FUNCTORIAL GNOMONICS: TEMPORAL PERCOLATION AND ALETHEIA

11.1 DISTINGUISHING BETWEEN SPATIAL AND TEMPORAL COVERING RELATIONS

The functioning of a *localization* schema in the physical continuum is based on the operational specification of an appropriate *covering* categorical environment consisting of varying reference loci for the determination of observables. As we have already made clear previously, the functional role of localization systems serves to guarantee an efficient *pasting code* of the observable information between different localizing domains, effectuating the compatible transition from the *local* to the *global* regime.

Until this stage, we have not established any particular interpretation of an abstract localization schema in a continuum of observable events in terms of *spatial* or *temporal* relations. From a physical viewpoint, since we have assumed that some localizing categorical environment admits an operational specification, we ought to expand on the functioning of a localization schema in spatiotemporal terms, so that, a reference to individuated observable events in these terms can be made possible. In this sense, it is necessary to struggle for a well defined notion of a category equipped with covering families admitting a viable interpretation in terms of spatial and temporal relations, which will in consequence be suitable to provide the necessary and sufficient means for the manifestation of some localization schema in the physical continuum in terms of observable events individuated from it in a spatial or temporal way.

Significantly, we do not assume any *spatialization* of *temporal* concepts, as is usually the case. Accordingly, it becomes unavoidable to disentangle the defining requirements characterizing *spatial covering* systems, namely families consisting of spatial reference domains, from those characterizing *temporal covering* systems, that is, families consisting of temporal reference domains. This strategy will prove fruitful if we manage to associate, at a later stage, a localization schema in the physical continuum with notions of spatially and temporally distinguished *events*. The difference between spatial and temporal covering systems will be based on the distinctive meaning that the notion of *extension* acquires, when referring to temporal loci, as compared to spatial ones.

If we consider a general categorical environment $\mathcal{B}$, and an object B in $\mathcal{B}$ to be interpreted as a *spatial* reference locus, then the extensional aspects of B are captured by the *contravariant* Hom-functor of generalized point-elements of B in $\mathcal{B}$, denoted by $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, which is a *representable presheaf* in $Sets^{\mathcal{B}^{op}}$. The functor $\mathbf{y}[B]$ gives a geometric form to the abstract extension of the spatial locus B in the environment of the category $\mathcal{B}$. In this sense, an arrow $C \to B$, such that C, B in $\mathcal{B}$, is interpreted as a *singular figure* of B. Stated differently, B is called a *spatial extension* of C in $\mathcal{B}$.

Note that, in this sense, B stands for the spatial extension of *many* different *spatial* loci, not necessarily constrained by relation to each other in any particular fashion, except that of the concrete conditions characterizing $\mathbf{y}[B]$ as a *presheaf functor*. Evidently, this is not the case if B is considered as a *temporal* reference domain. In this case, if B is considered as the *temporal extension* of C, then, any other locus D temporally extended by B, in the sense that an arrow $D \to B$ exists, is extended in this manner, by necessarily *factoring through* C. Equivalently stated, if a locus C is depicted in $\mathcal{B}$, and a locus B is considered to be a temporal extension of the specified locus C, then, in this temporal reference context, any other locus D for which an arrow exists with codomain B, must necessarily be a *proper part* of C, i.e. a *monic* arrow $D \infty C$, or a *singular part* of C.

Hence, in the case of temporal extension, if we specify the locus C, as temporally extended to B, then any other locus also temporally extended to B, is so extended by factoring through C. Of course, the definition of temporal extension of a locus C by some other locus B in the category *does not* depend on which locus C is specified as the one being extended by B, but once a particular C is depicted as a reference domain, then the *factorization* condition of any other D -extended also by B - through C, guarantees the satisfaction of the quality of temporal extension.

The definition of the notion of temporal extension as *distinguished* by that of spatial extension saves the underlying intuitions making up the idea of *generalized history* of a locus. From this perspective, if a locus B temporally extends a locus C, in the sense of being its generalized history, it can also serve as the temporal extension, i.e. the *history* of only what can be considered as being proper or singular parts of C, that is, generalized point-elements of C.

Note that it is the *distinct* quality of spatial or temporal extension of a locus, by virtue of its *relation* to other loci in a categorical environment that points to a corresponding interpretation of its *character* as such, meaning as being spatial or temporal, and *not* any *intrinsic ad hoc* postulated character. In this sense, a locus in a categorical environment can be the referent of *both* spatial and temporal connotations *depending on* the way that it is related with other loci. Consequently, the construction of some covering schema in the depicted categorical environment, utilized for the substantiation of some corresponding localization schema in the physical continuum, manifested in terms of spatial or temporal covering relations, will depend *only* on the *relational* characteristics of the loci of each covering family, in turn making requisite, a corresponding non-exclusive interpretation in spatial or temporal terms.

11.2 FUNCTORIAL SPATIAL LOCALIZATION SCHEMATA

We begin our exposition of the notions referring to functorial spatiotemporal localization by introducing, first of all, the conception of a spatial covering system. The general notion of a category $\mathcal{B}$, equipped with a *spatial covering system*, interpreted as a structured family of reference domains used for the spatial localization of physical continuum events in the environment of $\mathcal{B}$, is based on the definition of appropriate covering devices of a spatial character, called *spatial covering sieves*. Let us recapitulate the notion of a sieve, then examine the requirements for an intended interpretation of a family of sieves as a *spatial* covering schema.

For a locus B in $\mathcal{B}$, a B-*sieve* is a family S of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to S and $D \to C$ is any $\mathcal{B}$-morphism, then the composite $D \to C \to B$ belongs to S. We may think of a B-sieve as a right B-ideal. With reference to the functor of generalized points of B in $\mathcal{B}$, denoted by $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, we have already proved previously that a B-sieve is *equivalent* to a subfunctor $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$. Thus, epigrammatically, we state:

$$\langle \ B\text{-sieve: } S \ \rangle = \langle \ \text{Subfunctor of } \mathbf{y}[B]\text{: } \mathbf{S} \infty \mathbf{y}[B] \ \rangle$$

We recall that if S is a B-sieve and $h : C \to B$ is any arrow to the locus B, then:

$$h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\}$$

is a C-sieve, called the *pullback* of S *along* f. Consequently, we may define a presheaf functor $\mathbf{\Omega}$ in $Sets^{\mathcal{B}^{op}}$, such that its action on loci B in $\mathcal{B}$, is given by:

$$\mathbf{\Omega}(B) = \{S / S : B - sieve\}$$

and on arrows $h : C \to B$, by $h^*(-) : \mathbf{\Omega}(B) \to \mathbf{\Omega}(C)$, given by:

$$h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\}$$

We stress again that for a locus B in $\mathcal{B}$, the set of all arrows into B is a B-sieve, called the *maximal sieve* on B, and denoted by $t(B) := t_B$.

The natural question that arises in our context of enquiry is the following: How is it possible to restrict $\mathbf{\Omega}(B)$, that is the set of B-sieves for each locus B in $\mathcal{B}$, such that each B-sieve of the restricted set can assume the interpretation of a *spatial covering system* of B. In other words, we look for those appropriate conditions on the set of B-sieves, for each locus B in $\mathcal{B}$, so that the subset of B-sieves obtained, denoted by $\mathbf{\Omega}_\chi(B)$, respect the quality of spatial extension. In this way, the B-sieves of $\mathbf{\Omega}_\chi(B)$, for each locus B in $\mathcal{B}$, to be thought as spatial covering B-sieves, can legitimately be used for the definition of a spatial localization scheme in the physical continuum. The clue for an answer comes from the following observations:

1 We have seen in the discussion of the quality of spatial extension that it constitutes a relational property between reference loci B in $\mathcal{B}$. In this sense, an arrow $C \to B$, such that C, B in $\mathcal{B}$, is interpreted as a singular figure of B, and thus B, is interpreted as a spatial extension of C in $\mathcal{B}$. It is a natural requirement that the set of all figures of B should belong in $\mathbf{\Omega}_\chi(B)$ for each locus B in $\mathcal{B}$;

2 It is important to keep in mind that each spatial covering sieve on a locus B in $\mathcal{B}$, is going to serve as a model of a spatial localization system in the physical continuum, such that localized events are endowed with an interpretation in terms of spatial relations in the environment of $\mathcal{B}$. If we recall the relevant discussion about localization systems and their compatibility requirements, we realize that spatial covering sieves should be stable under pullback operations, and most importantly, the stability conditions should be expressed functorially;

3 Finally, it would be desirable to impose:

i a *transitivity* requirement on the specification of the spatial covering sieves, such that, spatial covering sieves of figures of a locus in spatial covering sieves of this locus should be spatial covering sieves of the locus themselves, and

ii a requirement of *common refinement* of spatial covering sieves.

If we take into account the above requirements we define a spatial covering scheme in the environment of $\mathcal{B}$ as follows:

A *spatial covering schema* on $\mathcal{B}$ is an operation J, which assigns to each locus B in $\mathcal{B}$, a collection $J(B)$ of B-sieves, called *spatial covering* B-sieves, such that, the following three conditions are satisfied:

1 For every locus B in $\mathcal{B}$ the maximal sieve $\{g : cod(g) = B\}$ belongs to $J(B)$ (maximality condition);

2 If S belongs to $J(B)$ and $h: C \to B$ is a figure of B, then $h^*(S) = \{f : C \to B, (h \cdot f) \in S\}$ belongs to $J(C)$ (stability condition);

3 If S belongs to $J(B)$, and if for each figure $h: C_h \to B$ in S there is a sieve R_h belonging to $J(C_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $J(B)$ (transitivity condition).

As a consequence of the conditions above, we verify that any two spatial covering sieves have a common refinement: if S, R belong to $J(B)$, then $S \cap R$ belongs to $J(B)$.

The operation J satisfying the aforementioned conditions, can be equivalently characterized in terms of a *Grothendieck topology* on the category $\mathcal{B}$, where the covering sieves implicate the requirements of spatial extension. A Grothendieck topology J may be thought of in the *shape* of a *presheaf* functor Ω_χ in $Sets^{\mathcal{B}^{op}}$, such that, by acting on loci B in $\mathcal{B}$, J gives the set of all spatial covering B-sieves, denoted by $\Omega_\chi(B)$, whereas by acting on figures $h : C \to B$, it gives a morphism of sets

$$h^*(-) : \Omega_\chi(B) \to \Omega_\chi(C),$$

expressed as:

$$h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\},$$

for $S \in \Omega_\chi(B)$. Clearly, Ω_χ is a *subobject* of Ω, that is, $\Omega_\chi \propto \Omega$.

11.3 FUNCTORIAL TEMPORAL LOCALIZATION SCHEMATA

Analogously with the conception of spatial covering schemata on the category $\mathcal{B}$, we can introduce the notion of temporal covering schemata on $\mathcal{B}$ consisting of temporal covering sieves. These covering systems should be construed in such a way that the relational quality of *temporal extension* between their reference domains is explicated properly. Accordingly, temporal covering sieves can be used for the *temporal localization* of physical continuum events in the environment of $\mathcal{B}$.

We have seen previously that the effectuation of spatial covering sieves required the imposition of certain restrictive conditions on the set of all B-sieves, for each locus B in $\mathcal{B}$, in order to qualify as denotations of spatial extension. In this sense, it is clear that an *analogous* interpretation of covering relations in terms of temporal extension between reference loci would require the satisfaction of all the relevant conditions. At a first stage, we may notice that the quality of temporal extension (in the sense of a generalized history of a locus) constitutes a *constrained* form of spatial extension, as discussed in detail previously. Thus, temporal covering B-sieves for each locus B in $\mathcal{B}$, should satisfy the conditions obeyed by spatial covering B-sieves, and additionally, a constraint signifying the *temporal* character of the relevant included extensive relations. Developing this line of reasoning, it is necessary to express the quality of temporal extension between reference domains in terms of sieves.

We consider a *Grothendieck topology* J on $\mathcal{B}$, such that, the maximality, stability and transitivity conditions are satisfied among covering sieves of reference loci B in $\mathcal{B}$. Hence, if S is a B-sieve that belongs to $J(B)$, we say that S is a *covering* B-sieve. It is also convenient to provide the following definition:

A B-sieve S *covers an arrow* $h : C \to B$ in $\mathcal{B}$, if and only if $h^*(S)$ is a *covering* C-sieve:

$$\langle S \triangleright [h : C \to B] \rangle \Leftrightarrow \langle h^*(S) \triangleright C \rangle$$

We notice that, as a consequence of the stability condition, if the arrow $h:C \to B$ belongs to S itself, then $id_C : C \to C$ belongs to $h^*(S)$, and thus, $h^*(S) = t_C$, i.e. $h^*(S)$ is the maximal covering C-sieve.

We may formulate this observation, given a covering B-sieve S and any arrow $h:C \to B$, as follows:

$$\langle h \in S \rangle \Leftrightarrow h^*(S) = t_C$$

This is a very convenient setting to explicate the notion of temporal extension among reference loci B in $\mathcal{B}$ in terms of covering sieves, if we further define;

A B-sieve is J*-closed*, if and only if, for all $\mathcal{B}$-arrows $h:C \to B$:

$$\langle\ S \text{ covers the } \mathcal{B}\text{-arrow } h:C \to B\ \rangle \ \Rightarrow\ \langle h \in S \rangle$$

In a suggestive notation, given J we say:

$$\langle S = [S] \rangle_J \Leftrightarrow \langle \langle S \rhd h \rangle \Rightarrow \langle h \in S \rangle \rangle_J$$

Given a Grothendieck topology J on $\mathcal{B}$, J*-closed sieves* constitute a *presheaf* functor Ω_κ in $Sets^{\mathcal{B}^{op}}$, such that, by acting on loci B in $\mathcal{B}$, Ω_κ gives the set of all J-closed B-sieves, denoted by $\Omega_\kappa(B)$, whereas by acting on arrows $h:C \to B$, it gives a morphism of sets $h^*(-):\Omega_\kappa(B) \to \Omega_\kappa(C)$, expressed as: $h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\}$, for $S \in \Omega_\kappa(B)$.

Indeed, we can immediately verify the following: For any B-sieve R and any $\mathcal{B}$-arrow $h:C \to B$;

$$\langle\ R:\ J\text{-closed } B\text{-sieve}\ \rangle \ \Rightarrow\ \langle\ h^*(R):\ J\text{-closed } C\text{-sieve}\ \rangle$$

In order to see that this is actually the case, we assume that $h^*(R)$ covers a $\mathcal{B}$-arrow $g:C \to B$. This means, by definition, that R covers the composition hg; taking into account that R is J-closed B-sieve, $hg \in R$, or equivalently, $g \in h^*(R)$. Hence, we obtain that $h^*(R)$ is J-closed C-sieve.

Clearly, Ω_k is a *subobject* of Ω, that is $\Omega_k \propto \Omega$.

The quality of *temporal extension* in terms of covering sieves is captured precisely by the defining requirement of J-*closed sieves*, if we re-express it as follows:

The locus B is a *temporal extension* of a locus C in $\mathcal{B}$, if a $\mathcal{B}$-arrow $h: C \to B$ exists, which is covered by a J-closed B-sieve.

Thus, *all* $\mathcal{B}$-arrows denoting the quality of temporal extension must necessarily be members of corresponding J-closed covering sieves. Consequently, a temporal covering scheme on $\mathcal{B}$, has to be properly expressed in the *restrictive* terms of J-closed covering sieves. These sieves obviously satisfy the maximality and transitivity conditions required. Moreover, since the property of being closed respects the stability condition under pullback operations, J-closed covering sieves also remain stable under pullback.

Now, by considering all the relevant requirements we can define a temporal covering schema in the environment of $\mathcal{B}$, as follows:

A *temporal covering schema* on $\mathcal{B}$ is an operation T, which assigns to each arrow $g: C \to B$ in $\mathcal{B}$ (interpreted as a *temporal extent* C - *irreducible duration* of B), a collection $T(g)$ of B-sieves, to be called *local time-forcing* B-sieves of temporal *resolution* unit $dom(g)$, such that, the following four conditions are satisfied:

i If S is B-sieve and $g \in S$, then S is a covering B-sieve (*maximality condition*);

ii If S covers an arrow $g: C \to B$, it also covers the composition $g \circ f$, for any arrow $D \to C$ (*stability condition*);

iii If S covers an arrow $g: C \to B$, and R is a B-sieve which covers all arrows of S, then R covers g (*transitivity condition*);

iv If S covers an arrow $g: C \to B$, then g belongs to S (*closure condition*).

We may again easily check that any two temporal covering sieves have a common refinement, that is: if R and S both cover $h: C \to B$, then $R \bigcap S$ covers h.

A temporal covering schema on $\mathcal{B}$, formulated in the above arrow-form, obviously satisfies the equivalent conditions [1]-[3] specifying in this form a spatial covering schema, and also additionally the closure constraint [4]. Due to this constraint, characteristic of temporal extension, we conclude that a J-closed B-sieve of temporal *resolution* unit B is necessarily the *maximal* B-sieve. This is clear, if we consider the identity arrow $id_B : B \rightarrow B$ and apply the closure constraint. The concept of temporal resolution unit will be analyzed in detail further on.

A *maximal* temporal covering scheme T on $\mathcal{B}$, exists as a presheaf functor Ω_τ in $Sets^{\mathcal{B}^{op}}$, such that: By acting on loci B in $\mathcal{B}$, T gives $\Omega_\tau(B)$, constituted only from the maximal B-sieve for each B, whereas by acting on arrows $h : C \rightarrow B$, it gives a morphism of sets $h^*(-) : \Omega_\tau(B) \rightarrow \Omega_\tau(C)$, expressed as:

$$h^*(t_B) = \{f / cod(f) = C, (h \cdot f) \in t_B\} = t_C$$

where $t_B \in \Omega_\tau(B)$. Clearly, Ω_τ is a subobject of Ω, that is $\Omega_\tau \propto \Omega$.

It is useful to notice that from *any* given B-sieve S we can construct a corresponding J*-closed* B-sieve, denoted by $[S]$, simply as follows:

$$[S] = \{f / cod(f) = B, S \triangleright f\}$$

where f is any $\mathcal{B}$-arrow with codomain B, and the notation $S \triangleright f$ denotes that S covers f. The above prescription means that the J-closed sieve $[S]$, corresponding to a given sieve S, is construed by *adding* in S *all* arrows that it *covers*. Furthermore, $[S]$ is the *smallest* closed sieve that contains S, named accordingly the *closure* of S.

11.4 PARADIGMATIC CATEGORICAL SPATIOTEMPORAL RELATIONS

After having explicated the defining requirements of spatial and temporal covering schemata in the environment of $\mathcal{B}$, bearing in mind that they are going to provide precise functorial concepts of spatial and temporal localization in a continuum of observable events, it is appropriate to concentrate our attention to some consequences of their functioning.

We suppose that $\mathcal{B}$ has a terminal object denoted by 1, that is for any locus B in $\mathcal{B}$ there exists a unique arrow $!_B : B \rightarrow 1$. Then we can view the locus B as a domain of irreducible durations (of temporal extent 1), if we define B as the temporal extension of 1, as follows:

The locus B is a domain of 1-irreducible durations in $\mathcal{B}$, if there exist $\mathcal{B}$-arrows $m:1 \to B$, such that, each one of them is covered by a J-closed B-sieve.

We notice that, according to the above definition, a locus B signifying a domain of *durations* by means of $\mathcal{B}$-arrows $1 \to B$ *cannot* be conceived *separately* from all J-closed B-sieves covering its point-durations. Put differently, an arrow $1 \to B$ obtains the semantics of an *irreducible duration* by being a *member* of a J-*closed* B-sieve that *covers* it, and subsequently, the locus B is interpreted as a domain of point-durations (*history of point-durations*). From a converse perspective, the definition of a point-duration of the reference domain B, forces an interpretation of the terminal object 1 as an *instantaneous* locus at point-duration m, denoted as 1_m, by means of the unique arrow $!_B : B \to 1$. Now, by definition, if we consider a J-closed B-sieve, covering m, its pullback along m is the maximal sieve on the instantaneous locus at m, denoted by t_{1_m}. This fact has the following consequences:

For every J-closed B-sieve covering m, where B stands for a domain of point-durations, 1_m consists of a *snapshot* of each and every locus in $\mathcal{B}$, a role which consolidates its interpretation as an instantaneous locus at point-duration m pretty clearly. Subsequently, the domain of point-durations B stands for the *temporal* extension of the instantaneous space at anyone of its specified durations $m:1 \to B$. It is important to notice that, all the above arguments are independent of any specific locus B used to illustrate them, since only *relational* properties, expressed in terms J-*closed* sieves, actually matter.

Thus, any reference locus in $\mathcal{B}$, equipped with arrows from the terminal object 1 of $\mathcal{B}$, being covered by J-closed sieves, acquires the status of a *temporal* domain of *point-durations*, and in each case, 1 becomes the *instantaneous* locus for some depicted duration. Subsequently, the instantaneous locus at any point-duration *cannot* signify the temporal extension of any other locus except of its own identity, identified with that point-duration, hence, it signifies only the pure quality of *spatial* extension.

This observation permits the characterization of 1_m as an *instantaneous space* at m. Concretely, the instantaneous space 1_m constitutes the *spatial* extension of all other loci C in $\mathcal{B}$, by means of the unique arrows $!_C : C \to 1_m$, contained in a J-closed sieve covering m. Precisely speaking, these unique arrows have been interpreted above as figurative snapshots at m corresponding to each and every locus in $\mathcal{B}$, displayed at 1_m.

11.5 SIMULTANEITY OF FIGURES AND LOCAL TIME DOMAINS OF DURATIONS

We have seen above that a domain of point-durations is interpreted as the temporal extension of the terminal object in $\mathcal{B}$, characterized as the instantaneous space at any depicted duration. The instantaneous space displays the quality of spatial extension in the purest sense, since it is the referent of diminished temporal extension, substantiated in the form of an irreducible point-duration.

In the intuitive sense, a domain of point-durations is the *generalized history* of the terminal locus, as instantiated at each duration in the form of an instantaneous space for that duration. Notice again that, by definition, an irreducible point-duration $m:1 \to B$ of a temporal domain B is covered by a *time-forcing* B-sieve of temporal *resolution* unit 1, such that, its pullback along m is the *maximal* sieve on the instantaneous space at m, denoted by t_{1_m}. Hence, in the perspective of a J-closed sieve covering $m:1 \to B$, the terminal 1_m is conceived as a *hole* of the sieve, such that every locus C is extended to B by *factoring* through the hole 1_m. More precisely, 1_m is the *maximal* hole of the J-*closed* B-sieve covering m, and since there exist unique arrows $!_C : C \to 1_m$, for every C in $\mathcal{B}$, all these C being extended to the temporal domain B, achieve this translation by passing through the maximal hole 1_m of this J-closed B-sieve. This observation permits the definition of the concept of *simultaneity* of figures with respect to a domain of irreducible point-durations B as follows:

Two figures $C \to B$ and $D \to B$ of a domain of point-durations B are *simultaneous* at a moment m, if and only if, they both *factor* through the *maximal* hole 1_m (temporal resolution unit) of *any* J-closed B-sieve covering m.

It is instructive again to clarify the truly *relational* spatiotemporal sense that a locus B acquires by means of the intended interpretation. In the case discussed above, the locus B is both; the denotator of temporal extension of the terminal 1 by its specification as a domain of irreducible point-durations, and also, the denotator of spatial extension of the figures $C \to B$ and $D \to B$ at one of their respective moments.

Furthermore, the same case concerning the concept of simultaneity, points to the conclusion that the decisive factor that determines *simultaneity*, with respect to a temporal domain of durations, is *factorization* through the *maximal hole* at a depicted duration of any J-closed sieve covering that duration. Nevertheless, it is important to clarify that temporal extension is *not* restricted exclusively to domains of point-durations; the latter should be considered only as *paradigmatic* cases. In this sense, it is possible to expand our argumentation and talk about generalized durations.

The locus B is a domain of *generalized* L*-durations* (temporal extent L-irreducible durations) in $\mathcal{B}$, if $\mathcal{B}$-arrows $m_L : L \to B$ exist, such that, each one of them is covered by a J-closed B-sieve. In that case, some $\mathcal{B}$-arrow $m_L : L \to B$ is interpreted as a temporal extent L-irreducible duration of B by means of being a member of a J-closed B-sieve covering it. Subsequently, the locus B is interpreted as a domain (history) of durations of temporal extent L.

Again, in the perspective of a J-closed B-sieve covering m_L, the locus L is considered a *hole*, such that every locus K that can be temporally extended to B, extends via factorization through the hole L. In this generalized sense, the hole L specifies the temporal *resolution* unit of a covering J-closed B-sieve, represented by the corresponding L-durations, that in turn, can be considered as denotations of simultaneity relations with respect to the applied temporal resolution unit of the domain B.

Note that *all* the J-closed B-sieves covering *all* the generalized durations of B contain *complete* information about all questions concerning *temporal* extension with respect to B, under *varying* temporal resolution units. In conclusion, the operational role of time in the present framework, is completely incorporated in the *functioning* of J-closed sieves. Thus, given a covering scheme J, for any J-closed B-sieve we define:

$$T_B(S) = \{h / cod(h) = B, S \triangleright h\}$$

or equivalently, since S stands for a J-closed B-sieve;

$$T_B(S) = \{h / cod(h) = B, h^*(S) = t_{dom(h)}\}$$

The set $T_B(S)$ is interpreted as the set of *generalized durations* of B covered by the J-closed sieve S, under varying temporal resolution units specified by the *holes* $dom(h)$. We can immediately verify that actually $T_B(S) = S$ for every J-closed B-sieve. Thus the sets S and $T_B(S)$ respectively, stand for the *active* and *passive* interpretation of the same entity, being the operation of local time on loci B in $\mathcal{B}$, transforming them into *local time domains* of generalized irreducible durations in the environment of $\mathcal{B}$. Due to this identification, J*-closed* sieves are interpreted as *local time-forcing* sieves for each locus B in $\mathcal{B}$.

Moreover, the temporal *resolution* unit of a J-closed sieve S is determined by the locus dom(h), specified as temporally extended to B by means of h, if the latter is covered by S. The locus $dom(h)$ can be thought of as a *hole* in the B-sieve S, that specifies the kind of generalized durations of a local time domain being covered by S, with respect to the relation of temporal extension between loci. In this sense, we may define a *local time operator* $\hat{T}_S$, associated with a J*-closed* B-sieve, for each locus B in $\mathcal{B}$ as follows:

$$\hat{T}_S \,|\, B\rangle = h \,|\, B\rangle$$

where $h \in T_B(S)$. The local time operator $\hat{T}_S$ acting on a locus B, denoted in the -so called- Dirac notation as the eigenstate $|\, B\rangle$, takes for eigenvalues the generalized durations being covered by S. Consequently, the locus B is interpreted as a local time domain endowed with generalized durations under *varying* temporal resolution units.

11.6 SIEVING SPATIAL FIGURES AT DURATIONS OF LOCAL TIME DOMAINS

A natural question that arises in this context of enquiry is the following: Given the means of functorial spatiotemporal localization, formulated in terms of *spatial* and *temporal* covering schemata, how is it possible to *spectrally classify spatial figures* of a reference locus B in $\mathcal{B}$ at *generalized durations* of that locus, considered as a *local time* domain?

In order to tackle this fundamental problem, we are going to use the notion of *subobject classifier* in a topos. First of all, it is useful to clarify the notion of *subobjects* in *any* categorical environment, since it is going to be the main conceptual tool in our argumentation.

A *subobject* of an object X in any category $\mathcal{X}$, is an *equivalence class* of *monic* arrows targeting X, denoted by $\mu : M \rightarrowtail X$. The set of all subobjects of X in the category $\mathcal{X}$, denoted by $\Theta_{\mathcal{X}}(X)$, is a *partially ordered* set under inclusion of subobjects.

The functor $\Theta_{\mathcal{X}}$ can be construed as a *presheaf* functor in the *topos* $Sets^{\mathcal{X}^{op}}$ by the operation of *pulling back* as follows; Given an arrow $g : Y \to X$ in $\mathcal{X}$, the pullback of any monic arrow $\mu : M \rightarrowtail X$ along the arrow g is a *new* monic arrow $\mu' : M' \rightarrowtail Y$, that is a subobject of Y, and obviously the assignment $\mu \mapsto \mu'$, defines a function $\Theta_{\mathcal{X}}(g) : \Theta_{\mathcal{X}}(X) \to \Theta_{\mathcal{X}}(Y)$.

An immediate question that arises here is related to the possibility of representing the subobject functor $\Theta_{\mathcal{X}}$ in the topos $Sets^{\mathcal{X}^{op}}$ by an *object* Ω in $\mathcal{X}$, considered as a category with pullbacks, such that for each X in $\mathcal{X}$, there exists a *natural isomorphism*:

$$\iota_X : \Theta_{\mathcal{X}}(X) \cong Hom_{\mathcal{X}}(X, \Omega)$$

If the subobject functor becomes *representable* with *representing* object Ω in $\mathcal{X}$, then we say that, the category $\mathcal{X}$ is equipped with a *subobject classifier*. By this term we mean a *universal* monic arrow:

$$\mathrm{T} := True : 1 \rightarrowtail \Omega$$

such that, to every monic arrow, $\mu : M \rightarrowtail X$ in $\mathcal{X}$, there is a *unique characteristic* arrow ϕ_μ, which, with the given monic arrow μ, forms a *pullback* diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ !\ } & 1 \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle T} \\ X & \xrightarrow{\ \phi_\mu\ } & \Omega \end{array}$$

This is equivalent to saying that every subobject of X in $\mathcal{X}$, is *uniquely* a *pullback* of the *universal* monic $\top$. Conversely, satisfaction of this property amounts to saying that the *subobject* functor $\mathbf{\Theta}_{\mathcal{X}}$ is *representable* by the object Ω, that is, isomorphic to $Hom_{\mathcal{X}}(-,\Omega)$. Note that the bijection ι_X sends each subobject $\mu : M \infty X$ of X to its unique characteristic arrow $\phi_\mu : X \to \Omega$ and conversely.

After these necessary introductory remarks, we turn to our main objective concerning the problem of *classification* of *figures* of reference loci B in $\mathcal{B}$, referring to B-sieves of corresponding spatial schemata, which provide the functorial means for spatial localization in a continuum of events. The starting point of our enquiry is determined by the realization that a spatial covering schema on $\mathcal{B}$ exists as a presheaf $\mathbf{\Omega}_\chi$ in $Sets^{\mathcal{B}^{op}}$.

Clearly, by the defining requirement of spatial covering sieves the following subobject relation holds: $\mathbf{\Omega}_\chi \infty \mathbf{\Omega}$, where, $\mathbf{\Omega}(B)$, denotes the set of all B-sieves for each locus B in $\mathcal{B}$. The connective link with our initial remarks appears if we recall that:

$$\langle \; B\text{-sieve: } S \; \rangle = \langle \; \text{Subfunctor of } \mathbf{y}[B]\text{: } \mathbf{S} \infty \mathbf{y}[B] \; \rangle$$

It is immediately evident that, because of the above correspondence, the presheaf functor $\mathbf{\Omega}$ may be used, at a first stage, for *classification* purposes in $Sets^{\mathcal{B}^{op}}$, since in particular, it could classify subobjects, that is, subfunctors of $\mathbf{y}[B]$ for each locus B in $\mathcal{B}$, according to the pullback square in $Sets^{\mathcal{B}^{op}}$:

$$\begin{array}{ccc}
\mathbf{S} & \xrightarrow{\;!\;} & \mathbf{1} \\
{\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\
\mathbf{y}[B] & \xrightarrow{\;\phi_S\;} & \mathbf{\Omega}
\end{array}$$

where:

$$\Theta_{Sets^{\mathcal{B}^{op}}}(\mathbf{y}[B]) := \Theta_{\hat{\mathcal{B}}}(\mathbf{y}[B]) = Hom_{\hat{\mathcal{B}}}(\mathbf{y}[B], \mathbf{\Omega}) = \mathbf{\Omega}(B)$$

by using the *Yoneda lemma*. The morphism $True : \mathbf{1} \to \mathbf{\Omega}$ is a natural transformation in $Sets^{\mathcal{B}^{op}}$, given by components $True_B$ for all reference loci B in $\mathcal{B}$. The functor $\mathbf{1}$ is given by the assignment $B \mapsto \{\varnothing\}$ with the obvious restriction morphisms. It is clear that $\mathbf{1}$ is the *terminal* object in $Sets^{\mathcal{B}^{op}}$, and the components of the morphism $True$ are defined by $True_B : \{\varnothing\} \to \mathbf{\Omega}(B)$; where $True_B(\{\varnothing\}) = t_B$, that is, the maximal B-sieve. Thus, $True_B$ is the map that picks out the *maximal* B-sieve.

For any subfunctor of $\mathbf{y}[B]$, $S : \mathbf{S} \infty \mathbf{y}[B]$, the classifying arrow ϕ_S is a natural transformation $\phi_S : \mathbf{y}[B] \to \mathbf{\Omega}$, given by components $(\phi_S)_C : \mathbf{y}[B](C) \to \mathbf{\Omega}(C)$, such that for any figure $\chi : C \to B$ of the locus B, belonging to the set $\mathbf{y}[B](C)$, we have:

$$(\phi_S)_C(\chi) = \{h / (\chi \cdot h) \in \mathbf{S}(dom(h))\}$$

where h is *any* $\mathcal{B}$-arrow with codomain C. Then, obviously, $(\phi_S)_C(\chi)$ is a C-sieve. Note that a $\mathcal{B}$-arrow with codomain C, for instance $h : D \to C$, determines a set theoretical morphism:

$$\mathbf{y}[B](h) : Hom_{\mathcal{B}}(C, B) \to Hom_{\mathcal{B}}(D, B)$$
$$\mathbf{y}[B](h)(\chi) = \chi \cdot h$$

that may *or* may not take the figure $\chi \in \mathbf{y}[B](C)$, by means of $\chi \cdot h$ into $\mathbf{S}(dom(h)) = \mathbf{S}(D) \subseteq \mathbf{y}[B](D)$. In this sense, the C-sieve $(\phi_S)_C(\chi) \in \mathbf{\Omega}(C)$, contains all, and only those, $\mathcal{B}$-arrows h that *actually* take the figure χ into the subobject $\mathbf{S}(dom(h))$. It is clear that $(\phi_S)_C(\chi) = t_C$, namely it is the *maximal* C-sieve in the set $\mathbf{\Omega}(C)$, if and only if the figure χ belonging to the set $\mathbf{y}[B](C)$, belongs to $\mathbf{S}(C)$ as well.

Furthermore, if we *replace* the representable functor $\mathbf{y}[B]$ by *any* presheaf functor $\mathbf{P}$, and the subfunctors $S : \mathbf{S} \infty \mathbf{y}[B]$ by corresponding subfunctors $Q : \mathbf{Q} \infty \mathbf{P}$, entirely analogous arguments lead to the conclusion that $\mathbf{\Omega}$ is the *subobject classifier* in the category of presheaves $\mathit{Sets}^{\mathcal{B}^{op}}$, by which we mean that the diagram:

$$\begin{array}{ccc} \mathbf{Q} & \xrightarrow{\ !\ } & \mathbf{1} \\ \Big\downarrow{\scriptstyle Q} & & \Big\downarrow{\scriptstyle T} \\ \mathbf{P} & \xrightarrow{\ \phi_Q\ } & \mathbf{\Omega} \end{array}$$

is a *pullback* diagram in $\mathit{Sets}^{\mathcal{B}^{op}}$, where $\mathbf{\Omega}(B) = \Theta_{\hat{\mathcal{B}}}(\mathbf{y}[B])$, namely the set of all B-sieves for each locus B in $\mathcal{B}$, or equivalently the set of all subfunctors of $\mathbf{y}[B]$. The set $\mathbf{\Omega}(B)$ for each locus B in $\mathcal{B}$, is a partially ordered set under the relation of inclusion of B-sieves, whereas the maximal element of the poset $\mathbf{\Omega}(B)$ is the maximal B-sieve.

The crucial observation for our purposes is related to the fact that a spatial covering schema on the categorical environment $\mathcal{B}$ bears thr status of a presheaf subfunctor of $\mathbf{\Omega}$, due to the subobject inclusion $\mathbf{\Omega}_\chi \infty \mathbf{\Omega}$, established previously, and consequently it can be characterized in terms of some *classifying arrow* into $\mathbf{\Omega}$.

Before we specify the description of spatial covering schemata in $\mathcal{B}$ in terms of characteristic arrows into $\mathbf{\Omega}$, it is convenient to introduce some terminology related to the semantics of the identity arrow, $id_{\mathbf{\Omega}_\chi} : \mathbf{\Omega}_\chi \to \mathbf{\Omega}_\chi$. Let us consider, a spatial covering B-sieve Σ, $\Sigma \in \mathbf{\Omega}_\chi(B)$. Then, we define the *identical* assignment;

$$\mathbf{\Omega}_\chi(B) \ni \Sigma \mapsto [id_{\mathbf{\Omega}_\chi}]_B(\Sigma) := \Sigma \in \mathbf{\Omega}_\chi(B)$$

According to the above, Σ acquires a dual interpretation, which can be expressed equivalently as both:

i In the active sense Σ is interpreted as a spatial covering B-sieve; that is as a device that acts on the locus B by covering it in spatial terms.

ii In the passive sense Σ, that is $[id_{\mathbf{\Omega}_\chi}]_B(\Sigma) := \Sigma$, is interpreted as a generalized variable spatial point of the locus B, where the latter has obtained a spatial reference by means of Σ, that covers it spatially.

Thus, by means of the identity arrow $id_{\Omega_\chi} : \boldsymbol{\Omega}_\chi \to \boldsymbol{\Omega}_\chi$, for each locus B in $\mathcal{B}$, what gets spatially covered is identified with what spatially covers.

Let us call $\chi : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$, the *classifying arrow* characterizing the subobject inclusion of a spatial covering scheme $J : \boldsymbol{\Omega}_\chi \infty \boldsymbol{\Omega}$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$

$$\begin{array}{ccc} \boldsymbol{\Omega}_\chi & \xrightarrow{\;!\;} & \mathbf{1} \\ \big\downarrow{\scriptstyle J} & & \big\downarrow{\scriptstyle True} \\ \boldsymbol{\Omega} & \xrightarrow{\;\chi\;} & \boldsymbol{\Omega} \end{array}$$

Obviously, the characteristic arrow $\phi_J := \chi$, such that $\chi : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$, determines the spatial covering scheme $J : \boldsymbol{\Omega}_\chi \infty \boldsymbol{\Omega}$ that classifies, and conversely, it is uniquely determined by that relation. In order to understand the semantics of χ, we consider a spatial covering scheme on the categorical environment $\mathcal{B}$, such that $J(B)$ denotes the set of spatial covering B-sieves for each locus B in $\mathcal{B}$. Next, we define the natural transformation $\chi : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$ as follows:

$$\chi_B(S) = \{h \,/\, cod(h) = B, S \triangleright h\}$$

or equivalently:

$$\chi_B(S) = \{h \,/\, cod(h) = B, h^*(S) \in J(dom(h))\}$$

that is, $\chi_B(S)$ denotes the set of all $h : C \to B$, such that S covers h. Thus, for any $h : C \to B$ being covered by S, h belongs to $\chi_B(S)$;

$$\langle S \triangleright h \rangle \Rightarrow h \in \chi_B(S)$$

Clearly, this prescription specifies $\chi_B(S)$ as a B-sieve, that is;

$$\Omega(B) \ni S \mapsto \chi_B(S) \in \Omega(B)$$

We notice that the classifying arrow at the locus B serves to specify *exactly* what each B-sieve *covers* with respect to a *spatial* covering scheme J. If we focus our attention on the definition of $\chi_B(S) \in \Omega(B)$, we notice the following:

For any $\mathcal{B}$-arrow $g: C \to B$, we obtain the logical conjugation relation:

$$\chi_C(g^*(S)) = g^*(\chi_B(S))$$

for any B-sieve, therefore χ is actually a natural transformation $\chi: \Omega \to \Omega$ as required. Furthermore, if in the definition of $\chi_B(S)$, we employ the maximal B-sieve $t_B \in \Omega(B)$, that by its specification is a covering sieve of all arrows with codomain B, we obtain:

$$\chi_B(t_B) = t_B$$

This relation holds for every locus B in $\mathcal{B}$, and thus in functional terms we further obtain:

$$\chi \circ True = True$$

Moreover, the classifying arrow χ_B clearly preserves order, in the sense that for B-sieves R, S:

$$R \subset S \Rightarrow \chi_B(R) \subset \chi_B(S)$$

Thus, for any B-sieves R and S we obtain:

$$\chi_B(R \bigcap S) \subseteq \chi_B(R) \bigcap \chi_B(S)$$

and conversely:

$$\chi_B(R) \bigcap \chi_B(S) \subseteq \chi_B(R \bigcap S)$$

Hence, for each B in $\mathcal{B}$ we obtain the equality:

$$\chi_B(R \bigcap S) = \chi_B(R) \bigcap \chi_B(S)$$

meaning that the operation of *spatial classification commutes* with the operation of *intersection* of sieves, expressed in suggestive functional terms as follows:

$$\chi \circ \wedge = \wedge \circ (\chi \times \chi)$$

Finally, since the classifying arrow χ_B preserves order, and also $R \subset \chi_B(R)$, if we operate on this inclusion by acting with χ_B, we obtain:

$$\chi_B(R) \subseteq \chi_B(\chi_B(R))$$

Conversely, if $h \in \chi_B(\chi_B(R))$, then by definition $\chi_B(R) \triangleright R$, that is $\chi_B(R)$ covers R. Moreover, for each $g \in \chi_B(R)$, by definition, $R \triangleright g$. If we bear in mind the transitivity property of a spatial covering scheme, then $R \triangleright h$, or else, $h \in \chi_B(R)$. Thus, if $h \in \chi_B(\chi_B(R))$, then $h \in \chi_B(R)$ or equivalently:

$$\chi_B(\chi_B(R)) \subseteq \chi_B(R)$$

Hence, for each B in $\mathcal{B}$ we obtain the equality:

$$\chi_B(\chi_B(R)) = \chi_B(R)$$

that is, the operation of classification is *idempotent*, expressed in functional terms simply as follows:

$$\chi \circ \chi = \chi$$

Furthermore, if we consider a spatial covering B-sieve R, then, any $h : C \to B$ covered by R, belongs to $\chi_B(R)$, that is, $h \in \chi_B(R)$. Now, let us assume that, $\chi_B(R)$ covers an arrow f. By definition of $\chi_B(R)$, R covers all arrows in $\chi_B(R)$. Thus, by the transitivity condition of covering sieves, we obtain that R covers the arrow f, and hence, $f \in \chi_B(R)$. We conclude, in this sense that:

$$\langle \chi_B(R) \triangleright f \rangle \Rightarrow \langle f \in \chi_B(R) \rangle$$

If we recall that for any $f : D \to B$:

$$\langle f \in \chi_B(R) \rangle \Leftrightarrow f^*(\chi_B(R)) = t_D$$

we conclude that:

$$\langle \chi_B(R) \triangleright f \rangle \Rightarrow f^*(\chi_B(R)) = t_D$$

The above condition means that, $\chi_B(R)$ is a J *-closed sieve*; Most importantly, $\chi_B(R)$ is the *closure* of R in the covering scheme J. This conclusion can be equivalently stated as follows:

$$\chi_B(R) = [R]$$

Returning to the interpretation given above, according to which the classifying arrow at the locus B serves to specify exactly what each B-sieve R covers, by means of the B-sieve $\chi_B(R)$, we conclude that:

Spatial generalized points of the locus B, where the latter has obtained a spatial reference with respect to a covering sieve R of a *spatial covering* scheme J operating on $\mathcal{B}$, are being *classified* with respect to the *generalized irreducible durations* of B, covered by $[R]$.

The B-sieve $\chi_B(R)$, identified as a *local time forcing* B-sieve, signifies the set of *generalized durations* of the locus B, *classifying spatial generalized points* of B depicted through R. Remarkably from this perspective, local time-forcing sieves have a dual role. On the one hand, they are the *constituents* of *temporal covering* schemata, and on the other, they are used as *devices* for *spatial classification*.

This interpretation reveals the two-fold operational role of local time forcing in the categorical environment of $\mathcal{B}$ as both, the *generator* of a *temporal covering* schema endowing loci B with a *relational* temporal reference in terms of generalized durations, and also, as the *generator* of a *classification* schema characterizing spatial relations in terms of the durations they cover.

Let us now concentrate our attention on the presheaf of time-forcing sieves Ω_κ, being the presheaf of J-closed sieves for a covering schema J. There is clearly a subobject inclusion $T:\Omega_{\tau} \infty \, \Omega$. The *classifying* arrow characterizing T, is denoted by $\phi_T := \kappa$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$:

$$\begin{array}{ccc} \Omega_\kappa & \xrightarrow{\ !\ } & 1 \\ \Big\downarrow T & & \Big\downarrow True \\ \Omega & \xrightarrow{\ \kappa\ } & \Omega \end{array}$$

Evidently, the characteristic arrow $\phi_T := \kappa$, such that, $\kappa : \Omega \to \Omega$ determines the subobject $T:\Omega_{k} \infty \, \Omega$ that classifies, and conversely, is uniquely determined by that relation. We define the natural transformation $\kappa : \Omega \to \Omega$ as follows:

$$\kappa_B(S) = \{h \,/\, cod(h) = B, h^*(S) = t_{dom(h)}\}$$

It is clear that $\kappa_B(S) = S$ for every J-closed B-sieve. The set $\kappa_B(S)$ is interpreted as the set of generalized durations of B covered by a J-closed sieve S, under varying temporal resolution units specified by the sieve holes $dom(h)$. We notice again that the classifying arrow at the locus B serves to specify exactly what each B-sieve covers with respect to T. In an analogous fashion we obtain the conditions:

$$\kappa \circ True = True$$
$$\kappa \circ \wedge = \wedge \circ (\kappa \times \kappa)$$
$$\kappa \circ \kappa = \kappa$$

It is instructive to define the map $B \mapsto \widehat{\kappa_R}(B)$ for each B-sieve R, given a covering scheme J, as follows:

$$\widehat{\kappa_R}(B) = \kappa_B(R) \qquad iff \qquad R = [R]_J$$

which, assigns to each locus B in $\mathcal{B}$, its set of generalized durations, if and only if the B-sieve R is J-covering them and is also J-closed.

This assignment endows the reference domain B precisely with the semantics of temporal relations, that is generalized durations, and also, permits the interpretation of the locus B as a local time domain.

Similarly, let us call $\tau:\Omega\rightarrow\Omega$, the classifying arrow characterizing the subobject inclusion of a maximal temporal covering schema $T:\Omega_\tau \infty \Omega$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$:

$$\begin{array}{ccc}
\Omega_\tau & \xrightarrow{\;!\;} & \mathbf{1} \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle True} \\
\Omega & \xrightarrow{\;\tau\;} & \Omega
\end{array}$$

We define the natural transformation $\tau:\Omega\rightarrow\Omega$ as follows:

$$\tau_B(S)=t_B$$

This prescription specifies $\tau_B(S)$ as the maximal B-sieve, and obviously, the analogous conditions for the characteristic arrow τ are trivially satisfied. The set $\tau_B(S)$ is interpreted as the maximal set of generalized durations of B, identified with the set of generalized elements of B being covered by a J-closed sieve covering the identity of B. Clearly, such a J-closed sieve possesses maximal temporal resolution capability.

Under these circumstances, we define a complete local time-domain as follows: If the generalized durations of a local time-domain are covered by a J-closed sieve that *covers* the *identity* of B, then we call it a *complete local time domain*. It is also evident from the explicit description of the classifying arrow referring to time-forcing sieves, that if such a sieve covers the identity of a locus B in $\mathcal{B}$, then it is necessarily the maximal temporal covering B-sieve, and the locus B acquires the interpretation of a complete local time-domain. In this sense, a maximal temporal covering scheme on $\mathcal{B}$ is equivalent to the specification of complete local time-domains in the environment of $\mathcal{B}$, described functorially by means of Ω_τ.

From the above detailed description of the cases considered, we conclude that we may form a unified framework of classification, according to which, an arbitrary subobject Ω_λ of $\mathbf{\Omega}$ amenable to a spatial or maximal temporal covering qualification, denoted by the inclusion $\mathbf{\Omega}_\lambda \infty \mathbf{\Omega}$ can be characterized by means of a classifying arrow $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$, defined by a pullback diagram in $Sets^{\mathcal{B}^{op}}$ as follows:

$$\begin{array}{ccc} \mathbf{\Omega}_\lambda & \xrightarrow{!} & \mathbf{1} \\ \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle True} \\ \mathbf{\Omega} & \xrightarrow{\lambda} & \mathbf{\Omega} \end{array}$$

where, the natural transformation $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$ is defined for each locus B in $\mathcal{B}$ by the set;

$$\lambda(S) = \{h / cod(h) = B, S \rhd h\}$$

If S is a *spatial covering* B-sieve we obtain its *closure* $[S]$ in the image of the *classifying* arrow λ, whereas, if S is a *time-forcing* B-sieve, which, also covers the identity of B, we obtain the maximal *temporal* resolution id_B-covering B-sieve t_B, identifying the locus B as a *complete local time-domain*. The classifying arrow λ is order preserving, idempotent and commutes with the operation of finite intersections of covering sieves.

It is important to emphasize that, from a *logical* point of view, the *subobject classifier* $\mathbf{\Omega}$ is interpreted as a domain of *truth values*, partially ordered by inclusion, where the maximal truth value, for each locus B in $\mathcal{B}$, is represented by the maximal B-sieve. Hence, $\mathbf{\Omega}$ extends and enriches the classical static, absolute, and rigid set-theoretic two-valued object of truth values.

Since these truth values are *sieves*, which operate in the context of a communication topos, the proper and precise interpretation of the *representing* object $\mathbf{\Omega}$ is in the logical terms of the ancient Greek term "*aletheia*", which bears the meaning of *unveiling* temporally through the *holes* of a *covering* sieve. We conceive of unveiling as a process of temporal *percolation*, which can be localized with respect to complete local time-domains. For this reason, "*aletheia*" is amenable to *localization* enunciated in terms of both *spatial* and *temporal sieving* relations, and thus acquires the characteristics of a logical *sheaf.*

From the preceding, we may draw the following conclusions:

i The classifying arrow $\lambda : \Omega \to \Omega$ can be interpreted as an *operator* of *localization* of "*aletheia*" with respect to a complete local time domain. Equivalently, the operator λ, together with the above prescribed properties, induces a *topology* on the *communication topos* $\widehat{\mathcal{B}}$.

ii Since the *maximal truth value* for each locus B in $\mathcal{B}$ is the *maximal* B*-sieve* t_B, and also, in case t_B operated as a time-forcing B-sieve would have been id_B-covering, the localization induced by the action of λ on t_B, encapsulates the association of a *complete local-time domain* with a local reference frame, where a *complete local description* of reality can be effectuated appropriately in the environment of $\mathcal{B}$, signified simultaneously in terms of the maximal truth value t_B.

In this sense, the maximal B-sieve t_B plays *three* interwoven roles; (a) it is a *maximal covering* B-sieve; (b) if considered as time-forcing, it forces the interpretation of the locus B as a *complete local time-domain* by means of its associated local time operator $\widehat{T}_{t_B}$; and (c) it *unveils* through its *holes* a *complete local description* of reality, being *forced* by $\widehat{T}_{t_B}$, with respect to the complete local time-domain B in $\mathcal{B}$, the latter being subsequently called a local reference frame.

Note that such a local reference frame has meaning *only in time* in its function as a complete local time-domain. It expresses the intuition that in a localization schema of the physical continuum events can be individuated *simultaneously* from the continuum only in time, that is, over *complete local-time domains* in the localizing environment, such that a complete local description of reality is legitimate in their own descriptive terms. It is worth stressing that the notion of simultaneity with respect to a complete local time-domain, $id_B : B \to B$, refers to B as a *totality*, since the maximal temporal resolution unit or hole of the time-forcing B-sieve covering id_B, is clearly extended to extent B.

In the following, we shall realize that the *effectuation* of a complete *local* description in *every* local reference frame can be implemented appropriately in terms of *spatial covering* sieves, that is, through variable generalized points of a locus B (interpreted *spatially* by means of each covering B-sieve), *classified* by the generalized durations of B (interpreted accordingly as a complete *local time-domain*). In order to substantiate this argument we need some further notions that we explicate in detail below.

First of all, we note that we have defined the classifying arrow $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$ as a localization operator on the functor $\mathbf{\Omega}$ of "*aletheia*". Of course, *besides* λ, we could obviously consider the *identity* arrow $id_{\Omega} : \mathbf{\Omega} \to \mathbf{\Omega}$. Thus, it is reasonable to ask for their *equalizer*, denoted by $\mathbf{\Omega}^{\Lambda}$, according to the diagram:

$$\mathbf{\Omega}^{\Lambda} \xrightarrow{\ \epsilon\ } \mathbf{\Omega} \underset{\lambda}{\overset{id_{\Omega}}{\rightrightarrows}} \mathbf{\Omega}$$

where, $\mathbf{\Omega}^{\Lambda}$ is defined, for each locus B in $\mathcal{B}$, as follows:

$$\mathbf{\Omega}^{\Lambda}(B) = \{S : S \in \mathbf{\Omega}(B) \wedge [id_{\Omega}]^{B}(S) = \lambda_{B}(S)\}$$

The condition imposed on the defining requirement of $\mathbf{\Omega}^{\Lambda}$, is satisfied *only for those* B-sieves that are *local time-forcing*; that is for the B-sieves S that are J*-closed*, $S = [S]_J$, with respect to a *spatial covering* scheme J on $\mathcal{B}$. It is also clear that the maximal B-sieve t_B belongs in $\mathbf{\Omega}^{\Lambda}(B)$. We furthermore notice that if a sieve R belonging to $\mathbf{\Omega}^{\Lambda}(B)$ is also id_B-covering, then it is necessarily the *maximal* B-sieve t_B, which forces and consolidates the interpretation of B as a *complete local time-domain*.

From these observations, we conclude that the *equalizer* $\mathbf{\Omega}^{\Lambda} \to \mathbf{\Omega}$ is actually the same as the *subobject* $T : \mathbf{\Omega}_{\kappa} \infty \; \mathbf{\Omega}$ consisting of *local time-forcing sieves* $R = [R]$, or equivalently, local time operators $\hat{T}_{[R]}$.

Furthermore, since the operator λ is *idempotent*, from the universal property of the equalizer $\mathbf{\Omega}^{\Lambda} \to \mathbf{\Omega}$ we derive the existence of a *unique* arrow, $\iota : \mathbf{\Omega} \to \mathbf{\Omega}^{\Lambda}$, such that $\mathbf{\Omega}^{\Lambda}$ is the *image* of the operator λ;

$$\mathbf{\Omega} \xrightarrow{\iota} \mathbf{\Omega}^{\Lambda} \xrightarrow{\text{Ú}} \mathbf{\Omega}$$

The above defines then the *epimorphic-monomorphic* factorization of the operator $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$.

Additionally, we may consider the following pullback diagram:

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\ !\ } & \mathbf{1} \\ \downarrow {\scriptstyle True^{\Lambda}} & & \downarrow {\scriptstyle True} \\ \mathbf{\Omega}^{\mathbf{\Lambda}} & \xrightarrow{\ \epsilon\ } & \mathbf{\Omega} \end{array}$$

Since we have that $\lambda \circ True = True$, the arrow *True factors through* $\mathbf{\Omega}^{\mathbf{\Lambda}}$ as

$$True^{\Lambda} : \mathbf{1} \to \mathbf{\Omega}^{\Lambda}$$

where the *equalizer* arrow $\text{Ú} : \mathbf{\Omega}^{\Lambda} \to \mathbf{\Omega}$ actually characterizes the *subobject* $\mathbf{1} \infty \ \mathbf{\Omega}^{\Lambda}$, according to the above pullback diagram.

Moreover, if $\Xi : \mathbf{\Omega}_{\Xi} \infty \ \mathbf{\Omega}$ denotes an arbitrary subobject of $\mathbf{\Omega}$, characterized by means of $\xi : \mathbf{\Omega}_{\Xi} \to \mathbf{\Omega}$ then its *closure* with respect to a covering schema Λ, denoted by $[\mathbf{\Omega}_{\Xi}]_{\Lambda}$ is characterized by $\lambda \circ \xi$, and also clearly, $\mathbf{\Omega}_{\Xi}$ is Λ *-closed*, if and only if $\lambda \circ \xi = \xi$, that is, equivalently, if and only if ξ *factors through* $\mathbf{\Omega}^{\Lambda} \infty \ \mathbf{\Omega}$.

The above description can be formulated suitably, in order to apply to *subobjects* of an *arbitrary* functor $\mathbf{P}$ in the *topos* $Sets^{\mathcal{B}^{op}}$ if we define that a *subobject* $Q : \mathbf{Q} \infty \ \mathbf{P}$ characterized by means of $\phi_Q : \mathbf{P} \to \mathbf{\Omega}$ is λ*-closed*, if and only if its λ*-closure*, specified by $\lambda \circ \phi_Q$ satisfies the condition $\lambda \circ \phi_Q = \phi_Q$.

Then we are able to claim that the functor $\mathbf{\Omega}^{\Lambda}$ *classifies* the λ-closed subobjects, in the sense that, for each functor $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$, there exists a *natural bijection*:

$$Hom_{\widehat{\mathcal{B}}}(\mathbf{P}, \mathbf{\Omega}^{\Lambda}) \cong [\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{P})]_{\lambda}$$

where $[\mathbf{\Theta}_{\hat{\mathcal{B}}}(\mathbf{P})]_{\lambda}$ denotes the set of λ-*closed* subobjects of $\mathbf{P}$, according to the *pullback* diagram:

$$\begin{array}{ccc} [\mathbf{Q}] & \xrightarrow{\;!\;} & \mathbf{1} \\ \downarrow{\scriptstyle [Q]} & & \downarrow{\scriptstyle True^{\Lambda}} \\ \mathbf{P} & \xrightarrow{\;\phi_{[Q]}\;} & \mathbf{\Omega}^{\Lambda} \end{array}$$

where $\mathbf{\Omega}^{\Lambda}(B) \equiv [\mathbf{\Theta}_{\hat{\mathcal{B}}}(\mathbf{y}[B])]_{\lambda}$, denotes the set of *all* λ-*closed* B-sieves for each locus B in $\mathcal{B}$, or equivalently, the set of all λ-closed subfunctors of $\mathbf{y}[B]$. In other words, this is precisely the set of *local time-forcing* B-sieves, or local time-forcing operators with respect to the *localization* operator λ associated with the *covering* schema Λ.

It is important to notice that the notions of *closure* with respect to the *localization* operator λ and the *covering* schema Λ respectively, are actually *equivalent*. The set $\mathbf{\Omega}^{\Lambda}(B)$, for each locus B in $\mathcal{B}$, is a *partially ordered* set under the relation of inclusion of λ-*closed* B-sieves. Moreover, the elements of $\mathbf{\Omega}^{\Lambda}(B)$ can be interpreted as *truth values* with respect to λ, where the maximal truth value is t_B, which is λ-closed.

Furthermore, $\mathbf{\Omega}^{\Lambda}(B)$, for each locus B in $\mathcal{B}$, can be endowed with the logical operations of conjunction, disjunction and implication, and thus, acquire the structure of an *Arend Heyting algebra* with respect to these operations. In this sense, we say that the functor $\mathbf{\Omega}^{\Lambda}$ is a Heyting algebra object in the topos $\mathcal{S}ets^{\mathcal{B}^{op}}$, which *classifies*, in particular, λ-*closed* subobjects of *any presheaf* functor in this category.

Here it is essential to consider the *pullback* diagram that describes the classification of λ-closed subobjects of the representable functor $\mathbf{y}[B]$ in $\mathcal{S}ets^{\mathcal{B}^{op}}$, being precisely those subobjects that play the role of *local time-forcing* B-sieves in the environment of $\mathcal{B}$. The characteristic arrow $\phi_{[S]}$, such that $\phi_{[S]}:\mathbf{y}[B] \to \mathbf{\Omega}^{\Lambda}$, determines the local-time forcing B-sieve $[S]:\mathbf{[S]} \infty \mathbf{y}[B]$ that classifies, and conversely, it is uniquely determined by that, as follows:

$$\begin{array}{ccc} [\mathbf{S}] & \xrightarrow{\;!\;} & \mathbf{1} \\ \Big\downarrow{\scriptstyle [S]} & & \Big\downarrow{\scriptstyle True^{\Lambda}} \\ \mathbf{y}[B] & \xrightarrow{\;\phi_{[S]}\;} & \mathbf{\Omega}^{\Lambda} \end{array}$$

We introduce the following terminology: An arbitrary B-sieve $R:\mathbf{R} \infty \mathbf{y}[B]$ is λ*-dense* in $\mathbf{y}[B]$, if $[\mathbf{R}]=\mathbf{y}[B]$, or equivalently, if $[R]=t_B$. This is the case if and only if R is an id_B-covering B-sieve. Thus, an arbitrary B-sieve is λ*-dense* in $\mathbf{y}[B]$, if and only if it belongs to the family of *spatial covering* B-sieves of a spatial covering schema Λ on $\mathcal{B}$.

Consequently, since *any spatial covering* B-sieve S in the schema Λ is λ*-dense*, in the sense that the induced local-time forcing B-sieve $[S]$ is an id_B-covering B-sieve, and thus, the *maximal local time-forcing* B-sieve t_B, we have the possibility of a *complete local description* of reality, identified as the *maximal truth value* in $\mathbf{\Omega}^{\Lambda}$. This expresses the completed temporal *percolation* of "*aletheia*" with respect to the *maximal* local time-forcing B-sieve t_B. The latter is effectuated *in terms* of *any* depicted *spatial covering* B-sieve S over the simultaneously substantiated - by means of $[S]=t_B$ - *complete local time-domain* B, identified previously with a local reference frame. Conclusively we say that:

A complete local description of reality is legitimate for any covering B-sieve of a spatial covering schema on $\mathcal{B}$, with respect to the correspondingly induced complete local time-frame B.

To slightly rephrase the above for reasons of clarity, we assert the following:

A complete local description of reality is forced by the action of the maximal local-time operator $\widehat{T}_{t_B}$ on a locus B, generated from any spatial covering B-sieve of a schema Λ by the process of λ-closure, that is, by the process of temporal percolation, and expressed in the descriptive terms of the spatially covering objects over the correspondingly induced complete local time-frame B.

At this stage, it is necessary to make clear the precise manner in which the *maximal truth value* t_B, for each locus B in $\mathcal{B}$, *reflects* a *complete local* description of reality, in case B stands for a complete local time-frame, being induced by a spatial covering B-sieve of a scheme Λ, or by its associated localization operator λ. The clarification of this issue necessitates the introduction of the concept of λ-sheaf.

In general, if $R : \mathbf{R} \infty \mathbf{y}[B]$ is a B-sieve, then a presheaf $\mathbf{P}$ is defined to be a λ-*sheaf* if and only if the induced map

$$R^* : Hom_{\hat{\mathcal{B}}}(\mathbf{y}[B], \mathbf{P}) \xrightarrow{\cong} Hom_{\hat{\mathcal{B}}}(\mathbf{R}, \mathbf{P})$$

is an *isomorphism* for every λ-*dense subfunctor* of $\mathbf{y}[B]$. We mention that if we imposed on the map R^* as above, the relevant requirement according to which R^* was just a *monomorphism*, then the presheaf $\mathbf{P}$ would be λ-*separated*.

Taking into account our previous remarks on the notion of subobjects being λ-dense, a presheaf $\mathbf{P}$ is a λ-sheaf, if and only if the above map R^* is an isomorphism for every covering B-sieve R of a spatial covering schema Λ on $\mathcal{B}$. The definition of a λ-sheaf essentially means that an arrow from a λ-*dense subfunctor* of $\mathbf{y}[B]$, namely a *spatial covering* B-sieve, to a functor Δ qualified as a λ-sheaf, can be extended *uniquely* to an arrow on all of $\mathbf{y}[B]$ targeting Δ, such that according to the diagram below $r = y \circ R$:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\ r\ } & \mathbf{\Delta} \\ \Big\downarrow R & & \Big\downarrow id_{\mathbf{\Delta}} \\ \mathbf{y}[B] & \xrightarrow{\ y\ } & \mathbf{\Delta} \end{array}$$

In this setting, we impose the condition that for a *spatial covering* schema Λ on $\mathcal{B}$ all *representable* presheaves on $\mathcal{B}$ are λ-*sheaves*. Thus, spatial covering schemata correspond to the so called *subcanonical* and *canonical* Grothendieck topologies.

Let $\hat{\mathcal{B}}_{\lambda} := \mathbf{Sh}[\mathcal{B}]$ be the full *subcategory* of $\hat{\mathcal{B}}$, the objects of which are the λ -sheaves, and let $I : \mathbf{Sh}[\mathcal{B}] \infty \hat{\mathcal{B}}$ be the *inclusion* functor. A convenient way to think about the category $\mathbf{Sh}[\mathcal{B}]$ is provided by the insight that it is the subcategory of $\hat{\mathcal{B}}$, which is *closed* under *isomorphisms* induced by the action of an *operator-functor* L, commuting with pull-back operations, on the objects and arrows of $\hat{\mathcal{B}}$, in the following sense:

If an object in $\hat{\mathcal{B}}$ is *isomorphic* to one in $\mathbf{Sh}[\mathcal{B}]$, then it is itself in $\mathbf{Sh}[\mathcal{B}]$. We express this idea, given a *localization* operator λ, by constructing an operator-functor $\mathbf{L} : \hat{\mathcal{B}} \to \mathbf{Sh}[\mathcal{B}]$ which is *left adjoint* to the inclusion functor $I : \mathbf{Sh}[\mathcal{B}] \infty \hat{\mathcal{B}}$ and also *preserves pull-backs*. The induced adjunction, i.e. the *encoding/decoding* functorial relations between the category of presheaves $\hat{\mathcal{B}}$ and the category of sheaves $\mathbf{Sh}[\mathcal{B}]$ establishes concretely the functorial schema of *metaphora* characterizing the *localization* of "*aletheia*" through temporal percolation.

By this specification we mean precisely that L *reflects* each functor in $\hat{\mathcal{B}}$ into the subcategory $\mathbf{Sh}[\mathcal{B}]$, such that any Δ in $\hat{\mathcal{B}}$ is a λ*-sheaf*, if and only if the map $\Delta \to \mathbf{L}\Delta$ is an *isomorphism*.

Furthermore we define the set:

$$\Psi = \{\boldsymbol{\psi} / \langle \boldsymbol{\psi} : \mathbf{Y} \to \mathbf{X} \rangle \in \hat{\mathcal{B}}_1 \wedge \langle \mathbf{L}\boldsymbol{\psi} : \mathbf{LX} \xrightarrow{\cong} \mathbf{LY} \rangle\}$$

where, $\hat{\mathcal{B}}_1$ denotes the set of arrows of $\hat{\mathcal{B}}$. We notice that an arrow $\psi \in \hat{\mathcal{B}}_1$ belongs to the set Ψ, if and only if it is taken by the *action* of L, that is $\mathbf{L}\boldsymbol{\psi}$, to an *isomorphism*. Thus, for every Δ in $\hat{\mathcal{B}}$ the map $\Delta \to \mathbf{L}\Delta$ belongs to the set Ψ. In this light we can also verify that:

i Any presheaf Δ in $\hat{\mathcal{B}}$ is a λ -sheaf if and only if for each $\langle \psi : \mathbf{Y} \to \mathbf{X} \rangle \in \Psi$, the induced map ψ^* is an isomorphism:

$$\psi^* : Hom_{\hat{\mathcal{B}}}(\mathbf{X}, \Delta) \xrightarrow{\cong} Hom_{\hat{\mathcal{B}}}(\mathbf{Y}, \Delta)$$

ii Any map $\langle \psi : \mathbf{Y} \to \mathbf{X} \rangle \in \widehat{\mathcal{B}}_1$ belongs to Ψ if and only if for each λ-sheaf Δ in $\mathbf{Sh}[\mathcal{B}]$, the induced map ψ^* is an isomorphism:

$$\psi^* : Hom_{\widehat{B}}(\mathbf{X}, \Delta) \xrightarrow{\cong} Hom_{\widehat{B}}(\mathbf{Y}, \Delta)$$

Taking into account the original definition of a λ-sheaf, in conjunction with propositions [i] and [ii] above, we conclude that the set Ψ is *completely determined* by its *restriction* to λ-*dense* subfunctors of $\mathbf{y}[B]$. In other words, Ψ is completely determined by its restriction to covering B-sieves, whereas the operator-functor $\mathbf{L}$ expresses the process of λ *-closure*, or equivalently, the process of *closure* with respect to a *spatial covering* schema Λ. From this perspective, the role of the localization operator λ is precisely incorporated in the complete determination of the set Ψ by its restriction to the set of covering B-sieves for each locus B in $\mathcal{B}$. Furthermore, for any monic arrow $\psi : \mathbf{Y} \rightarrowtail \mathbf{X}$ in $\boldsymbol{\Psi}$, we say that, the subobject $\mathbf{Y}$ is λ-dense in $\mathbf{X}$.

From the property of Λ-closure, referring to Λ-closed B-sieves, as an operation preserved by pulling-back, we conclude that the operator-functor $\mathbf{L}$ should *commute* with pull-backs. Thus, if we recall that Λ-closed B-sieves, for each locus B in $\mathcal{B}$, stand for local time-forcing B -sieves, or equivalently local time-forcing operators, we conclude that:

The λ *-sheaf reflection* functor $\mathbf{L} : \widehat{\mathcal{B}} \to \mathbf{Sh}[\mathcal{B}]$, which is *left adjoint* to the *inclusion* functor $I : \mathbf{Sh}[\mathcal{B}] \rightarrowtail \widehat{\mathcal{B}}$, and also *commutes* with pull-back operations, expresses precisely the functioning of *Time* as a process of λ*-closure*, which is to say a process of *temporal percolation* with respect to a *spatial covering* schema Λ on $\mathcal{B}$. Again, this is enacted by the generation of *local-time forcing* B-sieves, for each locus B in $\mathcal{B}$, qualified as *complete local time-frames* if and only if they are also id_B-covering.

In this sense, the process of λ*-closure* with respect to a *spatial covering* scheme Λ, for every *subobject* $\Delta : \mathbf{\Delta} \rightarrowtail \mathbf{E}$, and in particular, for every B -sieve $R : \mathbf{R} \rightarrowtail \mathbf{y}[B]$, is simply expressed by the following *pullback* diagram:

$$
\begin{array}{ccc}
[\mathbf{\Delta}] & \longrightarrow & \mathbf{L\Delta} \\
\Big\downarrow [\Delta] & & \Big\downarrow \mathbf{L}\Delta \\
\mathbf{E} & \longrightarrow & \mathbf{LE}
\end{array}
$$

We notice that since L *commutes* with pull-back operations, and thus preserves subobject inclusions, we obtain for each subfunctor $\Delta : \mathbf{\Delta} \infty\ \mathbf{E}$, a new subfunctor $[\Delta] : [\mathbf{\Delta}] \infty\ \mathbf{E}$, identified as the λ*-closure* of Δ, such that $[\Delta]$ contains Δ. In case that E is itself a λ-sheaf, then a subfunctor of E is also a λ-sheaf if and only if it is λ*-closed*. As a consequence, we reach the important conclusion that *local time-forcing* B-sieves, for each locus B in $\mathcal{B}$, are λ*-sheaves* themselves.

Let us now consider the *subobject classifier* of λ *-closed* subobjects, that is, the functor expressing the *sheaf-theoretic localization* of "*aletheia*" $\mathbf{\Omega}^{\Lambda}$. We recall that for each functor E in $Sets^{\mathcal{B}^{op}}$, the following natural bijection pertains:

$$Hom_{\widehat{\mathcal{B}}}(\mathbf{E},\mathbf{\Omega}^{\Lambda}) \cong [\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{E})]_{\lambda}$$

where, $[\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{E})]_{\lambda}$ denotes the set of λ-closed subfunctors of E, according to the pullback diagram:

$$
\begin{array}{ccc}
[\mathbf{\Delta}] & \xrightarrow{\ !\ } & \mathbf{1} \\
\Big\downarrow [\Delta] & & \Big\downarrow True^{\Lambda} \\
\mathbf{E} & \xrightarrow{\ \phi_{[\Delta]}\ } & \mathbf{\Omega}^{\Lambda}
\end{array}
$$

It is conceptually clear from the discussion above that the *functor* $\mathbf{\Omega}^{\Lambda}$ is actually a λ*-sheaf*, which remarkably plays the role of the *subobject classifier* in the *topos* of λ*-sheaves*, such that the *natural bijection* above takes the following form, if *restricted* to the subcategory $\mathbf{Sh}[\mathcal{B}]$:

$$Hom_{\mathbf{Sh}[\mathcal{B}]}(\mathbf{E},\mathbf{\Omega}^{\Lambda}) \cong \mathbf{\Theta}_{\mathbf{Sh}[\mathcal{B}]}(\mathbf{E})$$

where, $\Theta_{\mathbf{Sh}[\mathcal{B}]}(\mathrm{E})$ denotes the set of λ-subsheaves of the λ-sheaf E.

11.9 TEMPORAL GAUGES: *ALETHEIA* IN THE REFLECTION OF SHEAVES

We shall provide a simple argument, which actually proves that $\mathbf{\Omega}^{\Lambda}$ is a λ-*sheaf*, and also that it is the *subobject classifier* in $\mathbf{Sh}[\mathcal{B}]$. From the natural bijection referring to λ-closed subobjects of a functor E in $\widehat{\mathcal{B}}$ it is evident that if E is a λ-sheaf, then the natural transformations $\phi_{\Gamma}:\mathbf{E}\rightarrow\mathbf{\Omega}^{\Lambda}$ correspond to the subfunctors $\mathbf{\Gamma}$ which are also λ-sheaves, if and only if $\mathbf{\Omega}^{\Lambda}$ is a λ-sheaf itself.

For this purpose, we consider an arrow $\langle\xi:\Xi\rightarrow\mathbf{\Pi}\rangle\in\Psi$, i.e. a natural transformation $\langle\xi:\Xi\rightarrow\mathbf{\Pi}\rangle\in\widehat{\mathcal{B}}_1$, qualified as an isomorphism $\mathbf{L}\xi:\mathbf{L\Xi}\xrightarrow{\cong}\mathbf{L\Pi}$, when L acts upon it. Then, the presheaf $\mathbf{\Omega}^{\Lambda}$ in $\widehat{\mathcal{B}}$ is a λ-sheaf, if we show that for $\langle\xi:\Xi\rightarrow\mathbf{\Pi}\rangle\in\Psi$ the induced map ξ^{*} is an isomorphism:

$$\xi^{*}:Hom_{\widehat{\mathcal{B}}}(\mathbf{\Pi},\mathbf{\Omega}^{\Lambda})\xrightarrow{\cong}Hom_{\widehat{\mathcal{B}}}(\Xi,\mathbf{\Omega}^{\Lambda})$$

Equivalently, if we use the natural bijection characterizing $\mathbf{\Omega}^{\Lambda}$ as the subobject classifier of λ-closed subobjects, and also further consider λ-closed subobjects $[\mathbf{\Sigma}]$ and $[\mathbf{\Upsilon}]$ of Ξ and $\mathbf{\Pi}$ respectively, i.e. $[\mathbf{\Sigma}]\infty\ \Xi$; $[\mathbf{\Upsilon}]\infty\ \mathbf{\Pi}$, it will be enough to show that there exists a surjective and injective correspondence between them, concluding thereby that $\mathbf{\Omega}^{\Lambda}$ is actually a λ-sheaf. We consider the composition of pullback diagrams:

$$\begin{array}{ccccc}
[\mathbf{\Sigma}] & \longrightarrow & [\mathbf{\Upsilon}] & \longrightarrow & \mathbf{L}[\mathbf{\Sigma}]\cong\mathbf{L}[\mathbf{\Upsilon}] \\
\downarrow{\scriptstyle [\sigma]} & & \downarrow{\scriptstyle [\upsilon]} & & \downarrow{\scriptstyle \mathbf{L}[\sigma]} \\
\Xi & \longrightarrow & \mathbf{\Pi} & \longrightarrow & \mathbf{L\Xi}\cong\mathbf{L\Pi}
\end{array}$$

If $[\upsilon]:[\mathbf{\Upsilon}]\infty\ \mathbf{\Pi}$ is a λ-closed subfunctor of $\mathbf{\Pi}$, then upon restriction to Ξ along the arrow $\langle\xi:\Xi\rightarrow\mathbf{\Pi}\rangle\in\Psi$, it gives a λ-closed subfunctor of Ξ, that is, $[\sigma]:[\mathbf{\Sigma}]\infty\ \Xi$. Conversely, if $[\sigma]:[\mathbf{\Sigma}]\infty\ \Xi$ is a λ-closed subfunctor of Ξ, then, by pulling-back $\mathbf{L}[\sigma]$ along the arrow $\langle \Pi \to \mathbf{L}\Pi \cong \mathbf{L}\Xi \rangle \in \Psi$, we obtain a λ-closed subfunctor of Π, i.e. $[\upsilon]:[\Upsilon] \infty\ \Pi$. This completes the proof of the argument.

After having established that the *functor* $\mathbf{\Omega}^{\Lambda}$ is a λ*-sheaf,* and also operates as the *subobject classifier* in the category of λ-sheaves $\mathbf{Sh}[\mathcal{B}]$, it is instructive to consider the following *pullback* diagram, as reflected in $\mathbf{Sh}[\mathcal{B}]$:

$$\begin{array}{ccc} [\mathbf{S}] & \xrightarrow{\ \ !\ \ } & \mathbf{1} \\ {\scriptstyle [S]}\big\downarrow & & \big\downarrow{\scriptstyle True^{\Lambda}} \\ \mathbf{y}[B] & \xrightarrow{\ \phi_{[S]}\ } & \mathbf{\Omega}^{\Lambda} \end{array}$$

where, given a *spatial covering* schema Λ on $\mathcal{B}$, $[S]$ is a *local time-forcing* B-sieve, classified in the category of λ-sheaves by the *characteristic arrow* $\phi_{[S]}:\mathbf{y}[B] \to \mathbf{\Omega}^{\Lambda}$. Note that the classifier object $\mathbf{\Omega}^{\Lambda}$ comprehends *only* the *closed subobjects* $[S]$ of $\mathbf{y}[B]$, identified as local time-forcing B-sieves, actually being λ*-sheaves* themselves, and consequently characterizes them in terms of *truth values.*

Thus, if S is *any* B-sieve, $\mathbf{\Omega}^{\Lambda}$ perceives and classifies *only* its λ-closure $[S]$, given a covering scheme Λ, as this is precisely *reflected* in $\mathbf{Sh}[\mathcal{B}]$ by the action of the *left adjoint* operator L, providing a faithful manifestation of the temporal percolation process, that is, of the process of λ-closure. For reasons of clarity, we recall that from any given B-sieve S we can construct a corresponding λ-closed B-sieve, denoted by $[S]$, simply as follows:

$$[S] = \{ f / cod(f) = B, S \rhd^{\Lambda} f \}$$

where f is any $\mathcal{B}$-arrow with codomain B, and the notation $S \rhd^{\Lambda} f$ denotes that S covers f according to a covering scheme Λ. Note that $[S]$ is the smallest closed sieve that contains S, called accordingly the λ-closure of S.

Let us examine, precisely what is expressed by the *maximal truth value* in the universe of λ-sheaves, where "*aletheia*" is localized. We notice that, given a B-sieve S, if it has the property of being λ-dense, then $[S]=t_B$; in other words its λ-closure is assigned the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$. Of course, this is the case if S is an id_B-covering B-sieve, or simply, a spatial covering B-sieve, specified according to a spatial covering schema Λ. Stated equivalently, in the reflection of the subcategory of λ-sheaves, the B-sieve S is perceived as being the maximal B-sieve t_B through its λ-closure, and most importantly, this is the case if and only if S is a spatial covering B-sieve.

Thus, the maximal truth value t_B in $\mathbf{\Omega}^{\Lambda}(B)$, i.e. the maximal λ-closed B-sieve t_B, interpreted as a truth value, for each locus B in $\mathcal{B}$, expresses the fact that it is $\langle\lambda - True\rangle$, hence $\langle[true]\rangle^{\lambda}{}_B \equiv t_B$ in the λ-sheaves reflection, that S is all of $\mathbf{y}[B]$, if and only if S is a spatial covering B-sieve. Consequently, the truth-values object $\mathbf{\Omega}^{\Lambda}(B)$ in $\mathbf{Sh}[\mathcal{B}]$, for each locus B in $\mathcal{B}$, perceives every spatial covering B-sieve, as the maximal λ-closed B-sieve t_B.

We bear in mind now that the maximal λ-closed B-sieve, as the maximal id_B-covering local time-forcing B-sieve, forces the interpretation of the locus B as a complete local time-frame, where a complete local description of reality is feasible by means of its identification with the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$. Thus, we form the following conclusion:

In the "*aletheia*" localization-environment of the category of *sheaves* $\mathbf{Sh}[\mathcal{B}]$, we can substantiate a *complete local description* of reality, formulated in terms of *every spatial covering* B-sieve Σ of a spatial covering schema Λ, for each locus B in $\mathcal{B}$, qualified by the temporal λ-closure process of local-time forcing as a *complete local time-frame*. This is simply expressed by the equation:

$$\lambda \circ \phi_{\Sigma} = \langle[true]\rangle^{\lambda}$$

We also recall that the maximal λ-closed, id_B-covering B-sieve t_B, can equivalently be thought of, in operator form, as the maximal local time-operator $\widehat{T}_{t_B}$, which, by *acting* on the locus B *forces* the interpretation of B as a complete local time-frame. Moreover, every other λ-closed, h-covering B-sieve $[\Gamma]$, that is, time-forcing B-sieve covering durations $h: dom(h) \to B$, acts as a local-time operator $\widehat{T}_{[\Gamma]}$ on the locus B, identifying h in $[\Gamma]$, with irreducible durations of the so-forced time-domain B, according to the "eigenvalues" characteristic equation:

$$\widehat{T}_{[\Gamma]} | B\rangle = h | B\rangle$$

where, $h \in [\Gamma]$ are the generalized irreducible durations of the local time domain B, covered by Γ, or equivalently, forced by $\widehat{T}_{[\Gamma]}$. Moreover, each locus $dom(h)$ is thought of as a hole in the B-sieve $[\Gamma]$, which specifies the kind of generalized durations of a local time domain covered by Γ, under $dom(h)$-varying temporal resolution units, with respect to the relation of temporal extension between loci. Obviously, $\widehat{T}_{t_B}$, corresponding to the maximal B-sieve t_B, possesses the maximal set of "eigenvalues" h, since it is covering the id_B, being themselves elements of t_B, and thus, durations of the forced local time-frame B.

In this setting, it must be stressed that the notion of a *hole*, or a temporal resolution unit, of a covering sieve is the denotator of a relation of *simultaneity*. This is an important notion that will permit us to understand precisely the meaning of all the truth values contained in $\mathbf{\Omega}^{\Lambda}(B)$. First of all, note that the notion of simultaneity with respect to a complete local time-domain, $id_B : B \to B$, refers to B temporally as a simultaneous *totality*, since the maximal temporal resolution unit or hole of the time-forcing B-sieve covering id_B, is obviously extended to level B.

In this sense, concerning the relation of simultaneity, a complete local time-domain B is completely characterized by its *maximal* temporal resolution unit, which is B itself. Hence, in the corresponding local time-frame B, a spatial covering B-sieve, that is an id_B-covering B sieve, incorporating a complete description of reality in terms of the *maximal truth value* in $\mathbf{\Omega}^{\Lambda}(B)$, is perceived as a *simultaneously*-existing object in its λ*-closure* t_B. This is precisely what we mean by characterizing it as a λ*-dense* object in its λ-closure. Thus, the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$ encapsulates precisely the fact that an id - B

covering B sieve is a simultaneity in its λ- closure - forced complete local time-frame B by means of being λ-dense in its λ-closure.

As a result, we establish a threefold association in the reflections' topos of λ-sheaves for each locus B in $\mathcal{B}$: the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$ is associated with a id_B-covering B-sieve Σ as a simultaneous object in its λ-closure; Σ is associated in its λ-closure as a simultaneity with the maximal hole B of Σ; the maximal hole B of the id_B-covering B-sieve Σ is associated with the establishment of a complete local time-frame B in the λ-closure of Σ.

Of course, there exist B-sieves Φ that are *not necessarily* id_B-covering. For example, we may consider a B-sieve that spatially covers the arrow $h: C \to B$, meaning that $h^*(\Phi) \in \Lambda(C)$ for a spatial covering schema Λ on $\mathcal{B}$. In the reflection of λ-sheaves it shows up as the corresponding λ-closed B-sieve $[\Phi]$ covering h, which consequently is interpreted as a generalized $\langle \; dom(h)$-duration $\rangle$, denoted by $\langle \; h_{dom(h)} \; \rangle$ at temporal resolution unit $dom(h) = C$. By saying that B is a temporal extension of C, we fix the maximal temporal resolution unit of Φ at level C, and associate the hole C with the denotation of simultaneity at level C. Parenthetically, note that a generalized duration at temporal resolution unit B is a complete local time-domain B, and the term $\langle \; B$-duration $\rangle$ expresses precisely the signified simultaneity at maximal hole B, which is denoted by $\langle [true] \rangle^{\lambda}{}_{B}$.

From the perspective of the complete local time-frame B, the B-sieve Φ, comprehended through its λ-closure $[\Phi]$, is assigned a *truth value* $\langle [true] \rangle^{\lambda}{}_{B \downarrow C}$, which means that it expresses a simultaneity at maximum temporal resolution unit equal to the *hole* C. Hence, from the viewpoint of a complete local time-frame B the B-sieve Φ *spatially* covering $h: C \to B$, where C stands for the maximal hole of Φ, provides a *partial* description of reality up to simultaneity level, specified by the maximal hole $dom(h) = C$. Of course, from the perspective of the complete local time-frame C, the restriction of the B-sieve Φ to C, since it is id_C-covering by means of $h^*(\Phi) \in \Lambda(C)$, for $h: C \to B$, it provides a complete description of reality as perceived through its λ-

closure, denoted by the maximal truth value $\langle[true]\rangle^{\lambda}{}_{C}$, signifying simultaneity at maximal hole C. Thus, we form the conclusion:

The object of "*aletheia*", identified as the *truth-values* object in the universe of sheaves incorporates the precise means of *localization* of "aletheia", regarding *local* descriptions of reality with respect to *local time-frames*, on the basis of *simultaneity* relations, whose maximal extent is signified by the maximal *holes* of *spatial covering* sieves, perceived through their *closure*.

Finally, in order to emphasize the meaning of *simultaneity* associated with the maximal hole $\langle dom(h)\rangle_{max}$ of a local time-forcing B-sieve $[\Gamma]$ covering durations $h: dom(h) \to B$, we define for every local time operator $\widehat{T}_{[\Gamma]}$ on the locus B, identifying h in $[\Gamma]$, with a duration of the so-forced time-domain B, an associated local time-operator $\widehat{T}_{s[\Gamma]}$ of the maximal extent of simultaneity relations incorporated in the action of $\widehat{T}_{[\Gamma]}$ as follows:

$$\widehat{T}_{s[\Gamma]} \mid B\rangle = \langle dom(h)\rangle_{max} \mid B\rangle$$

where, $\langle dom(h)\rangle_{max}$ denotes the maximal temporal resolution unit of the local time-operator $\widehat{T}_{[\Gamma]}$.

The development of the ideas regarding the *gnomonic* and *functorial* conceptualization of "*aletheia*", as a process of temporal percolation, which can be localized sheaf-theoretically with respect to a variety of local time-frames, together with their logical relational descriptive rules, inspires the following claims:

We claim that neglecting the relational functioning of local time-frames in nature, together with their respective simultaneity relations and truth values assignments, is a source of paradoxes that subsequently generate defective epicyclic interpretations of various forms. It is enough to point out that reducing the temporal closure process to simultaneity relations at point-durations $1 \to B$, thus accepting only the existence of point time-frames in nature, or equivalently only instantaneous spaces through which motion can be qualified, is the source of serious conceptual and technical problems in attempts to reconcile classical theories, including general relativity, with quantum theories. From this standpoint, we emphasize that spatial covering, as well as time-forcing schemata, have been developed both for the purpose of explicating, and qualifying in logical terms, the process of localization of physical continuum events in relational spatiotemporal terms within a suitably specified categorical environment.

The further claim we wish to state concerns the ontology of localized events. We assert that it is precisely the local time-frame of simultaneity, or equivalently, the maximal hole of a local time-forcing covering sieve on a locus, forcing its former temporal interpretation, which determines the ontology of individuated events from the continuum by means of their localization over the locus. In this sense, continuum events are individuated as simultaneity-determined entities in the descriptive terms of local time-frames, where the maximal extent of simultaneity relations involved in the action of a time-forcing sieve $[\Gamma]$ on a locus B is determined by the maximal hole-"eigenvalue" of the corresponding local time-operator $\widehat{T}_{s[\Gamma]}$ acting on B. The ontology of individuated events, in this sense, is specified exactly by the nature of the maximal hole in the associated covering sieve.

Thus, localized events are not restricted in any way to point-events. The logical rules used for the understanding of the signified simultaneity-relations should comply with the sheaf-theoretic localization of truth-values assignments, expressing their spectral classification in local time-frames of corresponding temporal resolution units, and should never be reduced uncritically and exclusively to the descriptive terms of point time-frames. Otherwise, a variety of paradoxes of mixed ontological and logical inconsistencies arise, precisely as the net-effect of the reduction of the temporal percolation process to time-frames of merely point-durations.

11.10 NATURAL SPECTRAL SPATIOTEMPORAL OBSERVATION IN A TOPOS

Up to present, we have determined all the necessary spatial and temporal concepts needed for the individuation of observable events in the physical continuum through the localizing environment of a category of sheaves. Individuated events from the physical continuum become comprehensible through spatiotemporal observation that respects the norms of closed-sieve temporal percolation taking place over loci that have been qualified as local time-frames. In this manner, if we try to enunciate the means of functorial spatiotemporal localization in precise *spectral* terms, we can legitimately identify the above *loci* with the *spectra* of commutative observable algebras sheaf-theoretically.

From this perspective, spatiotemporal *observation* is the process that detects, and subsequently, organizes the set of generalized points of a spatial h-covering B-sieve R on a locus B, corresponding to *events* localized over that locus, by means of an A_B*-co-sieve* $\breve{O}$ consisting of *local* ς*-linear epimorphic* representations $A_B \to A_C$ of commutative, associative and unital algebras A_C, for $h : C \to B$, defined over an *algebraic number field* ς, whose elements are identified as *observables* taking co-final values in ς, as follows: in every A_B-co-sieve $\circlearrowleft$ a local epimorphisms of ς-algebras $A_B \to A_C$, corresponds to an A_C-state of A_B, interpreted as the A_C-state of an observable in A_B at duration $h: C \to B$ of the corresponding local time-frame B, forced by the temporal closure $[R]$ of R.

We call each commutative, associative and unital algebra without zero divisors, defined over an algebraic field ς, contained in an A_B-co-sieve $\circlearrowleft$, a *commutative* ς-*arithmetic* effectuating *observation* of events at duration $h: C \to B$ of a local time-frame B, or equivalently, at temporal resolution unit $dom(h)$ of the local time operator $\widehat{T}_{[R]}$ acting on the locus B. Without loss of generality, we may assume that the field ς is identical with the field R of the real numbers, or its algebraic closure C.

Note that if the spatial B-sieve is id_B-covering, and consequently, the corresponding time-forcing B-sieve is the *maximal* λ-*closed* B-sieve, making B a complete local time-frame, then the A_B-co-sieve $\circlearrowleft$ contains the identity R-linear representation $id_{A_B}: A_B \to A_B$. Thus, a complete local time-frame B is the *simultaneity* locus, or id_B-duration of A_B-evaluated observables in A_B. Furthermore, the R-states of observables in A_B, that is, the R-linear representations of A_B into the R-algebra R, namely $A_B \to \mathrm{R}$, correspond to *states* at point-durations $1 \to B$ of a local time-frame B, identified with spectrally observable point-figures or *spatial* 1-points of the corresponding *instantaneous* spaces 1_m at 1-*durations* $m: 1 \to B$. We call the set of all observable point-figures at all 1-durations of a local time-frame B, the R-*spectrum* of the commutative R-arithmetic A_B, meaning the set of individuated events at temporal resolution unit 1, which are precisely detectable by means of *evaluations* of observables into the R-algebra R.

In a completely analogous fashion, we define the A_C-*spectrum* of the commutative R-arithmetic A_B, where A_C is a commutative arithmetic in $\circlearrowleft$, as the set of individuated events at *temporal resolution* unit C, that are detectable as C*-figures* at *durations* $h: C \to B$, by means of *evaluations* of observables belonging to A_B into the R-algebra A_C.

In general algebraic terms, all A_C-states of the commutative R-arithmetic A_B, where $A_B \to A_C$ are local R-linear epimorphic representations of commutative R-arithmetics in the A_B-co-sieve $\mathcal{O}$, are in *bijective correspondence* with *prime ideals* of A_B, as follows:

$$\langle H : A_B \to A_C \rangle \xrightarrow{\cong} Ker(H)$$

The set of all prime ideals of the commutative arithmetic A_B, specified as above, constitutes the *prime spectrum* of A_B. Thus, the prime spectrum of A_B consists of all individuated events under varying temporal resolution units of the local time-frame B, being detectable as figures at the corresponding durations of B. We recall that the *maximal* spectrum of any commutative ς-arithmetic A_B is the set of *maximal ideals* of A_B, where an ideal α is maximal if and only if A_B / α is an algebraic field.

It is clear that spatiotemporal observation, understood as a process by means of which individuated events become *spectrally* detectable through evaluations of observables in local commutative arithmetics, constitutes a dual or opposite categorical perspective to the one corresponding to the localization of events in relational spatial and temporal terms. We may say that it constitutes the *algebraic encoding* of the information encapsulated in the spatial and temporal covering schemata, which in turn can be characterized accordingly as *geometrical*.

In this sense, taking into account the *duality* between sieves and co-sieves, spatiotemporal observation is categorically equivalent to *co-localization* of the *covariant* functor $\mathcal{O}$, which is by specification a *closed dense subobject* of $Hom_{\mathcal{A}}(A_B, -) := \bar{\mathbf{y}}[A_B]$, namely a $\bar{\lambda}$-co-sheaf locally isomorphic to the covariant representable functor $\bar{\mathbf{y}}[A_B]$ in the category $\overline{\mathbf{Sh}}[\mathcal{A}]$. Most importantly, from the *sheaf-theoretic* perspective the elements of the A_B-co-sieve $\mathcal{O}$, that is the observables, are identified with *local sections* of this $\bar{\lambda}$-co-sheaf $\mathcal{O}$. We may refer to all relevant functors as sheaves, under the condition that the distinction of the algebraic from the geometrical perspective becomes clear from the covariance or contravariance respectively of these functors.

In conclusion, natural spatiotemporal observation is essentially the algebraic manifestation of the temporal percolation process with respect to a spatial covering schema, or equivalently, the algebraic transcription of the action of local time-operators on loci B in a categorical environment $\mathcal{B}$ by means of information encoding referring to event-figures at durations $h: C \to B$ of local time-frames B, in terms of corresponding local commutative arithmetic evaluations of observables belonging to A_B into A_C, which take place at respective durations h of B.