remark 4.1 ([23]). Let R be a ring, let $\mathcal{H}_R$ be a class of modules, and let $\mathcal{H}\text{-dim}_R$ be a homological dimension such that $\mathcal{H}\text{-dim}_R$ maps $\mathcal{H}_R$ into $\mathbb{Z}$. The following properties should hold for $\mathcal{H}$-dim.

(i) If $M \in \mathcal{H}_R$ then $\mathcal{H}\text{-dim}_R\, M + \operatorname{depth} M = \operatorname{depth} R$.

(ii) Let x be an R- and M-regular element. If $M \in \mathcal{H}_R$, then $M/xM \in \mathcal{H}_{R/xR}$ and $\mathcal{H} \dim_R M = \mathcal{H}\text{-dim}_{R/xR}\, M/xM$.

(iii) If $M \in \mathcal{H}_R$, then $M_\mathfrak{p} \in \mathcal{H}_{R_\mathfrak{p}}$ and $\mathcal{H}\text{-dim}_R\, M \geq \mathcal{H} - \dim_{R_\mathfrak{p}} M_\mathfrak{p}$.

(iv) Given an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ of R-modules, if any two of the modules belongs to $\mathcal{H}_R$ then the third does as well.

If R is Noetherian, the following condition also holds.

(v) $k = R/\mathfrak{m} \in \mathcal{H}_R$ if and only if $M \in \mathcal{H}_R$ for all R-modules.

In this section, as well as in Sections 5 and 6, we discuss homological dimensions that have been explored in the coherent context. We begin the discussion with Gorenstein dimensions.

Definition 4.2 ([4]). Let R be a ring and let M be a finitely generated R-module.

(i) M is in the class $G(R)$ if

  (a) $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for all $i \geq 0$.

  (b) $M \cong M^{**}$.

(ii) M has Gorenstein dimension n, denoted $\operatorname{Gdim} M = n$, if there exists a minimal length exact resolution $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ such that $G_i \in G(R)$ for each i. If no finite resolution exists, then $\operatorname{Gdim} M = \infty$.

Projective modules are in $G(R)$, and have Gorenstein dimension zero; modules with finite projective dimension thus have finite Gorenstein dimension. Gorenstein dimension satisfies the properties of Remark 4.1 as shown in [4] and is thoroughly summarized by Christensen in [14] and extended to the finitely presented modules over a coherent ring by [41], [19], and [34].

McDowell [41] extends most of the results of Gorenstein dimensions to finitely generated modules over local coherent rings. Taking a different approach, Hummel and Marley [34] simply modify the assumption on the module, providing the following definition of Gorenstein dimension for coherent-like modules.

Definition 4.3 ([34]). Let R be a ring and let M be an R-module.

(i) M is in the class $\tilde{\mathrm{G}}(R)$ if

  (a) M and M^* are $(\mathrm{FP})^R_\infty$.

  (b) $\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(M^*, R) = 0$ for all $i \geq 0$.

  (c) $M \cong M^{**}$.

(ii) M has $\tilde{\mathrm{G}}$-dimension n, denoted $\tilde{\mathrm{G}}\mathrm{dim}\, M = n$, if there exists an exact resolution $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ of minimal length such that $G_i \in \tilde{\mathrm{G}}(R)$ for each i. If no finite resolution exists, $\tilde{\mathrm{G}}\mathrm{dim}\, M = \infty$.

The characteristics of modules in coherent rings leads to the following result.

Proposition 4.4 ([34]). *If* $(R, \mathfrak{m})$ *is a local coherent ring, then* $\mathrm{Gdim}\, M = \tilde{\mathrm{G}}\mathrm{dim}\, M$ *for every finitely presented* R*-module* M.

The properties of $\tilde{\mathrm{G}}\mathrm{dim}$, explored in [34] and [42], satisfy the first four properties of Remark 4.1 and are analogous to Gorenstein dimension.

Taking a different approach to generalizing Gorenstein dimension over finitely presented modules, Enochs and Jenda developed Gorenstein projective dimension.

Definition 4.5 ([19]). A complex $\mathbf{E} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ of modules is called acyclic if $H_i(\mathbf{E}) = 0$, where $H_i(\mathbf{E})$ is the ith homology module of $\mathbf{E}$.

Definition 4.6 ([19]). Let R be a ring.

(i) An R-module M is Gorenstein projective if there is an acyclic complex $\mathbf{P}$ of projective R-modules with $\mathrm{Coker}\, P^0 \longrightarrow P^1 \cong M$ and $\mathrm{Hom}(\mathbf{P}, Q) = 0$ for every projective R-module Q.

(ii) The Gorenstein projective dimension of a module M, denoted $\mathrm{Gpd}\, M$, is n if

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

is an exact resolution of minimal length such that G_i is Gorenstein projective.

Again, projective modules are Gorenstein projective, and modules with finite projective dimension have finite Gorenstein projective dimension. Gorenstein projective dimensions also satisfy the properties of Remark 4.1. Over Noetherian rings $\operatorname{Gdim} M = \operatorname{Gpd} M$ [7]; Hummel and Marley [34] show the equality holds in the coherent case for finitely presented modules. Analogous definitions can be given for Gorenstein injective and Gorenstein flat modules.

Definition 4.7 ([19]). Let R be a ring.

(i) An R-module M is Gorenstein injective if there is an acyclic complex

$$\mathbf{E} : \cdots \longrightarrow E^1 \longrightarrow E^0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

of injective R-modules with

$$\operatorname{Coker}(E^0 \longrightarrow E^1) \cong M \quad \text{and} \quad \operatorname{Hom}(N, \mathbf{E}) = 0$$

for every injective R-module N.

(ii) An R-module M is Gorenstein flat if there is an acyclic complex

$$\mathbf{F} : \cdots \longrightarrow F^1 \longrightarrow F^0 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots$$

of flat R-modules with

$$\operatorname{Coker}(F^0 \longrightarrow F^1) \cong M \quad \text{and} \quad N \otimes \mathbf{F} = 0$$

for every injective R-module N.

Gorenstein flat and injective dimensions are defined analogously to Gorenstein projective dimension. All three Gorenstein dimensions are being actively studied by many authors including Bennis and Mahdou [9, 10], and Mao and Ding [37, 39], among others.

More recently Iyengar and Krause [35], Christensen and Veliche [16], Sather–-Wagstaff, Sharif and White [44], and others have studied Gorenstein projective, injective, and flat modules in the context of totally acyclic complexes. A *totally acyclic* complex $\mathbf{M}$ is an acyclic complex that satisfies the following equivalent conditions.

Proposition 4.8 ([14, 15]). *Let R be a ring, and let $\mathbf{M}$ be an acyclic complex of finitely generated projective R-modules. Then the following conditions are equivalent.*

(i) $\operatorname{Hom}_R(\mathbf{M}, R)$ *is acyclic.*

(ii) $\operatorname{Hom}_R(\mathbf{M}, F)$ *is acyclic for every flat R-module F.*

(iii) $E \otimes_R \mathbf{M}$ *is acyclic for every injective R-module E.*

Using the definition above, we see that Gorenstein projective, injective and flat modules are kernels of totally acyclic complexes.

In any ring R, these three homological dimensions are related in the following way for any R-module M

$$\operatorname{Gdim}_R M \leq \operatorname{Gpd} M \leq \operatorname{pd}_R M,$$

where equality holds in the first spot if R is coherent and M finitely presented; equality holds in the second spot if M has finite projective dimension.

4.2 The Auslander–Bridger Formula

This section explores the several iterations of Remark 4.1(i) for Gorenstein dimension. We start with the Auslander–Buchsbaum formula, which relates projective dimension to Depth.

Theorem 4.9 ([5]). *Let R be a local Noetherian ring and let M be an R-module with* $\operatorname{pd} M < \infty$. *Then*

$$\operatorname{pd} M + \operatorname{Depth} M = \operatorname{Depth} R.$$

The Auslander–Bridger formula provides a link between Depth and Gorenstein dimension.

Theorem 4.10 ([4]). *Let R be a local Noetherian ring and let M be an R-module with* $\operatorname{Gdim} M < \infty$. *Then*

$$\operatorname{Gdim} M + \operatorname{Depth} M = \operatorname{Depth} R.$$

The Auslander–Bridger formula was first extended to coherent rings by McDowell [41], who considered a subclass of coherent rings called pseudo-Noetherian rings.

Definition 4.11 ([41]). Let R be a ring and let M be a nonzero R-module. R is pseudo-Noetherian if

(i) R is coherent, and
(ii) if for any finitely generated ideal I contained in set of zero-divisors of M, there exists a nonzero m in M with $Im = 0$.

Any R-module M satisfying the second condition is called pseudo-Noetherian.

Note that the property held by pseudo-Noetherian modules is a characteristic of modules over Noetherian rings (see [36]). However, not all coherent rings are pseudo-Noetherian, as seen in the following example.

Example 4.12 ([41]). Let K be the quotient field of $\mathbb{Z}[x]_{(2,z)}$, and let R be the power series ring $K[[t]]$. R is a coherent domain, but R/tR is not a pseudo-Noetherian R-module.

On the other hand, a ring whose modules are all pseudo-Noetherian may not be coherent; this is the case for any non-coherent generalized valuation ring [41].

McDowell [41] used the characterization of depth over coherent local rings from Proposition 2.5 as the definition of depth over pseudo-Noetherian rings. Over local pseudo-Noetherian rings, if M is a finitely presented module, $\operatorname{depth} M$ is the length of a maximal M-regular sequence [41]. In conjunction with McDowell's extension of Gorenstein dimension to coherent rings, the Auslander–Bridger formula can be generalized to pseudo-Noetherian rings.

Theorem 4.13 ([41]). *Let R be a local pseudo-Noetherian ring, and let M be a non-zero finitely presented R-module with* $\operatorname{Gdim} M < \infty$. *Then,*

$$\operatorname{Gdim} M + \operatorname{depth} M = \operatorname{depth} R.$$

Generalizing the Auslander–Bridger formula to any coherent rings is problematic. A key step in the proof of the coherent result requires both Gorenstein dimension and coherence to pass through faithfully flat extensions. However it is well known that coherence is not maintained under faithfully flat extensions; for instance $R[x]$ is not necessarily coherent even if R is coherent. In fact, much attention has been concentrated on the question of what conditions guarantee that coherence is maintained under faithfully flat extensions, and requires the assumption that the ring has finite weak dimension. The *weak dimension* of a ring R is defined as $\sup\{\operatorname{fd} M \mid M \text{ an } R\text{-module}\}$ (see [25] and [1]). As shown in [34], $(\mathrm{FP})^R_\infty$ is preserved under faithfully flat extensions. Thus the $(\mathrm{FP})^R_\infty$ assumption in the next result serves as a stepping stone to the coherent result desired.

Theorem 4.14 (Generalized Auslander–Bridger Formula [34]). *Let R be a local ring and let M be an R-module with* $\widetilde{\operatorname{G}}\operatorname{dim} M < \infty$. *Then*

$$\widetilde{\operatorname{G}}\operatorname{dim} M + \operatorname{depth} M = \operatorname{depth} R.$$

As finitely presented modules over a coherent ring are $(\mathrm{FP})^R_\infty$, the coherent case follows easily.

Corollary 4.15 ([34]). *Let R be a local coherent ring and let M be an R-module with* $\operatorname{Gdim} M < \infty$. *Then*

$$\operatorname{Gdim} M + \operatorname{depth} M = \operatorname{depth} R.$$

5 Gorenstein Rings and Injective Dimensions

Recall that over Noetherian and coherent rings, Gorenstein dimension and Gorenstein projective dimension coincide. Over Noetherian rings, Gorenstein rings have been characterized via Gorenstein dimensions.

Proposition 5.1 ([4, 19]). *Let $(R, \mathfrak{m}, k)$ be a Noetherian ring. The following conditions are equivalent.*

(i) *R is Gorenstein.*

(ii) $\operatorname{Gdim} M < \infty$ *for all R-modules M.*

(iii) $\mathrm{Gdim}\, k < \infty$.

(iv) $\mathrm{Gpd}\, M < \infty$ *for all* R*-modules* M.

(v) $\mathrm{Gpd}\, k < \infty$.

In light of the above characterization, Hummel and Marley [34] define Gorenstein as follows.

Definition 5.2 ([34]). A local ring R is Gorenstein if $\mathrm{Gdim}\, R/I < \infty$ for every finitely generated ideal I. An arbitrary ring R is Gorenstein if $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m}$.

While Gorenstein projective dimension and G-dimension are equivalent in coherent rings, note that all the (coherent) Gorenstein results below were first proved in the context of $(\mathrm{FP})_\infty$-modules and $\tilde{\mathrm{G}}$-dimensions. Since these results make use of the defining characteristics of G-dimension, they will be stated in those terms instead of Gorenstein projective dimension.

By the inequality $\mathrm{Gdim}\, M \leq \mathrm{pd}\, M$, it follows immediately that

Theorem 5.3 ([34]). *Coherent regular rings are Gorenstein.*

Using Corollary 4.15, it follows that

Theorem 5.4 ([34]). *A coherent Gorenstein ring is locally Cohen–Macaulay.*

While most Gorenstein results focus on coherent Gorenstein rings, the following example from [34] constructs a non-coherent Gorenstein ring. For the following example we define an R-module M to be linearly compact if every collection $\{N_i | i \in \mathcal{I}\}$ of cosets of M having the finite intersection property satisfies $\bigcap_{i \in \mathcal{I}} N_i \neq \emptyset$ [22]. A valuation ring R is *almost maximal* if for every ideal $I \neq 0$, R/I is linearly compact in the discrete topology.

Example 5.5 ([34, Example 5.2]). Let V be an almost maximal valuation domain with value group $\mathbb{R}$ (see Section II.6 of [22] for details). Let $\mathfrak{m}$ be the maximal ideal of V and let $a \in \mathfrak{m}$ be a nonzero element. Then $R = V/a\mathfrak{m}$ is a non-coherent Gorenstein ring.

Additional characterizations of Gorenstein rings include the following.

Proposition 5.6 ([34]). *Let* R *be a ring.*

(i) *If* R *is Gorenstein, then* R_p *is Gorenstein for any prime ideal* p.

(ii) R *is a local coherent Gorenstein ring if and only if* $R/(x)$ *is Gorenstein for any* R*-regular element* x.

(iii) *R is Gorenstein if and only if $R[x]$ is coherent and Gorenstein.*

(iv) *If $\{R_i\}$ is a family of coherent Gorenstein rings, then $R = \varinjlim R_i$ is coherent Gorenstein.*

While the definition of Gorenstein rings rests upon a characterization using G-dimension, one may ask how well this definition behaves in light the following characterization of Gorenstein rings:

Proposition 5.7 ([40]). *Let $(R, \mathfrak{m}, k)$ be an n-dimensional Noetherian local ring. The following conditions are equivalent.*

(i) *R is Gorenstein.*

(ii) $\operatorname{id} R \leq n$.

(iii) $\operatorname{Ext}^i_R(k, R) = 0$ *for $i < n$ and* $\operatorname{Ext}^n_R(k, R) \cong k$.

(iv) *R is a Cohen–Macaulay ring and* $\operatorname{Ext}^n_R(k, R) \cong k$.

(v) *R is a Cohen–Macaulay ring and every parameter ideal is irreducible.*

(vi) *R is a Cohen–Macaulay ring and there exists an irreducible parameter ideal.*

Hummel and Marley [34] provide a one-directional analogy of Proposition 5.7(v) and (vi) over coherent rings.

Proposition 5.8 ([34]). *If R is a local coherent Gorenstein ring with* $\operatorname{depth} R = n < \infty$*, then every n-generated ideal generated by a regular sequence is irreducible.*

Notice that while regular sequences of length $\operatorname{depth} R$ may not exist, if coherence can be preserved one may pass to a faithfully flat ring to obtain the necessary regular sequence [34]. It is currently unknown whether the reverse of Proposition 5.8 is true.

Hummel and Marley [34] have also made connections with characterization (ii), replacing injective dimension with FP-injective dimension. FP-injective modules, introduced by Stenström arises from a modification of the definition of injective modules.

Definition 5.9 ([46]). Let R be a ring and let M be an R-module.

(i) M is called FP-injective if $\operatorname{Ext}^1_R(F, M) = 0$ for all finitely presented modules F.

(ii) The FP-injective dimension of M is defined as

$$\text{FP-id}_R\, M = \inf\{n \geq 0 \mid \operatorname{Ext}^{n+1}_R(F, M) = 0 \forall \text{ finitely presented } R\text{-module } F\}.$$

FP-injective modules have also appeared in the literature as absolutely pure modules (for instance see [25] and [22]). FP-injective modules were connected to other homological dimensions by Ding and Chen [18] who explored FP-injectivity in conjunction with coherent rings. Below, some of the more salient properties of FP-injective dimension from Lemma 3.1 of [46] are summarized in the context of coherent rings.

Proposition 5.10 ([46]). *Let R be a coherent ring, let M be an R-module, and let n be a non-negative integer. The following conditions are equivalent.*

(i) $\text{FP-id}_R\, M \leq n$.

(ii) $\operatorname{Ext}^i_R(F, M) = 0$ *for all $i > n$ and all finitely presented R-modules F.*

(iii) $\operatorname{Ext}^{n+1}_R(R/I, M) = 0$ *for all finitely generated ideals I of R.*

(iv) *Given an exact sequence* $0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow 0$ *with E^i an FP-injective module for* $0 \leq i \leq n-1$, *then E^n is FP-injective.*

Additional details on FP-injective modules and coherent rings can be found in [34], [46], and [18].

The first characterization of Gorenstein rings via FP-injective dimension was by Ding and Chen in [18], who characterized local coherent rings of finite self-FP-injective dimension. They called local coherent rings with FP-id $R \leq n$ n-FC rings, later denoted Ding–Chen rings by Gillespie in [24]. The following theorem of Ding and Chen makes the initial connection between Gorenstein and n-FC rings.

Theorem 5.11 ([18]). *Let R be a local coherent ring. The following conditions are equivalent for $n \geq 1$.*

(i) *R is n-FC.*

(ii) Gpd $M \leq n$ *for all finitely presented R-modules M, that is, R is Gorenstein.*

However, there is an example of a local coherent Gorenstein ring with infinite FP-injective dimension.

Example 5.12 ([41]). Let k be a field, and let $R = k[[(x_n)_{n\in\mathbb{N}}]]$ be the power series ring in a countable infinite number of indeterminates over k such that only a finite number of indeterminates occur in the expansion of any element of R. R is a local coherent regular ring (and hence Gorenstein), but FP-id $R = \infty$.

Thus additional assumptions are needed for a Gorenstein ring to have finite FP-injective dimension. In light of the bound

$$\begin{aligned} \operatorname{depth} R &\leq \sup\{\operatorname{Gdim}(R/I) \mid I \text{ finitely generated ideal}\} \\ &= \sup\{n \mid \operatorname{Ext}^i_R(R/I, R) = 0, i \geq n, I \text{ finitely generated ideal}\}, \end{aligned}$$

for any ring R, the work of Hummel and Marley and Theorem 5.11 leads to the following result.

Theorem 5.13 ([34]). *Let R be a local coherent ring. The following conditions are equivalent for $n \geq 0$.*

(i) *R is n-FC.*

(ii) *R is Gorenstein with* depth $R = n$.

Rings satisfying this theorem will be denoted n-FC Gorenstein rings. In the case of n-FC Gorenstein rings, a result of Ding and Chen also provides a coherent equivalent to the following Noetherian result.

Proposition 5.14 ([13, Exercise 3.1.25]). *A Noetherian ring R is Gorenstein if and only if the set of modules with finite projective dimension is equal to the set of modules with finite injective dimension.*

In the coherent case, flat modules play the role of projective modules.

Proposition 5.15 ([17]). *Let R be a coherent ring with* FP-id $R \leq n$. *The following conditions are equivalent.*

(i) $\operatorname{fd} M < \infty$.
(ii) $\operatorname{fd} M \leq n$.
(iii) FP-id $M < \infty$.
(iv) FP-id $M \leq n$.

Thus, the following unpublished generalization of Proposition 5.14 follows easily.

Proposition 5.16. *Let R be a local coherent ring; the following conditions are equivalent.*

(i) *R is Gorenstein with* depth $R < \infty$.
(ii) *For any module M,* $\operatorname{fd} M < \infty$ *if and only if* FP-id $M < \infty$.

Proof. The forward direction holds by Proposition 5.15. The reverse direction holds trivially, as R is flat, and hence FP-id $R < \infty$. □

Note that Foxby [21] extended Proposition 5.14 by showing that a Noetherian ring with a single module of both finite projective and injective dimension is Gorenstein. It is unknown whether this result carries over to coherent rings.

The connection between FP-injective and flat modules is natural in light of the following duality between FP-injective dimension and flat dimension. In the following, the character module of M is denoted $M^+ = \operatorname{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$.

Lemma 5.17 ([20]). *Let R be a ring and let M be an R-module.*

(i) $\operatorname{fd} M = \operatorname{id} M^+ =$ FP-id M^+.
(ii) *If R is right coherent and M is a right R-module then* $\operatorname{fd} M^+ =$ FP-id M.

This relation is analogous to the relation between injective and flat modules over Noetherian rings.

Recall that an R-module M has *weak dimension n*, denoted $\operatorname{w.dim}_R M = n$, if there is a minimal length exact resolution of M, $0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, consisting of flat modules. The *weak dimension* of a ring R is defined $\operatorname{w.dim} R = \sup\{\operatorname{w.dim} M \mid M \text{ an } R\text{-module}\}$. By [29], coherent rings of finite weak dimension are regular coherent rings. One may ask whether there are Gorenstein rings with infinite weak dimension. The following result of Gillespie [24] yields a positive answer by providing examples of n-FC Gorenstein rings with infinite weak dimension.

Proposition 5.18 ([24]). *If R is an n-FC Gorenstein ring, then the group ring $R[G]$ is an n-FC Gorenstein ring for any finite group G.*

More work has been done recently with FP-injective dimension that has lead to further characterizations of n-FC, and hence coherent Gorenstein, rings. Mao and Ding [37–39] explore the existence of FP-injective (pre-) covers and flat (pre-)envelopes. Yang and Liu [48] extend the notion of FP-injectivity to complexes.

Given the connection between Gorenstein dimension and FP-injective dimension, and the fact that Gorenstein projective, injective and flat modules are kernels of totally acyclic complexes, a natural question arises of whether FP-injective dimension can also be viewed in terms of totally acyclic complexes. Mao and Ding [39] do this through the following definition of Gorenstein FP-injective modules, which are FP-injective modules that approximate the properties of Gorenstein injective modules.

Definition 5.19 ([39]). Let R be a ring, and let M be a left R-module. M is Gorenstein FP-injective if there is an exact sequence

$$\mathbf{E}: \quad \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective left R-modules with $M = \ker(E^0 \longrightarrow E^1)$ and $\operatorname{Hom}(F, \mathbf{E})$ exact for every FP-injective module F.

Clearly any kernel or cokernel of the sequence $\mathbf{E}$ above is Gorenstein FP-injective. Gorenstein FP-injectives are also closed under direct products [39]. In particular, Gorenstein FP-injective modules satisfy the properties of homological dimensions in Remark 4.1; see [39] for details. In addition, Gorenstein FP-injective and Gorenstein flat modules are related in the same way as FP-injective and flat modules are in Lemma 5.17.

Proposition 5.20 ([39]). *Let R be a coherent ring and let M be a right R-module. Then M is Gorenstein flat if and only if M^+ is Gorenstein FP-injective.*

Note that [39] shows the forward direction of this result holds for any ring.

Another variation on FP-injective and flat modules are FI-injective and FI-flat modules introduced by Mao and Ding.

Definition 5.21 ([38]). Let R be a ring.

(i) A left R-module is FI-injective if $\operatorname{Ext}^1_R(F, M) = 0$ for any FP-injective left R-module F.

(ii) A right R-module is FI-flat if $\operatorname{Tor}^R_1(N, F) = 0$ for any FP-injective left R-module F.

Note that Gorenstein FP-injective modules are also FI-injective.

Gorenstein FP-injective modules fit between injective and Gorenstein injective modules. Over a Noetherian ring, the classes of Gorenstein FP-injective and Gorenstein injective modules are identical. The following result illuminates the link between Gorenstein FP-injective and FP-injective modules.

Proposition 5.22 ([38]). *Let R be a coherent ring.*

(i) *R is left Noetherian if and only if every FP-injective left R-module is Gorenstein FP-injective.*

(ii) *If the class of Gorenstein FP-injective left R-modules is closed under direct sums, then R is left Noetherian.*

(iii) *If* FP-id $R \leq n < \infty$ *the following conditions are equivalent.*

   (a) $\mathrm{w.dim}\, R \leq n$.

   (b) *Every Gorenstein flat right R-module is flat.*

   (c) *Every Gorenstein FP-injective left R-module is FP-injective.*

   (d) *Every Gorenstein FP-injective left F-module is injective.*

The relation between these modules can be summarized as follows with the arrows indicating containment under the indicated conditions:

$$\begin{tikzcd}
 & FI\text{-injective} & \\
\begin{array}{c}\text{Gorenstein}\\ \text{injective}\end{array} \arrow[r, shift left, "\begin{array}{c}\text{left}\\ \text{Noetherian}\end{array}"] & \text{Gorenstein } FP\text{-injective} \arrow[l, shift left] \arrow[u] \arrow[dr, "\text{FP-id } R\leq n"'] & \text{injective} \arrow[l] \arrow[d, shift left, "\text{FP-id } R\leq n"] \\
 & & FP\text{-injective} \arrow[u, shift left]
\end{tikzcd}$$

Using Gorenstein FP-injective dimension, Theorem 3.4 in [39] provides an additional characterization of Gorenstein n-FC rings over perfect rings.

Theorem 5.23 ([39]). *Let R be a coherent perfect ring. The following conditions are equivalent.*

(i) *R is an n-FC ring.*

(ii) *For every exact sequence $0 \longrightarrow M \longrightarrow F^0 \longrightarrow \cdots \longrightarrow F^{n-1} \longrightarrow F^n \longrightarrow 0$ with F^i Gorenstein FP-injective for $0 \leq i \leq n-1$, then F^n is Gorenstein FP-injective.*

(iii) *For every exact sequence $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ with each F_i Gorenstein flat for $0 \leq i \leq n-1$, then F_n is Gorenstein flat.*

With multiple characterizations of Gorenstein rings that are compatible with the Noetherian case, we move on to a discussion of potential candidates for a theory of coherent complete intersection rings.

6 Foundations for Coherent Complete Intersections

Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be any R-module. Define the ith Betti number of R to be $\beta_i(R) = \dim_R \operatorname{Tor}_i^R(k,k)$, and the ith Betti number of M to be $\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M,k)$. The first formal definition of non-Noetherian complete intersections found by this author is by André [2]. This definition uses André–Quillen homology theory to extend the following "well-known" characterization of Noetherian complete intersection rings given by André.

Proposition 6.1 ([2]). *A local Noetherian ring is a complete intersection if its Poincaré series has the following form*

$$\sum \beta_i x^i = \frac{(1+x)^r}{(1-x^2)^s}$$

with the integer $r - s = \dim R > 0$.

In [2], André characterizes the rings satisfying the Poincaré equality given above without the restrictions on r and s, and defines rings satisfying this equality to be complete intersections. Since then, several authors have worked to characterize complete intersections via homological dimensions. The work of Avramov, Vesselin, Gasharov, and Peeva introduced complete intersection dimension.

Definition 6.2 ([6]). Let R and R' be local rings, and let M be an R-module.

(i) The map $R \longrightarrow R'$ is a (codimension c) deformation if it is a surjective local homomorphism with kernel generated by a (length c) regular sequence.

(ii) A quasi-deformation of R is a diagram of local homomorphisms $R \longrightarrow R' \leftarrow Q$, with $R \longrightarrow R'$ a flat extension and $R' \leftarrow Q$ a (codimension c) deformation. Given a quasi-deformation $R \longrightarrow R' \leftarrow Q$ and an R-module M, set $M' = M \otimes_R R'$.

(iii) For a nonzero R-module M, denote the complete intersection dimension of M to be $\text{CI-dim}_R M = \inf\{\operatorname{pd}_Q M' - \operatorname{pd}_Q R' | R \longrightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}$. For a module M over a Noetherian ring R,

$$\text{CI-dim}_R M = \sup\{\text{CI-dim}_{R_\mathfrak{m}} M_\mathfrak{m} | \mathfrak{m} \text{ a maximal ideal of } R\}.$$

Complete intersection dimension characterizes Noetherian complete intersection rings.

Theorem 6.3 ([6]). *Let* $(R, \mathfrak{m}, k)$ *be a local Noetherian ring. If R is a complete intersection, then every R-module has finite CI-dimension. If* $\text{CI-dim}_R k < \infty$, *then R is a complete intersection.*

Gerko [23] uses Gorenstein dimension and complexity to characterize Noetherian complete intersection rings. The *complexity* of an R-module is defined as $\mathrm{cx}_R M = \sup\{n | \beta_i^R(M) \leq \alpha x^{n-1}\}$.

In the Noetherian case, Gulliksen provides the following connection between complexity and complete intersection rings.

Proposition 6.4 ([30]). *If R is a local Noetherian complete intersection ring, then* $\mathrm{cx}\ M < \infty$ *for every R-module M.*

Gerko's investigation in [23] with PCI-dimension, also denoted lower CI dimension (*CI_*-dimension*), yields an easy definition of coherent complete intersection rings.

Definition 6.5 ([23]). Let R be a ring and let M be an R-module.

(i) Define PCI-dim $M = 0$ if Gdim $M = 0$ and cx $M < \infty$.

(ii) Define

$$\text{PCI-}\dim M = \inf\{n | 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \text{PCI-}\dim P_i = 0\ \forall i\}.$$

With this definition Gerko makes the following connection, which mirrors Theorem 6.3.

Proposition 6.6 ([23]). *If R is a Noetherian complete intersection,* PCI-dim $M < \infty$ *for every R-module M. Conversely, if* PCI-dim $k < \infty$ *then R is a complete intersection.*

Gerko also shows that PCI-dimension is related to CI-dimension.

Proposition 6.7 ([23]). *Let R be a ring, and let M be an R-module. Then* PCI-dim $M \leq$ CI-dim M, *with equality if and only if* CI-dim $M < \infty$.

However, Veliche showed the classes of modules with finite PCI-dimension and finite CI-dimension are not the same.

Proposition 6.8 ([47]). *Let R be a local Noetherian ring containing a field, with* depth $R \geq 4$. *Then there exists a perfect ideal I in R with* grade $I = 4$, *and a module M over R/I such that* PCI-dim$_{R/I}$ $M = 0$, *but* CI-dim$_{R/I}$ $M = \infty$.

Hence, using the following definition of complete intersection would allow complete intersections to be immediately Gorenstein.

Definition 6.9. Let $(R, \mathfrak{m})$ be a local ring. Define R to be a complete intersection if PCI-dim $R/I < \infty$ for all finitely generated ideals I. If R is a local coherent ring, R is a complete intersection if PCI-dim $M < \infty$ for all finitely presented modules M.

However this definition is unsatisfying in that it a priori assumes finite Gorenstein dimension, in particular that the ring is Gorenstein. Instead, via a suggestion to the author by Avramov, a preferable definition for coherent complete intersections may be the following.

Definition 6.10. A local coherent ring R is a complete intersection if $\operatorname{cx} M < \infty$ for every finitely presented R-module.

While $(\mathrm{FP})_\infty$-modules certainly have finite complexity, it is unclear whether finite complexity is sufficient to imply finite Gorenstein dimension. More work needs to be done to discover if complexity is a sufficient condition for finite Gorenstein dimension.

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