COMMUNICATION TOPOI: QUALIFYING THE PHYSICAL "CONTINUUM"

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The codomain of evaluation of physical attributes with respect to some measurement scale is usually identified with the concept of the "*physical continuum*". In standard approaches, the model used to represent these values is the real line $\mathbb{R}$ and its powers, specified as a set- theoretic structure of points, to be identified under instantiation as events, which are independent and distinguishable with precision.

The standard model of the "physical continuum" identifying events globally with the point-elements of the real line faces serious shortcomings when cases of *subjective uncertainty*, as in standard probability theory, or cases of *objective uncertainty* and *indistinguishability*, as in quantum mechanics, have to be taken into account. In these case, the notion of the physical "continuum" does not rely on an assumed preexisting set-theoretic structure of points on the real line. Rather, the evaluation of observables require the prior instantiation of well-defined *local* measurement *frames*, or even *local contexts* of observation, that depend on the prior *infiltration* or *percolation* of events through these pertinent local frames of measurement, or spectral observation. It is precisely in the extensive correlations among these local frames that continuity of observables can be assigned and part-whole or local-global relations can be meaningfully formulated.

In this sense, particular attention is needed in the clarification of what is meant by *localization*, and concomitantly, how it affects our model of the "physical continuum". The basic premise is that only through a consistent localization process does it become possible to *discern* observable events and assign an *individuality* to them. Generally, such a process should *not* depend on the existence of points, and moreover the standard notions of space and time should be *derivative* from localization, rather than the other way round.

Thinking in physical measurement terms, localization is tantamount to a process of *filtering* or *percolating* observables through an appropriate category of frames, such that the ordered structure of events emerging by their evaluation, *fibers over* the underlying local frames including their extensive correlations. In this sense, and from the reciprocal viewpoint, an *event* bears the *depth* of a *sieve* of local spectral frames through which it *percolates*, that resolve it compatibly at various frames of resolution through local observables. In turn, the latter defines a *homologous* physical procedure of observation or measurement.

Since it is assumed that frames together with their structural morphisms give rise to a *category*, the localization process should be understood in terms of an *action* of the category of frames on the global structure of observed events, according to the above. Then, the event structure is qualified in terms of a *partition spectrum*; it is partitioned into sorts parameterized by the objects of the category of frames. Thus, localization can be represented by means of a fibered structure, understood geometrically as a variable set over the base category of frames. The *fibers* are qualified, analogously to the case of the action of a group on a set of points, as the *generalized orbits* of the action of the category of frames. The notion of functional dependence incorporated in this action, forces the *ordered* structure of physical events to *fiber* over the base category of frames.

The partition spectrum emerging out of this action is characterized by *uniformity*. More precisely, for any two events observed over the same frame, the structure of all frames that relate to the first *cannot* be distinguished in any possible way from the structure of frames relating to the second. Given this uniformity, the *ordering* relation between events should be *induced* from the *base* category of frames, that is, by *lifting* relations between frames at the base to the fibers.

10.2 FUNCTORIAL LOCALIZATION: SHEAVES OF GERMS OF OBSERVABLES

In order to clarify the functioning of a localization process we will describe in detail the important case of *localization* of a commutative, unital $\mathbb{R}$-algebra of observables of a natural system over a base localizing category $\mathcal{O}(X)$, consisting of *open loci* U of a topological space X, the arrows between them being *inclusions*. In this case, the frames are defined in terms of the open loci U of X, *partially ordered* by inclusion.

Since observables are conceived as *global functions* on the $\mathbb{R}$-coordinatized state-space of this system, the process of localization forces the *replacement* of the algebra of observables $\mathcal{A}$ by an algebraic structure which will give us all *local* and *global* functional information together. All these functional elements should *interlock* compatibly together in an appropriate manner, which serves to respect the *extension* from local to global, as well as the *restriction* from global to local implied by the localization process. The structure at issue is precisely formalized by the notion of a *sheaf of germs* of a commutative, unital $\mathbb{R}$-algebra of observables, denoted by $\mathbf{A}$, which, incorporates all compatible local and global information together. Let us first introduce precisely the categorical notion of a sheaf on an abstract topological space, and then, examine its applicability in the current situation.

For this purpose, we consider the *category* of open sets $\mathcal{O}(X)$ in an abstract topological space, partially ordered by inclusion. If $\mathcal{O}(X)^{op}$ is the *opposite* category of $\mathcal{O}(X)$, and $\mathcal{S}ets$ denotes the scaffolding provided by the category of sets, we define:

A *presheaf* of sets on $\mathcal{O}(X)$ is a *contravariant* set-valued functor on $\mathcal{O}(X)$, denoted by $\mathbf{P}: \mathcal{O}(X)^{op} \to Sets$. For each base open set U of $\mathcal{O}(X)$, $\mathbf{P}(U)$ is a set, and for each arrow $F: V \to U$, $\mathbf{P}(F): \mathbf{P}(U) \to \mathbf{P}(V)$ is a set-function. If $\mathbf{P}$ is a presheaf on $\mathcal{O}(X)$ and $p \in \mathbf{P}(U)$, the *value* $\mathbf{P}(F)(p)$ for an arrow $F: V \to U$ in $\mathcal{O}(X)$ is called the *restriction* of p *along* F and is denoted by $\mathbf{P}(F)(p) := p \cdot F$. A presheaf $\mathbf{P}$ may be understood as a *right action* of $\mathcal{O}(X)$ on a set. This set is *partitioned* into sorts parameterized by the objects of $\mathcal{O}(X)$, and has the following property: If $F: V \to U$ is an *inclusion* arrow in $\mathcal{O}(X)$ and p is an element of $\mathbf{P}$ of sort U, then $p \cdot F$ is specified as an element of $\mathbf{P}$ of sort V. Such an action $\mathbf{P}$ is referred as an $\mathcal{O}(X)$-variable set. A variable set of this form is entirely determined by its category of elements.

The *category of elements* of a presheaf $\mathbf{P}$, denoted by $\int(\mathbf{P}, \mathcal{O}(X))$, is described as follows: The objects of $\int(\mathbf{P}, \mathcal{O}(X))$ are all *pairs* (U, p), with U in $\mathcal{O}(X)$ and $p \in \mathbf{P}(U)$. The arrows of $\int(\mathbf{P}, \mathcal{O}(X))$, that is, $(U', p') \to (U, p)$, are those morphisms $Z: U' \to U$ in $\mathcal{O}(X)$, such that $p' = \mathbf{P}(Z)(p) := p \cdot Z$. Notice that the arrows in $\int(\mathbf{P}, \mathcal{O}(X))$ are those morphisms $Z: U' \to U$ in the base category $\mathcal{O}(X)$, that *pull* a chosen element $p \in \mathbf{P}(U)$ *back* into $p' \in \mathbf{P}(U')$.

The category of elements $\int(\mathbf{P}, \mathcal{O}(X))$ of a presheaf $\mathbf{P}$, together with, the projection functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{O}(X)) \to \mathcal{O}(X)$ defines the *split discrete fibration* induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the *base* category of the fibration. We note that the fibers are categories in which the only arrows are identity arrows. If U is an open reference locus of $\mathcal{O}(X)$, the inverse image of U under $\int_{\mathbf{P}}$, is simply the set $\mathbf{P}(U)$, although its elements are written as pairs so as to form a disjoint union.

From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{O}(X)$, is the following: We identify an *element* of $\mathbf{P}$ of *sort* U, that is $p \in \mathbf{P}(U)$, with a *local* observable, which, can be observed by means of a measurement procedure over the reference locus U, being an open set of a topological space X. This identification forces the interrelations of local observables, over all reference loci of the base category $\mathcal{O}(X)$, to fulfill the requirements of a *uniform* and *homologous* fibered categorical structure. We recall that the latter is understood according to the following requirements:

a The reference loci used for observational purposes, together with, their structural morphisms, should form a mathematical *category*.

b For any two local observables, both amenable to a measurement procedure, over the same open domain of measurement U, the structure of all reference loci that relate to the first *cannot* be distinguished, in any possible way, from the structure of loci relating to the second. According to this, all the localized observables, within any particular reference locus, should be *uniformly equivalent* to each other.

The split discrete fibration induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the base category of the fibration, provides a well-defined notion of a uniform homologous fibered structure of local observables in the following sense: Firstly, by the arrows specification defined in the category of elements of $\mathbf{P}$, any local observable p, determined over the reference locus U, is *homologously* related with any other local observable p' over the reference locus U', and so on, by variation over all the reference loci of the base category. Secondly, all the local observables p of $\mathbf{P}$, of the same sort U, determined over the same reference locus U, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{P},\mathcal{O}(X))$ are induced by lifting arrows from the base category $\mathcal{O}(X)$, formed by partially ordering the reference loci. We conclude that the topological localization process is consistent with the physical requirement of uniformity.

The next crucial step of the construction, aims at the satisfaction of the following physical requirement: Since, we have assumed the existence of reference contexts (open observational domains) locally, according to the operational requirements of a corresponding physical procedure of measurement, the information gathered about local observables in different measurement situations should be *collated* by appropriate means. Mathematically, this requirement is implemented by the methodology of *completion* of the *presheaf* $\mathbf{P}$, or equivalently, *sheafification* of $\mathbf{P}$.

A *sheaf* is characterized as a presheaf $\mathbf{P}$ that satisfies the following condition: If $U = \bigcup_a U_a$, U_a in $\mathcal{O}(X)$, and elements $p_a \in \mathbf{P}(U_a)$, $a \in I$:index set, are such that for arbitrary a, $b \in I$, it holds:

$$p_a | U_{ab} = p_b | U_{ab}$$

where, $U_{ab} := U_a \bigcap U_b$, and the symbol $|$ denotes the operation of *restriction* on the corresponding open domain, then there exists a *unique* element $p \in \mathbf{P}(U)$, such that $p | U_a = p_a$ for each a in I . Then, an element of $\mathbf{P}(U)$ is called a *section* of the sheaf $\mathbf{P}$ over the open locus U . The sheaf condition means that sections can be *glued* together *uniquely* over the reference loci of the base category $\mathcal{O}(X)$. In particular, the sheaf-theoretic qualification of a uniform and homologous fibered structure of observables, as above, makes the latter also *coherent*, in terms of local-global compatibility of the information content it carries, under the operations of restriction and collation.

Thus, we form the following conclusion: The structure of a sheaf arises by imposing on the uniform and homologous fibered structure of elements of the corresponding presheaf the following two requirements:

i Compatibility of observable information under *restriction* from the *global to the local* level, and

ii Compatibility of observable information under *extension* from the *local to the global* level.

According to the first of the above requirements, a sheaf constitutes a *separated presheaf* (monopresheaf) of local observables over a global topological space, meaning that two observables are identical globally, if and only if, they are identical locally. In turn, according to the second requirement, locally compatible observables can be collated together in some global observable, which, is also *uniquely* defined because of the first requirement.

Furthermore, it is obvious that each set of sort U , $\mathbf{P}(U)$, can be endowed with the structure of an $\mathcal{R}$-algebra under pointwise sum, product, and scalar multiplication, denoted correspondingly by $\mathbf{A}(U)$; in that case, the morphisms $\mathbf{A}(U) \to \mathbf{A}(V)$ stand for $\mathcal{R}$-linear morphisms of $\mathcal{R}$-algebras. In this algebraic setting, the sheaf condition means that the following *sequence* of $\mathcal{R}$-algebras of local observables is *left exact*;

$$0 \to \mathbf{A}(U) \to \prod_a \mathbf{A}(U_a) \to \prod_{a,b} \mathbf{A}(U_{ab})$$

As an important example of the above, if $\mathbf{A}$ is the contravariant functor that assigns to each open locus $U \subset X$, the set of all real-valued continuous functions on U , then we will show that $\mathbf{A}$ is actually a sheaf.

Finally, it is important to explain the construction of the *inductive limit* (colimit) of sets (or rings, or $\mathcal{R}$ -algebras) $\mathbf{A}(U)$, denoted by $Colim[\mathbf{A}(U)]$, in order to explicate the physically important notions of stalks and germs of a sheaf. For this purpose, let us consider that x is a point of the topological measurement space X. Moreover, let K be a set consisting of open subsets of X, containing x, such that the following condition holds: For any two open reference domains U, V, containing x, an open set $W \in K$ exists, contained in the *intersection* domain $U \bigcap V$. We may say that K constitutes a *basis* for the system of open reference domains around x. We form the *disjoint union* of all $\mathbf{A}(U)$, denoted by;

$$\mathbf{D}(x) := \coprod_{U \in K} \mathbf{A}(U)$$

Then, we can define an *equivalence relation* in $\mathbf{D}(x)$, by requiring that $p \sim q$, for $p \in \mathbf{A}(U)$, $q \in \mathbf{A}(V)$, provided that, they have the *same restriction* to a smaller open set contained in K. Then we define;

$$Colim_K[\mathbf{A}(U)] := \mathbf{D}(x)/ \sim_K$$

Note that, if we denote, the inclusion mapping of V into U by;

$$i_{V,U} : V \infty U$$

and also, the restriction morphism of sets from U to V by;

$$\rho_{U,V} : \mathbf{A}(U) \to \mathbf{A}(V)$$

we can introduce well-defined notions of addition and scalar multiplication on the set $Colim_K[\mathbf{A}(U)]$, making it into an $\mathcal{R}$-module, or even, an $\mathcal{R}$-algebra, as follows:

$$[p_U] + [q_V] := [\rho_{U,W}(p_U) + \rho_{V,W}(q_V)]$$
$$\mu[q_V] := [\mu q_V]$$

where, p_U and q_V are elements in $\mathbf{A}(U)$ and $\mathbf{A}(V)$, and $\mu \in \mathcal{R}$.

Now, if we consider that K and Λ are two bases for the system of open sets domains around $x \in X$, we can show that there are canonical isomorphisms between $Colim_K[\mathbf{A}(U)]$ and $Colim_\Lambda[\mathbf{A}(U)]$. In particular, we may take all the open subsets of X containing x: Indeed, we consider first the case when K is arbitrary and Λ is the set of all open subsets containing x. Then $\Lambda \supset K$ induces a morphism;

$$Colim_K[\mathbf{A}(U)] \to Colim_\Lambda[\mathbf{A}(U)]$$

which is an isomorphism, since whenever V is an open subset containing x, there exists an open subset U in K contained in V. Since we can repeat that procedure for all bases of the system of open sets domains around $x \in X$, the initial claim follows immediately.

Then, the *stalk* of $\mathbf{A}$ at the point $x \in X$, denoted by $\mathbf{A}_x$, is precisely the inductive limit of sets $\mathbf{A}(U)$:

$$Colim_K[\mathbf{A}(U)] := \coprod_{U \in K} \mathbf{A}(U) / \sim_K$$

where K is a basis for the system of open reference domains around x, and $\sim_K$ denotes the equivalence relation of restriction within an open set in K. Note that the definition is *independent* of the chosen basis K. For an open reference domain W containing the point x, we obtain an morphism of $\mathbf{A}(W)$ into the stalk at the point x:

$$i_{W,x} : \mathbf{A}(W) \to \mathbf{A}_x$$

For an element $p \in \mathbf{A}(W)$ its image:

$$i_{W,x}(p) := p_x = germ_x p$$

is called the *germ* of p at the point x.

The fibered structure that corresponds to a sheaf of sets $\mathbf{A}$ is a *topological bundle* defined by the continuous mapping $\varphi : A \to X$, where;

$$A = \coprod_{x \in X} \mathbf{A}_x$$

$$\varphi^{-1}(x) = \mathbf{A}_x = Colim_{\{x \in U\}}[\mathbf{A}(U)]$$

The mapping φ is *locally* a *homeomorphism* of topological spaces. The topology in A is defined as follows: for each $p \in \mathbf{A}(U)$, the set $\{p_x, x \in U\}$ is open, and moreover, an arbitrary open set is a union of sets of this form. Obviously, the same arguments hold in the case of a sheaf of sets $\mathbf{A}$ endowed with some algebraic structure, for example rings or $\mathcal{F}$-algebras (where $\mathcal{F}$ is a field).

With respect to the physical interpretation, we remind that we have identified an *element* of $\mathbf{A}$ of sort U, that is a *local section* of $\mathbf{A}$, with a local observable p, which can be observed via a measurement procedure over the reference locus U. Then the equivalence relation, used in the definition of the stalk $\mathbf{A}_x$ at the point $x \in X$ signifies the following:

Two local observables $p \in \mathbf{A}(U)$, $q \in \mathbf{A}(V)$, induce the *same contextual information* at x in X, provided that they have the same restriction to a smaller open locus contained in the basis K. Then, the stalk $\mathbf{A}_x$ is the set containing all contextual information at x, that is, the set of all equivalence classes.

Moreover, the image of a local observable $p \in \mathbf{A}(U)$ at the stalk $\mathbf{A}_x$, that is, the equivalence class of this local observable p, is precisely the *germ* of p at the point x. Next, if we consider a local observable $p \in \mathbf{A}(U)$, it determines a function:

$$\dot{p} : x \mapsto germ_x p$$

whose domain is the open locus U and its codomain is the stalk $\mathbf{A}_x$, for each $x \in U$.

We may consider instead, the disjoint union $A = \coprod_{x \in X} \mathbf{A}_x$ as the codomain of the function $\dot{p}$. From this perspective, every local observable $p \in \mathbf{A}(U)$, gives rise to some *partial function*:

$$\dot{p} : U \to A$$

which, is defined on the open locus $U \subset X$. Hence, all local observables $p \in \mathbf{A}(U)$, admit a *functional representation*, established by means of the following correspondence:

$$\Delta(U) : p \mapsto \dot{p}$$

Stated equivalently, each local observable $p \in \mathbf{A}(U)$ can be legitimately considered a partial function:

$$\dot{p} : U \to A$$

defined over the open reference locus U, the value of which, at a point $x \in U$, that is, $germ_x p$, is the *contextual* observable information induced at x by the local observable p. Furthermore, such a partial function $\dot{p} : U \to A$ is identified with a *cross section* of the topological bundle of germs, defined by the *continuous* mapping $\varphi : A \to X$, such that,

$$\varphi^{-1}(x) = \mathbf{A}_x = Colim_{\{x \in U\}}[\mathbf{A}(U)]$$

Note that the mapping φ is locally a homeomorphism of topological spaces, and thus, the bundle is *étale*.

The previous discussion can be formalized categorically in terms of an *adjunctive correspondence*, defined fundamentally, between the category of *presheaves* of sets $Sets^{\mathcal{O}(X)^{op}}$ on the category of open loci $\mathcal{O}(X)$ of a topological space X, and the category of topological *bundles* $\mathcal{B}(X)$ over X, as follows:

$$\Gamma : \mathcal{B}(X) \rightleftarrows Sets^{\mathcal{O}(X)^{op}} : \Lambda$$

where, in the above adjunction, the functor $\Gamma : \mathcal{B}(X) \to Sets^{\mathcal{O}(X)^{op}}$, called *the cross sections-functor*, assigns to each bundle $\varphi : A \to X$ the sheaf of all cross-sections of A, while its left adjoint functor $\Lambda : Sets^{\mathcal{O}(X)^{op}} \to \mathcal{B}(X)$, called the *germs-functor*, assigns to each presheaf $\mathbf{A}$ the bundle of germs of $\mathbf{A}$. The adjunction is characterized completely by the *unit* and *counit* natural transformations, defined respectively as follows:

$$\eta_{\mathbf{A}} : \mathbf{A} \to \Gamma\Lambda\mathbf{A}$$
$$\acute{U}_A : \Lambda\Gamma A \to A$$

Moreover, if $\mathbf{A}$ is a *sheaf*, then, the unit $\eta_{\mathbf{A}}$ is an isomorphism, while, if A is *étale*, then, the counit $\acute{U}_A$ is an isomorphism. For these reasons, the above adjunction is restricted to a *natural equivalence* between the categories of *sheaves* $\mathbf{Sh}(X)$ on $\mathcal{O}(X)$, and the category of *étale topological bundles* $\mathcal{ET}(X)$ over X, as follows:

$$\Gamma : \mathcal{ET}\left(X\right) \rightleftarrows \mathbf{Sh}\left(X\right) : \Lambda$$

Note that the above adjunction (natural equivalence) is still valid if we consider instead of presheaves (sheaves) of sets, presheaves (sheaves) of *rings*, or $\mathbb{R}$ *-algebras*. Moreover, as a consequence of the categorical equivalence between sheaves on a topological space X and étale topological bundles over X, *every* sheaf can be considered as a sheaf of *cross-sections*.

It is also instructive to notice that the previous arguments can help us to understand the process of *completion* (or sheafification, or germification) of a presheaf. For this purpose, we realize that the notions of *germ*, *stalk* and *étale* bundle make sense for a general presheaf. More precisely, the germ at a point stands for an equivalence class of elements of the presheaf corresponding to open loci around that point, under the equivalence relation which follows from having the same germ. The stalk over this point is the set of all germs at this point. The étale bundle is the disjoint union of all stalks. The first crucial observation is that by the definition of a topology on the étale bundle, as described previously, it is legitimate to consider continuous sections of the *étale* bundle.

Stated equivalently, this procedure amounts to *transforming* the *elements* of the presheaf into partial continuous functions (*continuous sections*) *valued* into the *étale* space. Hence, we manage to *functionalize* the initial presheaf, by defining a *new* presheaf, called the *presheaf of sections* of the initial presheaf as follows: It is the presheaf, which associates to each open locus of the base topological space the set of continuous sections from that open locus into the *étale* space. Now, there is an obvious morphism from the initial presheaf to its presheaf of sections, which maps each element of the category of elements of the initial presheaf to the continuous section, which sends *each point* in an open locus of the base space to the *germ* of this element at that point.

The second crucial observation is that the associated functionalized presheaf of sections of a presheaf is actually also *localized* (locally determined), meaning that it is a sheaf, identified as the *sheaf of cross-sections* of the corresponding *étale* topological bundle. Thus, the latter sheaf is called the sheaf *associated* to the initial presheaf. Moreover, the process of completion of a presheaf into the sheaf of cross-sections of the corresponding *étale* topological bundle is *functorial*, meaning that for each presheaf there is a functor sending it to its completion, that is, to its associated sheaf of sections, called the *sheafification functor*. As a corollary, we conclude that a presheaf is a sheaf, which means a *complete* presheaf, if and only if the morphism to its associated functional presheaf of sections is an *isomorphism*.

Thus, the process of *completion* of a presheaf is equivalent to the combined processes of *functionalization* and *localization* of its elements. Consequently, the associated sheaf of sections of a corresponding presheaf, contains by its construction, the totality of local contextual information compatible with the one available from the initial presheaf due to its restriction property, and in this sense, it constitutes its completion.

Now, let us consider a sheaf of $\mathcal{R}$-algebras of local observables, identified as a sheaf of real-valued continuous *cross-sections* of the corresponding *étale* bundle. Then, the set of germs of all these sections at a point, the stalk at this point, is also an $\mathcal{R}$-algebra. Most importantly, the *stalk* at this point is a *local* $\mathcal{R}$-algebra, meaning that it has a *unique maximal ideal*. In turn, this maximal ideal consists of all germs vanishing at the point in question. The *quotient* of the stalk by this maximal ideal is *isomorphic* to the field of *real* numbers. Equivalently, this means that the morphism evaluating a germ of the stalk at a point to the real numbers, which provides a real value at the corresponding non-vanishing equivalence class of sections at the base point of interest, is a *surjective* morphism of $\mathcal{R}$-algebras taking as *kernel* the *maximal ideal* of the stalk at this point:

$$\mathbf{ev}_x : \mathbf{A}_x \to \mathbf{A}_x / \mu_x \cong \mathcal{R}$$
$$germ_x p \mapsto \mathbf{ev}_x(germ_x p) = p(x)$$

Thus the evaluation morphism of a germ of the stalk at a point of the base space is an $\mathcal{R}$-valued measurement of this observable germ, interpreted as an *observed event* of the corresponding natural system, and subsequently encoded by means of an $\mathcal{R}$*-state* of its topological state-space.

At a next stage of development of these ideas, the sheaf of germs of real-valued continuous functions on a topological space X is an object in the functor category of sheaves $\mathbf{Sh}(X)$ on varying reference loci U, being open sets of X, partially ordered by inclusion. The morphisms in $\mathbf{Sh}(X)$ are all *natural transformations* between sheaves. It is instructive to notice that a sheaf makes sense only if the base category of reference loci is specified, which is equivalent in our context to the determination of a topology on the space X. The *functor category of sheaves* $\mathbf{Sh}(X)$, provides an exemplary case of a construct known as *topos*.

A *topos* can be conceived as a *local* mathematical framework corresponding to a *generalized model* of set theory, or as a *generalized algebraic space*, corresponding to a categorical universe of variable observable information sets over the multiplicity of the reference loci of the base category. We recall that, formally a topos is a category, which has a terminal object, pullbacks, exponentials, and a subobject classifier, which in turn is understood as an object of *generalized truth values*. The particular significance of the sheaf of real-valued continuous functions on X, is due to the following *isomorphism*: The *sheaf* of germs of continuous real-valued functions on X, is isomorphic to the object of *Dedekind real* numbers in the topos of sheaves $\mathbf{Sh}(X)$. The aforementioned isomorphism validates the physical intuition which reads a local observable as a continuously variable real number over its locus of definition.

10.3 TOPOS-THEORETIC RELATIVIZATION OF REPRESENTABILITY

The transition in the semantics of the *physical continuum* from the topos of **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ is an instance of the principle of *topos-theoretic relativization* of physical *representability* referring to the interpretation of observed events. We initially notice that, in the former case, observables are identified with (continuous) functions determined *completely* by their values at *points*. In the latter case, observables are identified with local continuous sections of the *étale* space determined *completely* by their *germs*.

In order to analyze in more detail the *transition* in the *semantics*, we note that, in the former case, a continuous function from a base topological space X to the topological space $\mathcal{R}$ can be considered as a *continuous section* from X to the product space $X \times \mathcal{R}$. This product space is set-theoretically isomorphic to a space containing a *copy* of the coordinatizing frame-field $\mathcal{R}$ at each *point*, being the *inverse image* of the *projection* from $X \times \mathcal{R}$ to the base X. The value that is taken at a point is the value taken by the function. Thus, this type of modeling the notion of an *observable* is only appropriate in capturing its *point-properties*.

In contrast, in the latter case we obtain *local properties* of observables. This is due to the fact that in the sheaf-theoretic local environment, we associate *not* with the *value* that a section takes at a point of the base space, but its *germ*. In this sense, *instead* of the product total space $X \times \mathcal{R}$, we have the *étale* topological space, such that the *inverse image* of each point of the base space is *not* a copy of the coordinatizing frame-field $\mathcal{R}$, but the *stalk* at that point.

This essentially means that the transition of semantics from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ amounts to *shifting* the focus *from point-wise* behavior of observables *to local* behavior of observables. Obviously the *étale* topological space is a much *richer* and *bigger* space than the rigid space $X \times \mathcal{R}$, since the étale space provides information about the *local* behavior of observables around each point of the base space in terms of *germs*, instead of merely point-wise behavior of observables in terms of their values in the real numbers.

Thus, *observed events* are *not* determined by the *values* of continuous functions at points of the base space, but by the *evaluation* morphisms of *germs* at those points, according to our previous remarks. We conclude that the meaning of the principle of topos-theoretic relativization of physical representability as effectuated by the transition from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ amounts to a *relativization* with respect to the *local* behavior of physical observables as *opposed to* their *point* behavior.

It is worth explaining in some further detail the important notion of relativization of physical representability by shifting the semantics of observables from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$. The *absolute representability* principle is based on the *set-theoretic* conception of the *real line*, as a set of infinitely distinguished points coordinatized by means of the field of real numbers. Expressed categorically, this is equivalent to the interpretation of the algebraic structure of the reals inside the *absolute* universe of **Sets**, or more precisely inside the *topos of constant* **Sets**.

It is also well known that algebraic structures and mechanisms can admit a *variable* reference, formulated in category-theoretic jargon in terms of arrows-only specifications, *inside any* suitable *topos* of discourse. The *relativization* of physical representability with respect to the topos of sheaves $\mathbf{Shv}(X)$, amounts to the relativization of both the *notion* and the algebraic *structure* of the *real* numbers *inside* this *topos*. Regarding the notion of real numbers inside the topos $\mathbf{Shv}(X)$, this is equivalent to the notion of *continuously variable* real numbers over the open reference domains of X, or else, equivalent to the notion of real-valued continuous functions on X, when interpreted respectively inside the topos of **Sets**.

Equivalently stated, the internal object of *Dedekind reals* constructed within the *logic* of the topos $\mathbf{Shv}(X)$ is isomorphic to the *sheaf of germs* of continuous real-valued functions on the space X. Regarding the algebraic structure of the reals inside the topos $\mathbf{Shv}(X)$, they form *only* an algebra in this topos, which is identified with the *sheaf of commutative* $\mathbb{R}$-algebras of *germs* of continuous real-valued functions on X, where $\mathbb{R}$ corresponds in that case to the *locally constant* sheaf of germs of real numbers over X.

From a physical perspective, internally in the topos $\mathbf{Shv}(X)$ the *valuation* algebra of real numbers is *relativized* with respect to the base category of *open* sets of a topological space X. As a consequence it admits a description as a sheaf of of germs of continuous real-valued functions on X. In particular, for each open reference context U of X, we obtain a unital commutative algebra of continuous real-valued local sections. In this way, the semantics of localization of observables is transformed from a set-theoretic to a sheaf-theoretic one. More concretely, it is obvious that inside the topos $\mathbf{Sets}$ the *unique* localization measure of observables is a *point* of the $\mathbb{R}$-spectrum of the corresponding algebra of scalars, which is assigned a numerical identity.

By contrast, inside the topos $\mathbf{Shv}(X)$, the former is *substituted* by a variety of *localization measures*, dependent *only* upon the *open* sets in the topology of X. In the latter context, a point-localization measure, is identified precisely with the *ultrafilter* of all opens containing the point. This identification permits the conception of other *filters* owing their formation to admissible operations between opens as generalized measures of localization of observables. In a wider context, the relativization of representability effected in $\mathbf{Shv}(X)$ is physically significant, because the operational specification of measurement environments exists *only locally* and the underlying assumption is that the information gathered about local observables in different measurement situations can be collated together by appropriate means; a process that is precisely formalized by the notion of sheaf.

Conclusively, we assert that localization schemes referring to observables *may not depend* exclusively on the existence of *points*, and thus should *not* be tautosemous with the practice of conferring a numerical identity to them. Therefore, the relativization of representability with respect to the internal reals of the topos of sheaves $\mathbf{Shv}(X)$, amounts to the *substitution* of point-localization measures, represented numerically, with localization measures *fibering* over the base category of open reference loci, represented respectively by *local sections* in the sheaf of internal reals.

The *transition* in the semantics of physical *representability* under *relativization* from the topos $\mathbf{Sets}$ to the topos $\mathbf{Shv}(Y)$ can be formalized via the concept of an *admissible transformation* between *topoi*, called a *functorial geometric transformation*, or simply a *geometric morphism*. More concretely, a functorial geometric transformation from the topos $\mathbf{Sets}$ to the topos $\mathbf{Shv}(Y)$ is defined as a *pair* of *adjoint functors*:

$$\mathbf{F}^{\Leftarrow}: \mathbf{Sets} \to \mathbf{Shv}(Y)$$
$$\mathbf{F}_{\Leftarrow}: \mathbf{Shv}(Y) \to \mathbf{Sets}$$

where the functor $\mathbf{F}_{\Leftarrow}$ is *right adjoint* to the functor $\mathbf{F}^{\Leftarrow}$, which in turn is *left exact*. Then, the functor $\mathbf{F}_{\Leftarrow}$ is called the *direct image* part of the functorial geometric transformation, while the functor $\mathbf{F}^{\Leftarrow}$ is called the *inverse image* part.

The terminology arises from the simple realization that a *continuous* morphism between *topological spaces* X and Y, denoted by $h: X \to Y$, induces a *functorial geometric transformation* between the *categories of their sheaves* as follows:

$$\mathbf{H}^{\Leftarrow}: \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$$
$$\mathbf{H}_{\Leftarrow}: \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

such that:

$$\mathbf{Shv}(X) \rightleftarrows \mathbf{Shv}(Y)$$

form an *adjoint pair* of functors, where, if $\mathbf{A}$ is a *sheaf* on X, and U is an open locus in Y, then:

$$\mathbf{H}_{\Leftarrow}(\mathbf{A})(U) = \mathbf{A}((h^{-1})(U))$$

called the *direct image* of the sheaf $\mathbf{A}$ under the morphism h. On the opposite side, the *inverse image* of a sheaf $\mathbf{B}$ under the morphism h, denoted by $\mathbf{H}^{\Leftarrow}(\mathbf{B})$ is defined as the sheaf on X such that the stalk at any point $x \in X$ is the stalk at $h(x)$.

Most importantly, *any* functorial geometric transformation between the topos of sheaves on X and the topos of sheaves on Y *necessarily* arises from a *unique continuous* function between these spaces. The above is particularly useful, and can be demonstrated by a simple example as follows: the topos $\mathbf{Sets}$ can be considered as the topos of sheaves over the 1-point topological space, that is, $\mathbf{Shv}(1)$. Thus, a *point* y of a topological space, i.e. a continuous morphism $y: 1 \to Y$, gives rise to a *geometric morphism* described by:

$$\mathbf{y}^{\leftleftarrows}: \mathbf{Shv}(Y) \to \mathbf{Shv}(1)$$
$$\mathbf{y}_{\leftleftarrows}: \mathbf{Shv}(1) \to \mathbf{Shv}(Y)$$

such that:

$$\mathbf{Shv}(1) \rightleftarrows \mathbf{Shv}(Y)$$

form an *adjoint pair* of functors, where if $\mathbf{A}$ is a *sheaf* over the one-point topological space 1, which is just a set, and U is an open locus in Y, then:

$$\mathbf{y}_{\leftleftarrows}(\mathbf{A})(U) = \mathbf{A}((y^{-1})(U))$$

that is, the *direct image* of the set $\mathbf{A}$ under y, gives the value $\mathbf{A}$ if $y \in U$ and the value 1 otherwise. The sheaf $\mathbf{y}_{\leftleftarrows}(\mathbf{A})$ on Y is the *skyscraper sheaf* at the point y, which is a *totally discontinuous* sheaf on Y. Thus, if $\mathbf{A}$ is the sheaf of internal reals on 1, that is, the set of real numbers, then the skyscraper sheaf $\mathbf{y}_{\leftleftarrows}(\mathbf{A})$ on Y consists of a copy of the real numbers at the point y and is 1 at all other points. On the opposite side, the inverse image of a sheaf $\mathbf{B}$ under the morphism y, denoted by $\mathbf{y}^{\leftleftarrows}(\mathbf{B})$, is precisely the stalk of $\mathbf{B}$ at the point y. Thus, if $\mathbf{B}$ is the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$, then its inverse image under y is the stalk of the internal real numbers at y, that is, the set (local ring) of germs of continuous real-valued sections at y. This means in turn that the notion of a continuously variable real number over Y, which is a real number from the perspective of $\mathbf{Shv}(Y)$, is transformed via the inverse image functor $\mathbf{y}^{\leftleftarrows}$ corresponding to $y: 1 \to Y$, into the notion of a germ of continuous real-valued sections at y from the perspective of $\mathbf{Shv}(1)$.

This simple example illustrates the *first* fundamental aspect of the principle of relativization of physical representability with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space. More precisely, since the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$ is perceived via $\mathbf{y}^{\leftleftarrows}$ by the set (local ring) of germs of continuous real-valued sections at y in Y, this means that the transition in the semantics of observable representability at a point of a base topological space, as reflected within the topos of sets, amounts to the *substitution* of a *point-localization* measure of an observable, that is its real value at that point, *encoding point-wise information*, by its *germ* at the same point, *encoding local information.*

The *second* fundamental aspect of the principle of relativization of physical representability with respect to the internal real numbers of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space, is implemented by means of the following *functorial geometric transformation*:

$$\mathbf{Shv}(Y) \rightleftarrows \mathbf{Sets}$$
$$\mathbf{F}^{\leftrightarrows}: \mathbf{Sets} \to \mathbf{Shv}(Y)$$
$$\mathbf{F}_{\leftrightarrows}: \mathbf{Shv}(Y) \to \mathbf{Sets}$$

where,

$$\mathbf{F}_{\leftrightarrows} := \Gamma : \mathbf{Shv}(Y) \to \mathbf{Sets}$$

is the *global sections functor*, which assigns to a sheaf $\mathbf{A}$ in $\mathbf{Shv}(Y)$ its set of global sections (*global elements*) $\Gamma(\mathbf{A}) = \mathbf{Nat}(\mathbf{1}, \mathbf{A})$, where $\mathbf{1}$ is the *terminal* object in $\mathbf{Shv}(Y)$.

In particular, if $\mathbf{A}$ is the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$, then its set of global sections is the set of real-valued continuous functions on Y. This means that the notion of a continuously variable real number over Y, that is, a real number from the perspective of $\mathbf{Shv}(Y)$, is transformed by the inverse image functor $\mathbf{F}_{\leftrightarrows} := \Gamma$ to the notion of a real-valued continuous function on Y from the perspective of $\mathbf{Sets}$.

Consequently, the semantics of observable representability *globally*, as reflected within the topos of sets, remains *invariant* under the *relativization* with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$. Hence, the relativization of physical representability as above, forces the encoding of local observable information in terms of germs, and thus transforms the semantics of observables from the *point* level to the *local* level, while it leaves *invariant* their *global* interpretation.

In the opposite direction, the functor:

$$\mathbf{F}^{\leftrightarrows} := \Delta : \mathbf{Sets} \to \mathbf{Shv}(Y)$$

assigns to each set S the corresponding *constant* presheaf $\Delta(S) := \mathbf{\Delta S}$. This constant presheaf sends each open set U of Y to the same set S. In particular, if S is the set or real numbers $\mathbb{R}$, then the constant presheaf sends each open set U of Y to the set of real numbers $\mathbb{R}$. Thus, the corresponding *étale* space of the constant presheaf $\Delta\mathbb{R}$ is the *projection* $Y \times \mathbb{R} \to Y$. Therefore, for each open locus U of Y, $\Delta\mathbb{R}(U)$ is the set of continuous functions from U to the *discrete* space $\mathbb{R}$. This is exactly the set of *locally constant* functions from U to $\mathbb{R}$. In this sense, the sheaf $\Delta\mathbb{R}$ is called the *constant sheaf* corresponding to the set of real numbers.

A particularly interesting application of the above arises even in the case that the base topological space Y is *discrete*, considered as an *infinite* set. We may consider the *counit* natural transformation of the corresponding pair of adjoint functors:

$$\acute{U} : \mathbf{F}_{\leftrightarrows}\mathbf{F}^{\leftrightarrows} \to \mathbf{Id}_{\mathbf{Sets}}$$

$$\acute{U}_{S} : \mathbf{F}_{\leftrightarrows}\mathbf{F}^{\leftrightarrows} S \to S$$

If S is the set or real numbers $\mathbb{R}$, then we obtain:

$$\acute{U}_{\mathbb{R}} : \mathbf{F}_{\leftrightarrows}\mathbf{F}^{\leftrightarrows}\mathbb{R} \to \mathbb{R}$$

Note that the domain of the counit, is the set of global sections of the constant sheaf $\Delta\mathbb{R}$ in **Sets** as previously. This set is identified as the set of sections of the projection morphism from the cartesian product $Y \times \mathbb{R}$ (viewed as a set) to Y. It is denoted by $\mathbb{R}^Y$, while its elements $\chi \in \mathbb{R}^Y$ are mappings $\chi : Y \to \mathbb{R}$. Therefore, we obtain:

$$\acute{U}_{\mathbb{R}} : \mathbb{R}^Y \to \mathbb{R}$$

This is precisely the *evaluation* morphism of the set $\mathbb{R}^Y$ to the set of real numbers, that is, the morphism evaluating the set of global sections $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ at a point y of the base space Y to the set of real numbers. The result of evaluating the set $\mathbb{R}^Y$ at a point y of Y is equivalent to the process of *identification* of functions $\chi : Y \to \mathbb{R}$ in $\mathbb{R}^Y$ under the condition that their values at y are the *same*; in short, we may define an equivalence relation on the set $\mathbb{R}^Y$ as follows:

$$\chi \sim_y \psi$$

if and only if:

$$\chi(y) = \psi(y)$$

This means in turn that the set,

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\}$$

belongs to the *principal ultrafilter* of y, that is, to the set:

$$V_y = \{U \subseteq Y : y \in U\}$$

Now, if we *identify* the point y with its principal ultrafilter, we may reformulate the equivalence relation as follows:

$$\chi \sim_{V_y} \psi$$

if and only if:

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\} \in V_y$$

Consequently, the result of evaluating all the elements of the set $\mathbb{R}^Y$ at the point y, or equivalently, at the principal ultrafilter V_y, is the set of equivalence classes of $\mathbb{R}^Y$ modulo the equivalence relation $\sim_{V_y}$. This set is obviously isomorphic to the set of real numbers, that is:

$$\mathbb{R}^Y / \sim_{V_y} \simeq \mathbb{R}$$

and the evaluation morphism is actually the following:

$$Ú_{\mathbb{R}} : \mathbb{R}^Y \to \mathbb{R}^Y / \sim_{V_y} \simeq \mathbb{R}$$

The above procedure also give us the possibility of evaluating the set of global sections $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ at an *arbitrary ultrafilter* V of the base space Y, thought of as a *virtual point* of Y. More specifically, an ultrafilter V is a point of the *compactification* of Y. In this way, we may define the following equivalence relation on the set $\mathbb{R}^Y$:

$$\chi \sim_V \psi$$

if and only if:

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\} \in V$$

Similarly, the result of evaluating all the elements of the set $\mathbb{R}^Y$ at the *ultrafilter* (virtual point) V, is the set of equivalence classes of $\mathbb{R}^Y$ modulo the equivalence relation $\sim_V$. This set is *not* isomorphic to the set of real numbers, and is called the *ultrapower* of $\mathbb{R}^Y$ with respect to the *ultrafilter* V. Moreover, the set of real numbers can be naturally *embedded* in the ultrapower of $\mathbb{R}^Y$ with respect to the ultrafilter V. In this sense, the ultrapower of $\mathbb{R}^Y$ with respect to V, denoted by Υ_V, contains the *real* numbers and *additionally* contains *new* generalized *elements*.

Consequently we can imagine the elements of the ultrapower Υ_V as real numbers surrounded by a *cloud* of objective thickness. Of course, this procedure provides the possibility of a generalized interpretation of measurement states of the corresponding ring of global observables $\mathbb{R}^Y$, by means of the surjective morphism of rings:

$$\mathbb{R}^Y \to \Upsilon_V$$

Which is to say the legitimate consideration of Υ_V-states of $\mathbb{R}^Y$. Essentially, this means that the ring Υ_V can act as a *ring* of measurement *scales* for the evaluation of the observables in $\mathbb{R}^Y$. This is another indication of the fact that *absolute* representability with respect to $\mathbb{R}$-measurement scales should be *abandoned*, and instead a *covariance* principle referring to *all legitimate* rings of measurement scales should be *substituted* in its *place* for the evaluation of observables.

Based on the above conclusion, we may extend these ideas by taking into account the following: Firstly, the set of *global sections* $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ is actually a *commutative* $\mathbb{R}$-*algebra* as can easily be verified. Secondly, an injective correspondence exists between the *proper ideals* of the $\mathbb{R}$-algebra $\mathbb{R}^Y$ and the *filters* of the discrete space Y

(considered as an infinite set). More concretely, if we recall that the elements $\chi \in \mathbb{R}^Y$ are mappings $\chi : Y \to \mathbb{R}$, then we can define the *zero set* of an element χ as follows:

$$Z(\chi) = \{y \in Y \mid \chi(y) = 0\}$$

Furthermore, if we denote by $\mathcal{J}$ an *ideal* in the $\mathbb{R}$-algebra $\mathbb{R}^Y$, and by F a *filter* on the infinite set Y we obtain the *injective* correspondences:

$$\mathcal{J} \mapsto F_{\mathcal{J}} = \{Z(\chi) \mid \chi \in \mathcal{J}\}$$
$$F \mapsto \mathcal{J}_F = \{\chi \in \mathbb{R}^Y \mid Z(\chi) \in F\}$$

The above correspondences are *order-preserving* and *idempotent* under iteration. It follows that every *reduced power* algebra $\mathbb{R}^Y / \mathcal{J}$, where $\mathcal{J}$ is an ideal in the $\mathbb{R}$-algebra $\mathbb{R}^Y$ is of the quotient form:

$$\mathcal{A}_F = \mathbb{R}^Y / \mathcal{J}_F$$

for a unique *generating filter* F on the index set Y.
Next, we note that reduced power algebras of the above form can be related to each other as follows: For two filters F, G on Y, such that $F \subseteq G$, we obtain the surjective morphism of $\mathbb{R}$-algebras

$$\mathcal{A}_F \twoheadrightarrow \mathcal{A}_G$$
$$\chi + \mathcal{J}_F \mapsto \chi + \mathcal{J}_G$$

from which we conclude that the algebras $\mathcal{A}_G$ and $\mathcal{A}_F / (\mathcal{J}_G / \mathcal{J}_F)$ are *isomorphic*. A degenerate case refers to the power algebras obtained when a filter F on Y is generated by a non-empty subset Σ of Y, that is, in case that ${}_{\Sigma}F = \{K \subseteq Y \mid K \supseteq \Sigma\}$. Then, we obtain the power algebras of the form:

$$\mathcal{A}_{{}_{\Sigma}F} = \mathbb{R}^{\Sigma}$$

Further, if Σ is a *finite* subset of Y having $n \geq 1$ elements we obtain:

$$\mathcal{A}_{\Sigma F} = \mathbb{R}^n$$

Consequently, the n-dimensional *Euclidean spaces* are *power algebras* of $\mathbb{R}^Y$ of the form $\mathbb{R}^\Sigma$, where Σ is a finite subset of Y having n-elements. The exclusion of all the degenerate cases, leading to the formation of power algebras, amounts to *restricting* the generating filters to those that are supersets of the *Maurice Frechet filter* on Y. The *Frechet* filter on Y, denoted by Fr is a *cofinite* filter on Y, where a cofinite filter on Y consists of all subsets of Y having finite complement in Y. Thus, non-degenerate reduced power algebras of $\mathbb{R}^Y$ are of the form:

$$\mathcal{A}_F = \mathbb{R}^Y / \mathcal{J}_F$$

for a *unique* generating filter F on the index set Y, such that $F \supseteq Fr$. Moreover, because of the relation $Fr \subseteq F$, every non-degenerate reduced power algebra $\mathcal{A}_F$ of $\mathbb{R}^Y$ is the *surjective image* of the reduced power algebra $\mathcal{A}_{Fr}$ corresponding to the *Frechet* filter on Y, and moreover, it is *isomorphic* to $\mathcal{A}_{Fr} / (\mathcal{J}_F / \mathcal{J}_{Fr})$, that is:

$$\mathcal{A}_F \simeq \mathcal{A}_{Fr} / (\mathcal{J}_F / \mathcal{J}_{Fr})$$

The two most important properties of all the non-degenerate reduced power algebras of $\mathbb{R}^Y$ are that they have *zero divisors*, unless the dividing ideal is prime, and that they are *non-Archimedean*. Still, they can legitimately act as rings of measurement scales for the evaluation of the observables in $\mathbb{R}^Y$. Thus, once again the absolute representability with respect to $\mathbb{R}$-measurement scales should be abandoned in favour of a covariance principle with respect to all legitimate rings of measurement scales for the evaluation of observables.

10.4 DIFFERENTIAL RINGED SPACES OF STATES

The transition in the semantics of the physical continuum from the topos of $\mathbf{Sets}$ to the topos of sheaves $\mathbf{Sh}(X)$ as an instance of the principle of topos-theoretic relativization of physical representability, entails a transition in the *semantics of observables* from (continuous) functions determined completely by their values at points, to local continuous sections of the *étale* space of a sheaf determined completely by their germs. Thus, in the latter case of localization within the environment of the topos of sheaves $\mathbf{Sh}(X)$ we are naturally led to the introduction of the notion of a *commutative ringed space* of states. We note that the notion of ringed spaces is used extensively in Algebraic Geometry, in the theory of Abstract Differential Geometry, and in the theory of $\mathcal{C}^{\infty}$-differentiable spaces.

A *commutative ringed space* of states is a pair $(X,\mathbf{R})$ consisting of a topological space X and a *sheaf of commutative rings* of observables $\mathbf{R}$ on X. The space X is called the underlying *state-space* of the ringed space, while the sheaf $\mathbf{R}$ is called the *structure sheaf* of observables. Now, for any open locus $U \subset X$, the pair $(U,\mathbf{R}|_U)$ is also a ringed space, called an open subspace of states of $(X,\mathbf{R})$. A morphism of ringed spaces $H=(h,\phi)$ from $(X,\mathbf{R})$ to $(Y,\mathbf{Q})$ consists of a continuous morphism of topological spaces $h: X \to Y$ and also a morphism of sheaves of rings $\phi : \mathbf{H}^{\leftarrow}(\mathbf{Q}) \to \mathbf{R}$.

We recall that if $\mathbb{R}$ is the field of real numbers, then an $\mathbb{R}$-algebra of observables $\mathcal{A}$ is a ring $\mathcal{A}$ together with a morphism of rings $\mathbb{R} \to \mathcal{A}$ (making $\mathcal{A}$ into a vector space over $\mathbb{R}$) such that, the morphism $\mathcal{A} \to \mathbb{R}$ is a linear morphism of vector spaces. Notice that the same holds if we substitute the field $\mathbb{R}$ with any other field, for instance, the field of complex numbers $\mathbb{C}$.

Next, we introduce the notion of a commutative (locally) $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states as a pair $(X,\mathbf{A})$ consisting of a topological space X and a sheaf of commutative $\mathbb{R}$-algebras of observables $\mathbf{A}$ on X, *such that* the *stalk* $\mathbf{A}_x$ of *germs* is a *(local) commutative* $\mathbb{R}$-algebra for *any point* $x \in X$. A morphism of $\mathbb{R}$-ringed spaces $H=(h,\phi)$ from $(X,\mathbf{A})$ to $(Y,\mathbf{B})$ consists of a continuous morphism of topological spaces $h: X \to Y$ and also a morphism of sheaves of $\mathbb{R}$-algebras $\phi : \mathbf{H}^{\leftarrow}(\mathbf{B}) \to \mathbf{A}$, such that for every $x \in X$ the induced morphism of stalks at x, that is, $\phi_x : \mathbf{B}_{h(x)} \to \mathbf{A}_x$ is a morphism of $\mathbb{R}$-algebras. We denote the category of commutative (locally) $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) spaces of states by $\mathcal{A}_{\mathbb{R}}$.

At this stage, we are able to introduce the notion of a *category of models* as a subcategory of the category $\mathcal{A}_{\mathbb{R}}$, denoted by $\mathcal{M}_{\mathbb{R}}$, which satisfies the following conditions:

i The base locus $\mathbb{U}$ of an object $(\mathbb{U}, \mathbf{A})$ of $\mathcal{M}_{\mathbb{R}}$ is some *model* topological space;

ii If $(\mathbb{U}, \mathbf{A})$ is an object of $\mathcal{M}_{\mathbb{R}}$, and $\mathbb{V} \subseteq \mathbb{U}$ an open locus, then $(\mathbb{V}, \mathbf{A} \,|\, \mathbb{V})$ is *also* an object of $\mathcal{M}_{\mathbb{R}}$, and the *injection* $(\mathbb{V}, \mathbf{A} \,|\, \mathbb{V}) \infty\ (\mathbb{U}, \mathbf{A})$ is a morphism in $\mathcal{M}_{\mathbb{R}}$.

We say that given a category of models $\mathcal{M}_{\mathbb{R}}$, an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{B})$ is an $\mathcal{M}_{\mathbb{R}}$-*manifold* if the following conditions are satisfied:

i Every point $y \in Y$ has an open locus U together with an *isomorphism* of $\mathbb{R}$-ringed spaces, that is:

$$H = (h, \phi) : (U, \mathbf{B} \,|\, U) \to (\mathbb{U}, \mathbf{A}) \in \mathcal{M}_{\mathbb{R}}$$

We call the above isomorphism an $\mathcal{M}_{\mathbb{R}}$-*coordinate chart*, or *reference frame* of $(Y, \mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$;

ii For any pair of $\mathcal{M}_{\mathbb{R}}$-coordinate charts:

$$H_a = (h_a, \phi_a) : (U_a, \mathbf{B} \,|\, U_a) \to (\mathbb{U}_a, \mathbf{A}_a)$$
$$H_b = (h_b, \phi_b) : (U_b, \mathbf{B} \,|\, U_b) \to (\mathbb{U}_b, \mathbf{A}_b)$$

with $U_a \cap U_b \neq \varnothing$, the induced isomorphism:

$$H_{ab} := H_a \circ H_b^{-1} : (\mathbb{U}_{ba}, \mathbf{A}_b \,|\, \mathbb{U}_{ba}) \to (\mathbb{U}_{ab}, \mathbf{A}_a \,|\, \mathbb{U}_{ab})$$

is a *morphism* in the category of models $\mathcal{M}_{\mathbb{R}}$, where:

$$\mathbb{U}_{ba} := h_b(U_a \cap U_b)$$
$$\mathbb{U}_{ab} := h_a(U_a \cap U_b)$$

The isomorphism H_{ab} in the category of models $\mathcal{M}_{\mathbb{R}}$ is called a *gluing datum* between overlapping $\mathcal{M}_{\mathbb{R}}$-coordinate charts of $(Y,\mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$;

iii The following *cocycle relations* are satisfied whenever they are defined:

1 $H_{aa} = 1$

2 $H_{ab} = H_{ba}^{-1}$

3 $H_{ab} \circ H_{bc} = H_{ac}$

From the above, we conclude the following: Given a *gluing datum* between overlapping $\mathcal{M}_{\mathbb{R}}$-coordinate charts of $(Y,\mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$, we consider the disjoint union:

$$\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$$

with its natural structure as an $\mathcal{M}_{\mathbb{R}}$ *-manifold*, where I is a corresponding indexing set. Then, we introduce on $\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$ a relation defined as follows:

$$(\mathbb{U}_a,\mathbf{A}_a) \not\ni (y_a,s_a) \sim (y_b,s_b) \in (\mathbb{U}_b,\mathbf{A}_b)$$

if and only if:

i $(y_a,s_a) \in \mathbb{U}_{ab} \subset \mathbb{U}_a$

ii $(y_b,s_b) \in \mathbb{U}_{ba} \subset \mathbb{U}_b$

iii $(y_a,s_a) = H_{ab}((y_b,s_b))$

Then, according to the *cocycle relations* of the given gluing datum we obtain that: Because of (1) the relation is *reflexive*, because of (2) the relation is *symmetric*, and because of (3) the relation is *transitive*. Hence the relation $\sim$ defined on $\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$ is an *equivalence relation*, giving rise to a groupoid. The *quotient* space

$$\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)/\sim$$

with respect to this equivalence relation has the induced structure of an $\mathcal{M}_{\mathbb{R}}$-*manifold*, such that the natural projection

$$p : \coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a) \to \coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a)/\sim$$

is a morphism of $\mathcal{M}_{\mathbb{R}}$-manifolds. Moreover, the set $\{p(\mathbb{U}_a, \mathbf{A}_a)\}_{a \in I}$ is defined as an $\mathcal{M}_{\mathbb{R}}$-*coordinate atlas* on $\coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a)/\sim$.

An immediately obvious application of the notion of an $\mathcal{M}_{\mathbb{R}}$-manifold becomes clear if we consider as a category of models $\mathcal{M}_{\mathbb{R}}$ the smooth category of models, denoted by $\mathcal{M}_{\mathbb{R}}^{sm}$. The category $\mathcal{M}_{\mathbb{R}}^{sm}$ has as objects pairs of the form $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$, where $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ denotes the sheaf of real-valued smooth functions of class $\mathcal{C}^{\infty}$ on $\mathbb{R}^n$. We notice that in this case morphisms $(\mathbb{R}^m, \mathcal{C}^{\infty}{}_{\mathbb{R}^m}) \to (\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$ are just smooth maps $\mathbb{R}^m \to \mathbb{R}^n$.

Then, given the category of smooth models $\mathcal{M}_{\mathbb{R}}^{sm}$, an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of observables $(Y, \mathbf{B})$ is called a *smooth* $\mathbb{R}$-*manifold* if it satisfies the conditions [I], [II], [III] given previously. We also notice that the structure of a smooth $\mathbb{R}$-manifold is obtained by the *equivalence relation* on *the disjoint union* of its *coordinate charts* induced by the corresponding *gluing datum* with respect to the category of smooth models. We denote a smooth $\mathbb{R}$-manifold by the pair $(Y, \mathbf{O}_Y)$, where, for every open subset U of Y, $(\mathbf{O}_Y)(U) = \mathcal{C}^{\infty}{}_Y(U)$ is the ring of smooth functions on U.

A morphism between smooth $\mathbb{R}$-manifolds is called a *diffeomorphism* when it is an *isomorphism* of the corresponding $\mathbb{R}$-*ringed spaces*. In this sense, some smooth functions $u_1, \ldots u_n \in \mathcal{C}^{\infty}{}_Y(U)$ define a smooth coordinate system on U if the corresponding morphism $(u_1, \ldots, u_n) : U \to \mathbb{R}^n$ induces a diffeomorphism of U onto an open locus of the model smooth topological $\mathbb{R}^n$ for some appropriate $n \in \mathbb{N}$.

An interesting observation has to do with the fact that the definition of a smooth $\mathbb{R}$-manifold as an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{O}_Y)$ obtained by means of a gluing datum with respect to the category of smooth models $\mathcal{M}_{\mathbb{R}}^{sm}$, takes into account the principle of *relativization* of physical *representability* with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space. We emphasize the significance of this principle concerning its dual aspect referring to both the local and the global behavior of observables.

From the other side, we know that a smooth $\mathbb{R}$-manifold, is completely determined by the ring ($\mathbb{R}$-algebra) of all global smooth functions (assuming that it is a Hausdorff topological space with a countable basis). This means that the semantics of a smooth $\mathbb{R}$-manifold is completely determined by the information encoded in the global observables only, or equivalently, its semantics is completely understood with respect to the constant topos of sets **Sets**.

We consider this fact as a serious *drawback* which affects the interpretation of our current physical theories to a very significant degree, on which we shall expand later. Most importantly, the *localization* properties of observables expressed in terms of *observable germs* are actually overlooked. In this sense, it is necessary to understand the procedure by means of which the $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{O}_Y)$ can be *reconstituted* from the ring ($\mathbb{R}$-algebra) ${\mathcal{C}^\infty}_Y$ of global real-valued smooth functions on Y.

Firstly, the *set* Y is recovered as the $\mathbb{R}$-*spectrum* of *global* observables (global real-valued smooth functions) ${\mathcal{C}^\infty}_Y$, that is the set of all surjective morphisms of $\mathbb{R}$-algebras:

$$\gamma : {\mathcal{C}^\infty}_Y \to \mathbb{R}$$

Thus, we have that, set-theoretically:

$$Y = \mathbb{R}\mathbf{S}pec({\mathcal{C}^\infty}_Y) := \mathbf{Hom}_{\mathbb{R}-alg}({\mathcal{C}^\infty}_Y, \mathbb{R})$$

We recall that the $\mathbb{R}$-algebra (field) $\mathbb{R}$ is called the *coordinatizing frame* of each state of the formed state-space $\mathbb{R}\mathbf{S}pec({\mathcal{C}^\infty}_Y)$. The geometric semantics of this connotation denotes the set of elements which can be $\mathbb{R}$-observed by a measurement procedure on the ring of observables ${\mathcal{C}^\infty}_Y$. Eventually, that set of elements, constituting the $\mathbb{R}$-spectrum of ${\mathcal{C}^\infty}_Y$, are identified with the $\mathbb{R}$-coordinatized *points* of a geometric state-space that can be observed by means of the ring of global real-valued functions ${\mathcal{C}^\infty}_Y$. The next task is to endow the set (state-space) $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ with an appropriate topology, so that it bears the structure of a topological space. We consider the *Israel Gelfand topology* on $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$, which is defined by the requirement that it is the smallest topology such that:

$$\hat{f}:\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})\rightarrow\mathbb{R}$$
$$\hat{f}(\gamma):=\gamma(f)\in\mathbb{R}$$

is *continuous* for any $f\in\mathcal{C}^{\infty}{}_{Y}$. Thus, we get a morphism of $\mathbb{R}$ -algebras:

$$\mathcal{C}^{\infty}{}_{Y}\rightarrow\mathbf{Hom}_{cstates}(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}),\mathbb{R})$$
$$f\mapsto\hat{f}$$
$$\hat{f}(\gamma):=\gamma(f)\in\mathbb{R}$$

where the **Hom** -set contains morphisms between continuous state spaces. Now, according to the *Gelfand representation theorem*, if Y is a separated smooth $\mathbb{R}$ -manifold whose topology has a countable basis, then the above morphism of $\mathbb{R}$ -algebras is an *isomorphism*, and the *morphism*:

$$\lambda:Y\rightarrow\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$$

is a *homeomorphism* of topological state-spaces. Note that if Y is a *smooth* $\mathbb{R}$ *-manifold* each point $y\in Y$ defines a morphism of of $\mathbb{R}$ -algebras ($\mathbb{R}$ -evaluation):

$$\lambda_{y}:\mathcal{C}^{\infty}{}_{Y}\rightarrow\mathbb{R}$$
$$\lambda_{y}(f)=f(y)$$

Equivalently stated, each point $y\in Y$, defines a *maximal ideal* μ_{y} of $\mathcal{C}^{\infty}{}_{Y}$, that is:

$$\mu_{y}:=Ker\lambda_{y}=\{f\in\mathcal{C}^{\infty}{}_{Y}:f(y)=0\}$$

Consequently, we obtain a natural morphism

$$\lambda : Y \to \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$$
$$\lambda(y) = \lambda_y \simeq \mu_y$$

which is a *homeomorphism* of topological spaces. Thus, identifying the topological space Y with the topological space $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ via λ, the topology of Y coincides with the *Gelfand topology*, the morphism of $\mathbb{R}$-algebras $\mathcal{C}^{\infty}{}_{Y} \to \mathbf{Hom}_{states}(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbb{R})$, where $f \mapsto \hat{f}$, such that, $\hat{f}(\gamma) := \gamma(f) \in \mathbb{R}$ is an *isomorphism*, and the *evaluation* morphism λ_y is the same as γ, being identified with the *maximal ideal* μ_y for each $\mathbb{R}$-state y in Y.

Furthermore, if U is any open set in $Y = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$, then we require that $\mathbf{O}_Y(U) = \mathcal{C}^{\infty}(U)$, where:

$$\mathcal{C}^{\infty}(U) = \mathcal{C}^{\infty}{}_{Y} \,|\, U$$

is the *algebraic localization* of $\mathcal{C}^{\infty}{}_{Y}$, or *ring of fractions* of $\mathcal{C}^{\infty}{}_{Y}$ with respect to the *multiplicative set* of all global real-valued smooth functions without zeros in U, that is:

$$\mathcal{C}^{\infty}(U) = \mathcal{C}^{\infty}{}_{Y} \,|\, U = \mathcal{C}^{\infty}(Y) \,|\, U := \{f / g : f, g \in \mathcal{C}^{\infty}{}_{Y} \,|\, g(y) \neq 0, \forall y \in U\}$$

The localization of $\mathcal{C}^{\infty}{}_{Y}$ with respect to the multiplicative set of all global real-valued smooth functions without zeros in U corresponds to the *localization* of $\mathcal{C}^{\infty}{}_{Y}$ at the *ideal* μ_K of all real-valued smooth functions vanishing at a *closed subset* K of Y (*zero-set* of the ideal μ_K), *complementary* to U, that is:

$$K = \{y \in Y : g(y) = 0, \forall g \in \mu_K\}$$

In this way, given the ring ($\mathbb{R}$-algebra) $\mathcal{C}^{\infty}{}_{Y}$ of global real-valued smooth functions on Y, we finally obtain a *topological space* $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ *identified* with Y, viz. the $\mathbb{R}$-*spectrum* of $\mathcal{C}^{\infty}{}_{Y}$, together with a *structure sheaf* of rings ($\mathbb{R}$-algebras) $\mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}$, identified with the *completion* (sheafification) of the presheaf of rings ($\mathbb{R}$ -algebras) $U \to \mathcal{C}^{\infty}U(U)$, denoted by $\mathbf{C}^{\infty}{}_{Y} = \mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}$, such that:

$$\Gamma\mathbf{C}^{\infty}{}_{Y} = \Gamma\mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})} = \mathcal{C}^{\infty}{}_{Y}$$

Consequently, we have constructed the $\mathbb{R}$ -*ringed* (or $\mathbb{R}$ -algebraized) space of observables

$$(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}) := (\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{C}^{\infty}{}_{Y})$$

from the *ring* ($\mathbb{R}$ -algebra) $\mathcal{C}^{\infty}{}_{Y}$ of *global* real-valued smooth functions on Y, where $Y = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$. If we apply this procedure to the ring ($\mathbb{R}$ -algebra) $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$, we obtain the *local smooth model* $\mathbb{R}$ -ringed space $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$, as an object in the smooth model category $\mathcal{M}_{\mathbb{R}}{}^{sm}$, by means of which we have defined the notion of a smooth $\mathbb{R}$ -manifold previously within the topos of sheaves.

An extremely significant observation has to do with the fact that the *procedure* of *reconstitution* of the $\mathbb{R}$ -algebraized space of states $(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{C}^{\infty}{}_{Y})$ from the ring ($\mathbb{R}$ -algebra) of observables $\mathcal{C}^{\infty}{}_{Y}$, where Y is a smooth $\mathbb{R}$ -manifold, can be applied for an *arbitrary* $\mathbb{R}$ -algebra of observables $\mathcal{A}$. Of course, in this case the $\mathbb{R}$ -spectrum of $\mathcal{A}$, that is, $\mathbb{R}\mathbf{S}pec\mathcal{A}$ is not a smooth $\mathbb{R}$ -manifold any more, meanong that it is *not* modeled on the local smooth model $\mathbb{R}$ -ringed space $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$. Still, the *associated* $\mathbb{R}$ -*ringed* space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ *can be* an $\mathcal{M}_{\mathbb{R}}$ -manifold for an appropriate *choice* of a category of models $\mathcal{M}_{\mathbb{R}}$.

A natural issue that arises in this setting is the conceptualization of an $\mathbb{R}$ -ringed space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ as an $\mathcal{M}_{\mathbb{R}}$ -manifold, where the category of models has objects (local models of an $\mathcal{M}_{\mathbb{R}}$ -manifold) of the form $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$, where $\mathcal{D}$ is a *quotient* of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ by an *ideal* ξ of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$, that is:

$$\mathcal{D} := \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$$

for some natural number n. Obviously, $\mathbf{D}$ refers to the *completion of the presheaf* U Œ $\mathcal{D}(U)$ for U open locus in $\mathbb{R}\mathbf{S}pec(\mathcal{D})$ obtained by the procedure of *algebraic localization* of the $\mathbb{R}$-algebra $\mathcal{D}$. We think of the ideal ξ as the ideal of all elements of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ vanishing at a closed subset $[\xi]_0$ of $\mathbb{R}^n$, that is:

$$[\xi]_0 = \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in \xi\}$$

Then, evidently $\mathbb{R}\mathbf{S}pec(\mathcal{D})$ is identified with the *zero-set* of the ideal ξ, that is:

$$\mathbb{R}\mathbf{S}pec(\mathcal{D}) = \mathbb{R}\mathbf{S}pec((\mathcal{C}^{\infty}{}_{\mathbb{R}^n}) / \xi) = [\xi]_0 = \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in \xi\}$$

We notice that, by analogia to the local smooth model case, we may call $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$ a *local differential model*. This is due to the fact that, although $\mathbb{R}\mathbf{S}pec(\mathcal{D}) = [\xi]_0$ is *not* a smooth manifold, it *can be* interpreted as a *differential space*, whose global algebra of differentiable functions is $\mathcal{D} := \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$. In this sense, differentiable functions on $\mathbb{R}\mathbf{S}pec(\mathcal{D}) = [\xi]_0$ are thought of as *restrictions* of smooth functions on $\mathbb{R}^n$. This is due to the existence of the surjective morphism of $\mathbb{R}$-algebras:

$$\circlearrowleft : \mathcal{C}^{\infty}{}_{\mathbb{R}^n} \leq \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$$

which is interpreted as a *restriction* morphism, that is for any $f \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ the equivalence class $[f] \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$ is said to be the restriction of the smooth function f to the differential function $[f]$, or else, the restriction of f to the *differential subspace* $[\xi]_0$ of the smooth space $\mathbb{R}^n$. In this manner, the notion of *differentiability* induced by local differential models of the form considered, and subsequently, the notion of an $\mathcal{M}_{\mathbb{R}}$-differential manifold obtained, is much more *general* than the notion of *smoothness* induced by local smooth models and the associated smooth manifolds. Moreover, the notion of differentiability *supersedes* the notion of smoothness, which in turn, is obtained as a *special* case of the former.

Motivated by the previous observation, we say that an *ideal* of a differentiable algebra is *closed* if and only if its *quotient* by this ideal is *also* a differentiable algebra. In particular, since $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ is a differentiable algebra, and the quotient algebra $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$ is also a differentiable algebra by *restriction*, the ideal ξ is *closed*. We also say that an $\mathbb{R}$-algebra $\mathcal{Z}$ is *simple* if 0 is the *unique* element of $\mathcal{Z}$ *vanishing* at any point of $\mathbb{R}\mathbf{S}pec(\mathcal{Z})$. This is equivalent to saying that the morphism of $\mathbb{R}$-algebras:

$$\mathcal{Z} \to \mathbf{Hom}_{cstates}(\mathbb{R}\mathbf{S}pec(\mathcal{Z}), \mathbb{R})$$
$$f \mapsto \hat{f}$$

is *injective*, so that any simple $\mathbb{R}$-algebra is *isomorphic* to an algebra of real-valued continuous functions on the topological space $\mathbb{R}\mathbf{S}pec(\mathcal{Z})$. It is clear that if $\mathcal{Z}$ is of the form $\mathcal{Z} = \mathcal{D} = \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$ where ξ denotes the *closed ideal* of all elements of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ vanishing at a closed subset $[\xi]_0$ of $\mathbb{R}^n$, then $\mathcal{Z}$ is *simple*. Now, we are ready to consider an *extension* of differentiable algebras to *non-simple* ones.

Intuitively, these algebras are *still* of the form $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$, where ξ is a *closed ideal*, but now they contain *nilpotent* elements. This equivalently means that the ideal is of the form $\xi = \zeta^r$ for some *power* r. For example, if μ_x is the (maximal) ideal of all smooth functions of $\mathcal{C}^{\infty}_{\mathbb{R}^n}$ vanishing at x in $\mathbb{R}^n$, then $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x^{\ 2}$ is considered a differential algebra containing *nilpotent* elements. Note that:

$$\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x^{\ 2}) = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x) = \{x\}$$

but $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x^{\ 2}$ contains nilpotent elements. In this sense, we can consider *local differential model spaces* of the form

$$(\{x\}, \mathbf{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x^{\ (r+1)})$$

interpreted as the [$r-th$ *infinitesimal region of* x] differential space, being a differential subspace of $(\mathbb{R}^n, \mathbf{C}^{\infty}{}_{\mathbb{R}^n})$. These differential spaces have an interesting physical interpretation, since the restriction of a smooth function $f \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ to a differential space of the above form is equivalent to the $r-th$ *Taylor expansion* of f at x. This is called the $r-th$-*jet* of f at x and we obtain:

$$j_x^r = [f] \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x^{(r+1)}$$

According to the preceding analysis we can now conceptualize an $\mathbb{R}$-ringed space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ as an $\mathcal{M}_{\mathbb{R}}$-*manifold*, called a *differential* $\mathbb{R}$-*ringed space*, where the category of models has objects (local models of an $\mathcal{M}_{\mathbb{R}}$-manifold) of the form $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$.

In this context, $\mathcal{D}$ stands for a differentiable algebra, since it is a quotient of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ by a closed ideal ξ of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$, that is:

$$\mathcal{D} := \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$$

for some natural number n. In order to achieve the greatest generality we give the following preliminary definition:

A (locally) $\mathbb{R}$-ringed space $(Y, \mathbf{O}_Y)$ is called an *affine differential* $\mathbb{R}$-*ringed space* of states if it is *isomorphic* to a differential $\mathbb{R}$-ringed space of the form $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$, where $\mathcal{D}$ is a differentiable algebra of observables. Then, we consider as a category of models $\mathcal{M}_{\mathbb{R}}^{diff}$ the category of affine differential $\mathbb{R}$-ringed spaces.

Then, a differential $\mathbb{R}$-ringed space of states $(X, \mathbf{O}_X)$ is defined as an $\mathcal{M}_{\mathbb{R}}^{diff}$-*manifold*. This means that any point $x \in X$ has an open neighbourhood U in X, called an *affine open locus*, such that $(U, \mathbf{O}_X | U)$ is an affine differential $\mathbb{R}$-ringed space of states. Such open loci of X are called *affine open sets* and they naturally define a *basis* for the topology of X. The sections of $\mathbf{O}_X$ on an open locus U in X are said to be *differentiable* (observable) functions on U. The value of a differentiable function at a point $x \in U$ is identified with the *residue class* of its germ f_x at x, that is, with $f(x) \in \mathcal{O}_{X,x} / \mu_x = \mathbb{R}$. Then, any differentiable function $f \in \mathbf{O}_X(U)$ determines a *continuous* morphism $\breve{f} : U \to \mathbb{R}$, where $\breve{f}(x) = f(x)$, although f is *not* determined by $\breve{f}$.

An immediate consequence is the following: If ζ is a *closed ideal* of a differentiable algebra $\mathcal{D}$, then $(\mathbb{R}\mathbf{S}pec(\mathcal{D}/\zeta), \mathbf{D}/\zeta)$ is a *closed differential subspace* of $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$. Inversely, any *closed differential subspace* of $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$ is defined by a *unique closed ideal* of $\mathcal{D}$.

For brevity of notation, if a differentiable algebra is denoted by $\mathcal{D}$, then we denote the corresponding affine differential $\mathbb{R}$-ringed space by $\hat{\mathcal{D}}$. Now, there exists a categorical *natural equivalence* (duality) between the category of (commutative and unital) differentiable $\mathbb{R}$-algebras of observables $\mathcal{D}$, and the category of affine differential (commutative) $\mathbb{R}$-ringed spaces of states $\hat{\mathcal{D}}$, which is defined functorially, as follows:

$$\mathbb{R}\mathbf{Spec} : \mathcal{D} \rightleftarrows \hat{\mathcal{D}} : \Gamma$$

where, $\mathbb{R}\mathbf{Spec} : \mathcal{D} \to \hat{\mathcal{D}}$ is the *real spectrum* functor, which assigns to a commutative and unital differentiable $\mathbb{R}$-algebra of observables $\mathcal{D}$ its dual *affine differential* $\mathbb{R}$ *-ringed space* of states $(\mathbb{R}Spec\mathcal{D}, \mathbf{D})$, and, $\mathbf{\Gamma} : \hat{\mathcal{D}} \to \mathcal{D}$ is the *global sections* functor in the opposite direction, which assigns to the *structure sheaf* $\mathbf{D}$ over the real spectrum topological space $\mathbb{R}Spec\mathcal{D}$, its $\mathbb{R}$-algebra of *global sections* $\mathbf{\Gamma}(\mathbb{R}Spec\mathcal{D}, \mathbf{D}) := \mathbf{D}(\mathbb{R}Spec\mathcal{D})$, identified with the *differentiable* $\mathbb{R}$ *-algebra* of differentiable functions $\mathcal{D}$.

The conceptual importance of this natural equivalence lies in the fact that the notion of a *differential model* object, incorporates a fundamental *categorical duality*, which, when interpreted physically, *unifies* the *algebraic encoding* of differentiable observable information expressed in terms of (commutative and unital) differentiable $\mathbb{R}$-algebras of observables, together with, the *geometric-topological representation* of these model objects in terms of affine differential (commutative) $\mathbb{R}$-ringed spaces of states. Moreover, both, the algebraic, and, the equivalent dual geometric representation of differential model objects, implement and reciprocally respect faithfully the bidirectional *localization-globalization* process of observation within the topos of sheaves.

10.5 DIFFERENTIAL OBSERVABILITY AND LOCALLY FREE MODULES

In order to clarify the meaning of the observation process within the localization environment of a topos of sheaves it is instructive to use again the principle of *relativization* of physical *representability*. Let us consider a commutative and unital coordinatization ring, thought of as a $\mathbb{R}$-algebra of observables, whose representability is relativized with respect to the topos of sheaves $\mathbf{Shv}(Y)$, Then, the open sets of the global topological space Y, play the role of a category of extensional reference contexts of observation, partially ordered by inclusion. Consequently, the right action of this category on the ring of observables, partitions it into sorts parameterized by the base local contexts, inducing the uniform homologous fibered information structure of a presheaf of local observables over the base category of localizing contexts. In this sense, the *geometric classification* of the total information content included in the *representation* of the corresponding natural system by a formal system of the above form, through observation, which in addition respects the requirements of a coherent localization-globalization process, can be performed in a *dually equivalent* manner.

More precisely, observable information organized in the form of a *presheaf* can firstly be *contextualized* at base points by application of the *germs*-functor, and then, *glued* together (by taking the disjoint union of the fibers and topologizing the formed space as an *étale* bundle), such that, the set of cross sections constitutes a *sheaf* obtained by the subsequent application of the *cross-sections* functor. Alternatively, observable information organized in the form of a *presheaf* can firstly be made compatible with respect to *restriction* from the global to the local, as well as, *extension* from the local to the global, by application of the *sheafification* functor, and then, be *contextualized* at the base points of the equivalent *étale* bundle, by subsequent application of the *germs*-functor on the sheaf of local observables formed from the *sheafification* of the presheaf we started with.

Thus, observation of natural systems preserving the requirements of a coherent *localization-globalization* process, can be implemented equivalently, given the initial *organization* of observable information in the form of a presheaf of rings of local observables, either, by firstly *contextualizing* information at base points of a topological space in terms of observable germs, and then, *gluing* appropriately, or, by firstly making locally observable information *compatible* with respect to *restriction* and *extension*, and then, *contextualizing* at those perspectives. Again, there is a *key concept* that *unlocks* the meaning of this *equivalence*, referring to the sequence of the operations needed, in order to make observation of natural systems preserving the requirements of a coherent localization-globalization observation process; It is that, *any sheaf* of local observables can be conceived as a *sheaf of cross-sections* of its corresponding equivalent by duality *étale* bundle.

The formalization of the above takes place, when we first recollect that there is a categorical natural equivalence between the categories of *sheaves* $\mathbf{Sh}(Y)$ on $\mathcal{O}(Y)$, and the category of *étale topological bundles* $\mathcal{ET}(Y)$ over Y, as follows:

$$\Lambda : \mathbf{Sh}\left(Y\right) \rightleftarrows \mathcal{ET}\left(Y\right) : \Gamma$$

Now, the crucial fact that a sheaf of local observables is identified as a sheaf of cross-sections of the corresponding étale bundle, is expressed *functorially* by the requirement that the *unit* natural transformation of this adjunction (natural equivalence) is an *isomorphism* $\eta_{\mathbf{A}} : \mathbf{A} \to \Gamma\Lambda\mathbf{A}$. Moreover, in the case at issue, the *counit* $\text{Ú}_A : \Lambda\Gamma A \to A$ is also an *isomorphism*. Thus, the *cross sections-functor* $\mathbf{\Gamma} : \mathcal{ET}(Y) \to \mathbf{Sh}(Y)$ and the *germs-functor* $\mathbf{\Lambda} : \mathbf{Sh}(Y) \to \mathcal{ET}(Y)$ can be conceived as inverses to each other.

Let us now extend this line of thought, by considering the above $\mathbb{R}$ -algebra of observables as a *differentiable* $\mathbb{R}$ -algebra $\mathcal{D}$ corresponding to a *model differential space* of states $\hat{\mathcal{D}}$. We intend to conceptualize the procedure of model differentiable observable information acquisition constituting a corresponding model differential observation process. The reason that we focus our attention on the notion of a model differential observation process is that we can extend it for the case of an $\mathcal{M}_{\mathbb{R}}^{diff}$ -*manifold* by means of a *gluing datum*.

We claim that a model differential observation process should be conceived categorically as a *natural transformation of the identity* of $\mathcal{D}$, enacted through the multitude of *all variable* localization domains of its *real spectrum* topological space, which classifies its total information content. This natural transformation of the identity of $\mathcal{D}$ is expressed concretely, by the *counit* of the *adjunction* between the category of (commutative and unital) differentiable $\mathbb{R}$ -algebras of observables $\mathcal{D}$, and the category of affine differential (commutative) $\mathbb{R}$ -ringed spaces of states $\hat{\mathcal{D}}$.

Moreover, since the *adjunction* is actually a categorical *natural equivalence* (duality), the *counit* natural transformation is an *isomorphism*. In this sense, a model differential observation process, enacted through the multitude of all variable localization domains of its real spectrum topological space classifying its total information content, according to the localization-globalization process, is conceived as the *operational implementation* of the counit isomorphism:

$$\text{Ú}_{\mathcal{D}} : \Gamma\mathbb{R}\mathbf{Spec}\mathcal{D} \to \mathcal{D}$$

In this sense, the composite endofunctor:

$$\mathbf{\Pi} := \mathbf{\Gamma}\mathbb{R}\mathbf{Spec} : \mathcal{D} \to \mathcal{D}$$

may be called a model differential observation. Note that *any* differentiable $\mathbb{R}$-algebra of observables can be considered as a *fixed point* of a model differential observation functor $\mathbf{\Pi}$.

Next, let us consider a commutative (locally) $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states denoted by the pair $(X, \mathbf{A})$. An $\mathbf{A}$-*module* E is called a *locally free* $\mathbf{A}$-module of finite rank m, if for any point $x \in X$ there exists an open locus U of X such that:

$$E|U = (\mathbf{A}|U)^m$$

where $(\mathbf{A}|U)^m$ denotes the $m-th$-*direct sum* of the sheaf of $\mathbb{R}$-algebras of observables $\mathbf{A}$ restricted to U, for some $m \in \mathbb{N}$. Furthermore, if $\mathbf{A}$ is a constant sheaf of $\mathbb{R}$-algebras, then any locally free $\mathbf{A}$-module of finite rank m, for some $n \in \mathbb{N}$, stands for a *local system* of coefficients.

In case that we view a smooth $\mathbb{R}$-*manifold* as an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{O}_Y = \mathbf{C}^{\infty}{}_Y)$ obtained by means of a *gluing datum* with respect to the category of smooth models $\mathcal{M}_{\mathbb{R}}{}^{sm}$, then a (smooth) *locally free* $\mathbf{C}^{\infty}{}_Y$-module of *finite* rank corresponds *bijectively* to a (smooth) *vector bundle* on Y.

It is worth explaining the above stated bijective correspondence between locally free $\mathbf{C}^{\infty}{}_Y$-modules of finite rank and vector bundles on Y as follows:

Firstly, we notice that since sections of $\mathbf{C}^{\infty}{}_Y$ are smooth functions on Y, we may think of the corresponding sections of the $\mathbf{C}^{\infty}{}_Y$-module $(\mathbf{C}^{\infty}{}_Y)^m$ as *vector-valued* functions.

Secondly, we notice that if we have a general locally free $\mathbf{C}^{\infty}{}_Y$-module E of finite rank, then by definition, it is *locally isomorphic* to $(\mathbf{C}^{\infty}{}_Y)^m$ for some $m \in \mathbb{N}$. We bear in mind that, if y is a point of Y, and μ_Y is *the ideal* of $(\mathbf{C}^{\infty}{}_Y)_y$ consisting of *germs* of smooth functions at y, vanishing at y, we may consider for any $f \in (\mathbf{C}^{\infty}{}_Y)_y$ its image in

$(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$. Then, the *evaluation* morphism $(\mathbf{C}^{\infty}{}_{Y})_y \to \mathbb{R}$, which takes any smooth function f to its real value $f(y)$, provides an *isomorphism* of $(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$ with $\mathbb{R}$. Hence, the *evaluation* morphism of a smooth function f is *equivalent* to considering the *image* of f in $(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$. This observation leads us to the conclusion that, by analogia, the $\mathbb{R}$-vector space $\varepsilon_y = E_y / \mu_y E_y$ is the *vector space* in which the sections of E take values at the point $y \in Y$.

Thirdly, we notice that the essential difference between the case referring to $\mathbf{C}^{\infty}{}_{Y}$ and the case referring to a locally free sheaf E lies in the fact that in the latter case the vector space $\varepsilon_y = E_y / \mu_y E_y$ *associated with* $y \in Y$ *depends on the point* y in comparison to the former case where such a dependence is not existent. Put differently, there is *no natural isomorphism* of these $\mathbb{R}$-vector spaces at *two different points* y_1 and y_2 of Y.

Of course, we may consider the set-theoretic union of all ε_y, viz., $\varepsilon = \bigcup_y \varepsilon_y$. We notice that ε has a *natural projection* morphism π into Y, such that $\pi^{-1}(y) = E_y$. Moreover, for any $y \in Y$ there exists an open locus U such that $E|U$ is isomorphic to $(\mathbf{C}^{\infty}{}_{Y})^m$, so that $\pi^{-1}(U)$ may be identified with $U \times \mathbb{R}^m$, just as we may identify the *morphism* $\pi^{-1}(U) \to U$ with the *projection* $U \times \mathbb{R}^m \to U$. In particular, the set $\pi^{-1}(U)$ can be endowed with the smooth structure of the *product space* $U \times \mathbb{R}^m$. Then it is straightforward to construct an appropriate *gluing datum* and thus obtain a corresponding $\mathcal{M}_{\mathbb{R}}$-*manifold* if we consider as a category of models $\mathcal{M}_{\mathbb{R}}$ the *smooth* category of *vector models*, denoted by $\mathcal{M}_{\mathbb{R}}{}^{smv}$. The category $\mathcal{M}_{\mathbb{R}}{}^{smv}$ has as objects pairs of the form:

$$(\mathbb{U} \times \mathbb{R}^m \subset \mathbb{R}^{n+m}, \mathbf{C}^{\infty}{}_{\mathbb{R}^n}(\mathbb{U}) \otimes \mathbf{C}^{\infty}{}_{\mathbb{R}^m})$$

Then, given the category of smooth vector models $\mathcal{M}_{\mathbb{R}}{}^{smv}$, we may construct a *smooth* $\mathcal{M}_{\mathbb{R}}{}^{smv}$-*manifold* satifying the conditions [I], [II], [III] given previously. Again, we note that the *structure* of a smooth $\mathcal{M}_{\mathbb{R}}{}^{smv}$-manifold is obtained by the *equivalence relation* on the disjoint union of its coordinate charts induced by the corresponding gluing datum with respect to the category of smooth vector models.

Now, the datum (ε, Y, π), consisting of a *smooth* $\mathcal{M}_{\mathbb{R}}^{smv}$*-manifold* ε (which is an $n+m$ -dimensional $\mathbb{R}$ -manifold), a smooth n-dimensional $\mathbb{R}$ -manifold Y and a *smooth projection* morphism $\pi : \varepsilon \rightarrow Y$ with the above described properties instantiates a *smooth vector bundle over* Y of rank m.

Thus, for any *locally free* $\mathbf{C}^{\infty}{}_{Y}$-module E of *finite* rank we can construct an *associated smooth vector bundle* (ε, Y, π) of the same rank. Inversely, the *sheaf of sections* of any (smooth) vector bundle on Y is a *locally free* $\mathbf{C}^{\infty}{}_{Y}$*-module* of finite rank. Thus, the correspondence is *bijective*, and consequently, we can identify locally free sheaves of $\mathbf{C}^{\infty}{}_{Y}$-modules with sheaves of (smooth) sections of their associated vector bundles on Y.

This *bijective* correspondence between locally free sheaves of $\mathbf{C}^{\infty}{}_{Y}$-modules with sheaves of (smooth) sections of their associated vector bundles on Y, where $(Y, \mathbf{O}_{Y} = \mathbf{C}^{\infty}{}_{Y})$ is a smooth $\mathbb{R}$ -manifold, has received such close attention in order to provide the necessary geometric intuition underlying the general notion of a locally free sheaf of $\mathbf{A}$ -modules of finite rank.

Note that the definition of a locally free sheaf of $\mathbf{A}$ -modules (of finite rank) holds with respect to an arbitrary commutative (locally) $\mathbb{R}$ -ringed (or $\mathbb{R}$ -algebraized) space of states denoted by the pair $(X, \mathbf{A})$. Thus, it can be applied for example to the case of an affine differential $\mathbb{R}$ -ringed space of states, which is by definition, isomorphic to a differential $\mathbb{R}$ -ringed space of the form $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$, where $\mathcal{D}$ is a differentiable algebra of observables, or even to the case of a general differential $\mathbb{R}$ -ringed space of states $(X, \mathbf{O}_{X})$, since it is definition an $\mathcal{M}_{\mathbb{R}}^{diff}$-manifold, but one which equally retains the analogous geometric intuition by means of associated differential vector bundles.

Next, let us consider the *category* of *differential* $\mathbb{R}$ *-ringed spaces* of states $(X, \mathbf{O}_{X})$, and let $\hat{X}$ be an object in this category. Then, the functor represented by $\hat{X}$ is the *contravariant functor* $\mathbf{y}^{\hat{X}} : \hat{\mathcal{X}} \rightarrow \mathbf{Sets}$, defined as follows:

i For all differential spaces $\hat{Y}$ in $\hat{\mathcal{X}}$, $\mathbf{y}^{\hat{X}}(\hat{Y}) := Hom_{\hat{\mathcal{X}}}(\hat{Y}, \hat{X})$.

ii For all differential spaces-morphisms $f:\hat{Z}\to\hat{Y}$ in $\hat{\mathcal{X}}$,

$$\mathbf{y}^{\hat{X}}(f): Hom_{\hat{\mathcal{X}}}(\hat{Y},\hat{X})\to Hom_{\hat{\mathcal{X}}}(\hat{Z},\hat{X})$$

is defined as pre-composition with f, that is, $\mathbf{y}^{\hat{X}}(f)(g):=g\circ f$.

The functor $\mathbf{y}^{\hat{X}}:\hat{\mathcal{X}}\to\mathbf{Sets}$ represented by the differential spaces of states $\hat{X}$, is called the *Grothendieck functor* of generalized points (states) of $\hat{X}$. Moreover, the information contained in the differential space of states $\hat{X}$ is classified completely by its functor of generalized points (states) $\mathbf{y}^{\hat{X}}$.

In this sense, it is possible to extract geometric information without knowing whether there is actually a differential space in possession of a functor of the above form as its functor of generalized points. Pursuing the functorial approach one step further, we notice that the functor of generalized points of an $\mathbb{R}$-ringed differential space of states is completely determined by its *restriction to the subcategory* of $\mathbb{R}$*-ringed affine differential spaces* of states, together with, the *gluing datum* between any two $\mathbb{R}$-ringed affine differential spaces belonging to the covering family of that differential space.

In this manner, it is specified by means of the contravariant representable functor from the category of $\mathbb{R}$-ringed affine differential spaces to the category of sets, or equivalently, via the covariant representable functor from the category of differentiable $\mathbb{R}$-algebras of observables to the category of sets (modulo the compatibility conditions), thus admitting a well-defined operational determination in terms of *model differential observation* processes, as previously outlined. Furthermore, the appropriate implementation of the corresponding gluing conditions, should again respect the localization-globalization process, conceived in this generalized categorical context.

Hence, there is every need to secure compatibility of information under the operations of restriction (from the global to the local level) and extension (from the local to the global level), where, the notions of local and global receive a meaning only with respect to a *suitable notion* of topology (categorical Grothendieck topology) defined on the category of differential $\mathbb{R}$-ringed spaces of states. Conclusively, at this stage, we may say that a differential $\mathbb{R}$-ringed space of states constitutes the *sheafification* of the model differential observable information *encoded* in its *functor of generalized points* (restricted to affine differential $\mathbb{R}$-

ringed spaces of states), with respect to a topology that explicates the localization-globalization process in a categorical context.

Further on, we are going to examine in detail the semantic role of the notion of a *categorical Grothendieck topology*. At the moment, we should take note that, in relation to the functorial viewpoint on differential $\mathbb{R}$-ringed spaces of states, the *Gelfand topology*, which has enebled the *sheafification* of observables (taking values in the affine differential $\mathbb{R}$-ringed spaces of states), gives rise to a *Grothendieck topology* on the category of differential $\mathbb{R}$-ringed spaces of states.

10.6 TOPOLOGY OF SIEVING: COMMUNICATION SITES

The notion of a categorical Grothendieck topology requires, first of all, the *abstraction* of the constitutive properties of *localization* in appropriate *categorical* terms, and then, the effectuation of these properties for the definition of *localization* systems. The crucial observation has to with the fact that the concept of sheaf, in terms of coverings, restrictions, and collation, can be defined and used *not just* in the *spatial* sense, namely on the usual topological spaces, but in a generalized spatial sense, on more *general topologies* (Grothendieck topologies). In the usual definition of a sheaf on a topological space we use the open neighbourhoods U of a point in a space X; such neighbourhoods are actually *injective* topological maps $U \infty X$. The neighbourhoods U in topological spaces can be replaced by morphisms $V \to X$ *not* necessarily injective, and this can be done in *any* category with *pullbacks*.

In effect, a covering by open sets can be replaced by a *new* notion of *covering* provided by a family of morphisms satisfying certain conditions. These conditions abstract the constitutive properties of a well-defined *localization* process in appropriate *categorical* terms. Our presentation applies to *any* small category $\mathcal{B}$, consisting of *base* reference categorical objects B, with structure-preserving morphisms between them, as arrows. Of course, in the classical topological case, $\mathcal{B}$ is tautosemous with $\mathcal{O}(X)$ and the reference contexts B are tautosemous with the open sets U of X, partially ordered by inclusion.

For an object B of $\mathcal{B}$, we consider *indexed families*:

$$\mathbf{S} = \{\psi_i : B_i \to B, i \in I\}$$

of maps to B, and we assume that for each object B of $\mathcal{B}$ we have a set $\mathbf{\Lambda}(B)$ of certain such families satisfying conditions to be specified later. These families play the role of coverings of B under those conditions. For the coverings provided, it is possible to construct the *analogue* of the *topological* definition of a sheaf, where as presheaves on $\mathcal{B}$ we consider the functors $\mathbf{P}: \mathcal{B}^{op} \to Sets$.

According to the topological definition of a *sheaf* on a space we demand that for each open cover $\{U_i, i \in I\}$ of some U, every *family* of elements $\{p_i \in \mathbf{P}(U_i)\}$ that satisfy the *compatibility* condition on the *intersections* $U_i \cap U_j, \forall i, j$, are *pasted together* as a *unique element* $p \in \mathbf{P}(U)$. Imitating the above procedure for any covering $\mathbf{S}$ of an object B, and *replacing* the *intersection* $U_i \cap U_j$ by the *pullback* $B_i \times_B B_j$ in the general case, according to the diagram;

$$
\begin{array}{ccc}
B_i \times_B B_j & \xrightarrow{g_{ij}} & B_j \\
{\scriptstyle h_{ij}}\downarrow & & \downarrow{\scriptstyle \psi_j} \\
B_i & \xrightarrow{\psi_i} & B
\end{array}
$$

we effectively obtain for a given *presheaf* $\mathbf{P}: \mathcal{B}^{op} \to Sets$ a *diagram* of sets as follows:

$$
\begin{array}{ccc}
\mathbf{P}(B_i \times_B B_j) & \xrightarrow{\mathbf{P}(g_{ij})} & \mathbf{P}(B_j) \\
{\scriptstyle \mathbf{P}(h_{ij})}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(\psi_j)} \\
\mathbf{P}(\mathbf{B_i}) & \xrightarrow{\mathbf{P}(\psi_i)} & \mathbf{P}(B)
\end{array}
$$

In this case the compatibility condition for a sheaf takes the form: if $\{p_i \in \mathbf{P}_i, i \in I\}$ is a family of compatible elements, namely satisfy $p_i h_{ij} = p_j g_{ij}, \forall i, j$, then a *unique* element $p \in \mathbf{P}(B)$ is determined by the family such that $p \cdot \psi_i = p_i, \forall i \in I$, where the notational convention $p \cdot \psi_i = \mathbf{P}(\psi_i)(p)$ has been used . Equivalently this condition can be expressed in the categorical terminology by the requirement that in the diagram:

$$\prod_{i,j}\mathbf{P}(B_i \times_B B_j) \leftleftarrows \prod_i \mathbf{P}(B_i) \xleftarrow{e} \mathbf{P}(B)$$

the arrow e, where:

$$e(p)=(p \cdot \psi_i, i \in I)$$

is an equalizer of the maps:

$$(p_i, i \in I) \to (p_i h_{ij}; i, j \in I \times I)$$

and

$$(p_i, i \in I) \to (p_i g_{ij}; i, j \in I \times I)$$

correspondingly.

The specific *conditions* that the elements of the set $\Lambda(B)$, or else the *coverings* of $\mathcal{B}$, have to satisfy naturally lead to the notion of a *Grothendieck pretopology* on the category $\mathcal{B}$ as follows:

A *Grothendieck pretopology* on $\mathcal{B}$ is an operation Λ which assigns to each object B in $\mathcal{B}$ a set $\Lambda(B)$. Each $\Lambda(B)$ contains indexed families of $\mathcal{B}$-morphisms with codomain B:

$$\mathbf{S}=\{\psi_i : B_i \to B, i \in I\}$$

such that, the following conditions are satisfied:

i For each B in $\mathcal{B}$, $\{id_B\} \in \Lambda(B)$;

ii If $C \to B$ in $\mathcal{B}$ and $\mathbf{S}=\{\psi_i : B_i \to B, i \in I\} \in \Lambda(B)$ then:

$$\{\psi_1 : C \times_B B_i \to B, i \in I\} \in \Lambda(C)$$

Note that ψ_1 is the pullback in $\mathcal{B}$ of ψ_i along $C \to B$;

iii If $\mathbf{S}=\{\psi_i : B_i \to B, i \in I\} \in \Lambda(B)$, and for each $i \in I$:

$$\{\psi_{ik}{}^{i} : C_{ik} \to B_i, k \in K_i\} \in \Lambda(B_i)$$

then

$$\{\psi_{ik}{}^{i} \circ \psi_i : C_{ik} \to B_i \to B, i \in I; k \in K_i\} \in \Lambda(B)$$

Note that C_{ik} is an example of a *double indexed* object rather than the intersection of C_i and C_k.

The notion of a Grothendieck pretopology on the category $\mathcal{B}$ is a *categorical generalization* of a system of set-theoretical *covers* on a topology T, where a cover for $U \in T$ is a set $\{U_i : U_i \in T, i \in I\}$ such that $\cup U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows of a pretopology contained in $\Lambda(B)$ is a cover as well.

Now, the notion of a Grothendieck topology on a small category $\mathcal{B}$, consisting of base reference categorical objects B, can be presented in terms of appropriate covering devices admitting a *functorial* interpretation. We emphasize that the notion of a Grothendieck topology requires, first of all, the abstraction of the constitutive properties of localization in appropriate categorical terms, and then, the effectuation of these properties for the definition of localization systems. Regarding these objectives, the sought abstraction is implemented by means of covering devices on the base category of reference contexts, called in categorical terminology *covering sieves*. The constitutive properties of *localization* abstracted categorically in terms of sieves, qualified as covering ones, satisfy the following basic requirements, as we will see subsequently:

- i The covering sieves are *covariant* under *pullback* operations, that is, they are *stable* under *change* of a *base* reference context. Most importantly, the stability conditions are *functorial*;
- ii The covering sieves are *transitive*.

From a physical perspective, we benefit from thinking of covering sieves as generalized *measures* of *localization* of states. The operation assigning to each reference context of the base category a collection of covering sieves satisfying the closure conditions stated previously, gives rise to the notion of a *Grothendieck topology* on the *base* category of contexts. The construction of a suitable Grothendieck topology on the base category of contexts is significant for the following reasons: Firstly, it lays out precisely and unambigously the conception of the *local* in a *categorical* measurement environment, such that this conception becomes *detached* from its restricted *spatial* connotation in terms of geometric point-spaces, rather finding expression exclusively in *relational information* terms. Secondly, it permits the *collation* of local information into global by utilization of the notion of a *sheaf* for a suitable Grothendieck topology.

Firstly, we shall explain the *general* notion of *sieves*, and afterwards, we shall focus more narrowly on the notion of *covering sieves*.

A B-*sieve* with respect to a reference *context* B in $\mathcal{B}$, is a *family* S of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to S and $D \to C$ is any $\mathcal{B}$-morphism, then the composite $D \to C \to B$ belongs to S. We may think of a B-sieve as a *right* B-*ideal*. We notice that, in the case of $\mathcal{O}(X)$, since $\mathcal{O}(X)$-morphisms are *inclusions* of open loci, a right U-ideal equates to a *downwards closed* U-*subset*.

It is important to realize that a B-sieve is equivalent to a *subfunctor* $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$, where $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, denotes the *contravariant representable* functor of the reference locus B in $\mathcal{B}$.

More specifically, given a B-sieve S, we define:

$$\mathbf{S}(C) = \{g / g : C \to B, g \in S\} \subseteq \mathbf{y}[B](C)$$

This definition yields a functor $\mathbf{S}$ in $Sets^{\mathcal{B}^{op}}$, which is obviously a subfunctor of $\mathbf{y}[B]$. Conversely, given a subfunctor $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$, the set:

$$S = \{g / g : C \to B, g \in \mathbf{S}(C)\}$$

for some reference loci C in $\mathcal{B}$, is a B-sieve. Thus, epigramatically, we state:

$$\langle \; B\text{-sieve: } S \; \rangle = \langle \; \text{Subfunctor of } \mathbf{y}[B]\text{: } \mathbf{S} \infty \mathbf{y}[B] \; \rangle$$

We notice that if S is a B-sieve and $h : C \to B$ is any arrow to the locus B, then:

$$h^*(S) = \{f / cod(f) = C, (h \circ f) \in S\}$$

is a C-sieve, called the *pullback* of S *along* h, where, $cod(f)$ denotes the codomain of f. Consequently, we may define a *presheaf* functor $\mathbf{\Omega}$ in $Sets^{\mathcal{B}^{op}}$, such that its *action* on loci B in $\mathcal{B}$, is given by:

$$\mathbf{\Omega}(B) = \{S / S : B - sieve\}$$

and on arrows $h : C \to B$, by $h^{*}(-) : \mathbf{\Omega}(B) \to \mathbf{\Omega}(C)$, given by:

$$h^{*}(S) = \{f / cod(f) = C, (h \circ f) \in S\}$$

We notice that for a context B in $\mathcal{B}$, the set of all arrows into B, is a B-sieve, called the *maximal sieve* on B, and denoted by, $t(B) := t_B$.

At a next stage of development, the key conceptual issue to be settled is the following: How is it possible to *restrict* $\mathbf{\Omega}(B)$, that is the *set* of all B-sieves for each reference context B in $\mathcal{B}$, such that *each* B-sieve of the restricted set can acquire the interpretation of a *covering* B-sieve, with respect to a generalized covering system?

Equivalently stated, we wish to impose the satisfaction of appropriate *conditions* on the set of B-sieves for each context B in $\mathcal{B}$, such that, the subset of B-sieves obtained, denoted by $\mathbf{\Omega}_{\chi}(B)$, implement the *constitutive* properties of *localization* in *functorial* terms. In this sense, the B-sieves of $\mathbf{\Omega}_{\chi}(B)$, for each locus B in $\mathcal{B}$, to be thought as covering B-sieves, can legitimately be used for the implementation of localization processes. The appropriate conditions depicting the covering B-sieves from the set of all B-sieves, for each reference context B in $\mathcal{B}$, are the following:

i We interpret an arrow $C \to B$, where C, B are contexts in $\mathcal{B}$, as a *figure* of B, and thus, we interpret B as an *extension* of C in $\mathcal{B}$. It is a natural requirement that the set of all figures of B should belong in $\mathbf{\Omega}_{\chi}(B)$ for each context B in $\mathcal{B}$.

ii The covering sieves should be *stable* under *pullback* operations, and most importantly, the stability conditions should be expressed *functorially*. This requirement means, in particular, that the *intersection* of covering sieves should also be a covering sieve, for each reference context B, in the base category $\mathcal{B}$.

iii Finally, it would be desirable to impose: (i) a *transitivity* requirement on the specification of the covering sieves, such that, intuitively stated, covering sieves of figures of a context in covering sieves of this context, should also be covering sieves of the context themselves, and (ii) a requirement of *common refinement* of covering sieves.

If we take into account the above requirements we can define a generalized covering system, called a Grothendieck topology, in the environment of $\mathcal{B}$ as follows:

A *Grothendieck topology* on $\mathcal{B}$ is an operation $\mathbf{J}$, which assigns to each reference context B in $\mathcal{B}$, a collection $\mathbf{J}(B)$ of B-sieves, called *covering* B*-sieves*, such that the following three conditions are satisfied:

i For every reference context B in $\mathcal{B}$, the maximal B-sieve $\{g : cod(g) = B\}$ belongs to $\mathbf{J}(B)$ (*maximality condition*).

ii If S belongs to $\mathbf{J}(B)$ and $h : C \to B$ is a figure of B, then $h^*(S) = \{f : C \to B, (h \circ f) \in S\}$ belongs to $\mathbf{J}(C)$ (*stability condition*).

iii If S belongs to $\mathbf{J}(B)$, and if for each figure $h : C_h \to B$ in S, there is a sieve R_h belonging to $\mathbf{J}(C_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $\mathbf{J}(B)$ (*transitivity condition*).

As a consequence of the definition above, we can easily check that any two B-covering sieves have a common refinement, that is: if S, R belong to $\mathbf{J}(B)$, then $S \cap R$ belongs to $\mathbf{J}(B)$.

It is important to notice that given a pretopology Λ we can define a topology $\mathbf{J}$ giving rise to the *same sheaves* on $\mathcal{B}$. More specifically, we say that for any $B \in \mathcal{B}$, we have R belongs to $\mathbf{J}(B)$ if and only if R contains a pretopology covering belonging to $\Lambda(B)$.

As a first application we may consider the partially ordered set of open subsets of a topological space X, viewed as the base category of *open* reference domains, $\mathcal{O}(X)$. Then we specify that S is a covering U-sieve if and only if U is contained in the *union* of open sets in S. The above specification fulfills the requirements of covering sieves posed above, and consequently, defines a topological covering system on $\mathcal{O}(X)$.

Obviously, a categorical covering system, i.e. a Grothendieck topology $\mathbf{J}$ exists as a *presheaf* functor $\mathbf{\Omega}_{\chi}$ in $Sets^{\mathcal{B}^{op}}$, such that: by acting on contexts B in $\mathcal{B}$, $\mathbf{J}$ gives the set of all covering B-sieves, denoted by $\mathbf{\Omega}_\chi(B)$, whereas by acting on figures $h: C \to B$, it gives a morphism $h^*(-): \mathbf{\Omega}_\chi(B) \to \mathbf{\Omega}_\chi(C)$, expressed as:

$$h^*(S) = \{f \,/\, cod(f) = C, (h \circ f) \in S\}, \text{ for } S \in \mathbf{\Omega}_\chi(B).$$

A small category $\mathcal{B}$ together with a Grothendieck topology $\mathbf{J}$, is called a *site*, denoted by, $(\mathcal{B}, \mathbf{J})$.

A *sheaf on a site* $(\mathcal{B}, \mathbf{J})$ is a *contravariant functor* $\mathbf{P}: \mathcal{B}^{op} \to Sets$, satisfying an *equalizer* condition, expressed, in terms of covering B-sieves S, as in the following diagram in **Sets**:

$$\prod_{f \circ g \in S} \mathbf{P}(dom(g)) \leftleftarrows \prod_{f \in S} \mathbf{P}(dom(f)) \xleftarrow{\;e\;} \mathbf{P}(B)$$

If the above diagram is an equalizer for a particular covering sieve S, we obtain that $\mathbf{P}$ satisfies the *sheaf* condition *with respect* to the *covering sieve* S. The theoretical advantage of the above relies on the fact that it provides a description of sheaves entirely in terms of objects of the category of presheaves.

A *Grothendieck topos* over the small category $\mathcal{B}$ is a category which is equivalent to the *category of sheaves* $Sh(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$. The site can be conceived as a system of generators and relations for the topos. We note that a category of sheaves $Sh(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$ is a full subcategory of the functor category of presheaves $Sets^{\mathcal{B}^{op}}$.

The basic properties of a Grothendieck topos are the following:

i It admits *finite projective limits*; in particular, it has a terminal object, and it admits fibered products.

ii If $(K_i)_{i \in I}$ is a family of objects of the topos, then the *sum* $\coprod_{i \in I} K_i$ exists and is *disjoint.*

iii There exist *quotients* by equivalence relations and have the *same* good properties as in the category of sets.

10.7 QUANTUM-CLASSICAL COMMUNICATION: MODULATION VIA BOOLEAN FRAMES

The appropriate mathematical structure associated with the modelling of events is an *ordered* structure. In the Hilbert space formulation of quantum mechanics, events are considered as *closed subspaces* of a separable, complex Hilbert space. In this case, the quantum event structure is identified with the *lattice* of closed subspaces of the Hilbert space, ordered by inclusion, and carrying an *orthocomplementation* operation which is given by the orthogonal complement of the closed subspaces. In consequence, the quantum event structure is modelled in terms of a complete, atomic, *orthomodular* lattice.

The quantum event structure is isomorphic to the *partial Boolean* algebra of closed subspaces of the Hilbert space of the system or, alternatively the partial Boolean algebra of *projection* operators of the system. It models the *event structure* of a quantum mechanical system, just as the event structure of a classical system is modelled in terms of a *Boolean algebra* isomorphic to the Boolean algebra of Borel subsets of the phase space of the system or, equivalently the Boolean algebra of characteristic functions on the Borel subsets of the phase space.

The notion of an *event* is considered to be equivalent to a *proposition* regarding the behaviour of a physical system. The quantum logical formulation of Quantum theory is based on the *identification* of propositions with projection operators on a complex Hilbert space. Furthermore, the order relations and the lattice operations of the lattice of quantum propositions are associated with the logical implication relation and the logical operations of conjunction, disjunction, and negation of propositions. In effect, a *non-classical*, globally non-Boolean logical structure is induced which has its origins in Quantum theory.

On the contrary, the propositional logic of classical mechanics is Boolean logic. This means that the class of models over which validity and associated semantic notions are defined for the propositions of classical mechanics is the class of *Boolean logic* structures. We stress that Boolean logic refers to a Boolean algebra of propositions in which the Boolean lattice operators of join, meet, and complement, correspond to the logical operations of disjunction, conjunction and negation respectively. Moreover, the ordering in the lattice is interpreted as a logical relation of implication between the propositions of the algebra, and also, 1 and 0 are used to denote the greatest and lower elements of the lattice respectively.

It is a standard practice pertaining to the quantum-logical formalizations of quantum event structures that, due to the identification of quantum events with projection operators on a complex Hilbert space, the derived non-Boolean lattice structure is contrasted with the Boolean lattice structure referring to classical events. Although this is indeed the case *globally*, the pertinent problem is that every *single* observed event in the quantum domain requires taking explicitly into account the *complete Boolean algebra* of projection operators, which *spectrally resolves* the observable that this event refers to. Most significantly, such a complete Boolean algebra bears the status of a logical structural *invariant* characterizing a *whole algebra* of observables commuting with the one in question, and thus, potentially may give rise to the same event.

In other words, a complete Boolean algebra of projections is the logical structural *invariant* of a *commutative subalgebra* of observables, meaning that it instantiates the *simultaneous* spectral resolution of all these commuting observables. Given that observables exist which *do not commute* with any particular commutative subalgebra of observables, we encounter a multiplicity of possible Boolean algebras of orthogonal projections, which play the role of *an invariant* only in the *context* of all commuting observables that can be *simultaneously* resolved spectrally by this invariant.

In sum, the conceptual peculiarity characterizing a quantum event structure pertains to the fact that in the quantum domain there is *no* such thing as a *unique* and *universal* logical structural *invariant* with respect to which all possible observables can be spectrally resolved simultaneously. On the contrary, there exists a *multiplicity* of *spectral invariants* associated with *commutative* subalgebras of observables. Therefore, although a quantum event structure is globally non-Boolean, it can be spectrally qualified *only* in terms of Boolean event structures *attached* to it, in their *function* as logical structural invariants of co-measurable families of observables. Since these spectral invariants are *not* global, we consider them local, where the *locality* requirement refers precisely to the physical *context* of all commuting observables that can be simultaneously resolved spectrally by an invariant of this form.

We conclude that a complete Boolean algebra of projection operators in its function as a local spectral invariant of a commutative subalgebra of quantum observables plays the role of a *Boolean frame* with respect to which a quantum event can be qualified, and thus, lifted to the *empirical* level. Equivalently, each local Boolean frame serves as the local *pre-conditioning invariant* logical structure for the *evaluation* of events of all co-measurable observables in this frame. Due to the *non-availability* of a global uniquely defined Boolean frame, we must by default take into account *all possible* local Boolean frames *together* with their interrelations. The crucial problem is whether it is possible to identify a *universal* way to specify a quantum event structure through the literal *adjunction* of local Boolean spectral invariants to it, objectified in terms of *local Boolean* logical frames.

The existence of a universal solution essentially renders the global orthomodular lattice structure of quantum events physically and empirically vacuous without the *gnomonic adjunction* of local spectral invariants to it, effecting the *quantum-classical* communication through observability and measurement. The role of each locally adjoined Boolean frame is the instantiation of a *partial* or *local structural congruence* with a Boolean event structure pertaining to a context of measurement. The multiplicity of applicable local Boolean frames effects the *filtration*, *percolation*, and *separation* of several *resolution* sizes and types of quantum observable *grain* depending on the character of the corresponding spectral orthogonal projections.

The *objective* of a universal solution is the *derivation* of the *non-directly* accessible *quantum kind* of event structure by means of *all* possible partial or local structural *congruences* with the *directly* accessible *Boolean kind* of event structure, forced by means of *adjoining local spectral invariants* as probing frames to the former. In this setting, the major role is subsumed by all possible structural *relations* allowed *among* the probing *Boolean frames*, the spectra of which may be disjoint or nested or overlapping and interlocking together non-trivially. It is the realization that distinct Boolean frames may have non-trivial intersections, or more generally, *non-trivial pullback* compatibility relations that it is at the heart of the *so* called "quantum paradoxes".

The *universality* of the required solution poses the need to formulate the problem in *functorial* terms, to *circumvent* dependence on the artificial choice of particular Boolean frames. In turn, it requires a category-theoretic framework of interpretation, based on the aforementioned notion of partial or local structural congruence, which *locally* allows the *conjugation* of a *quantum* event structure by means of the *spectral invariants* adjoined as frames to it.

A *Boolean categorical* event structure is a *small* category, denoted by $\mathcal{B}$, which is called the *category of Boolean event algebras*. The objects of $\mathcal{B}$ are *complete* Boolean algebras of events, and the arrows are the corresponding Boolean algebraic *homomorphisms*.

A *quantum categorical* event structure is a *locally small co-complete* category, denoted by $\mathcal{L}$, which is called the *category of quantum event algebras*. The objects of $\mathcal{L}$ are the non- directly accessible global *quantum event algebras*, and the arrows are the corresponding quantum algebraic *homomorphisms*.

We consider a *Boolean shaping* functor of $\mathcal{L}$, $\mathbf{M}:\mathcal{B}\to\mathcal{L}$, which is to say a *forgetful* functor assigning to each Boolean event algebra the underlying quantum event algebra and to each Boolean homomorphism the underlying quantum algebraic homomorphism. Because of the fact that an *opposite*-directing functor from $\mathcal{L}$ to $\mathcal{B}$ is *not* feasible, since a quantum event algebra *cannot* be realized within *any* Boolean event algebra, we seek for an *extension* of $\mathcal{B}$ into a *larger* categorical environment, where *any* such realization becomes possible.

This *extension* should conform to the intended physical semantics adopted in *adjoining* a multiplicity of Boolean spectral invariants to a quantum event algebra, objectified as probing Boolean frames of the latter. For this reason, it is necessary to extend the *probes* from the categorical level of $\mathcal{B}$ to the categorical level of *diagrams* in $\mathcal{B}$, such that the initial probes can be *embedded* in the latter extended category.

This is accomplished by means of the *Yoneda embedding* $\mathbf{y}:\mathcal{B}\rightarrow Sets^{\mathcal{B}^{op}}$, which constitutes the *free completion* of $\mathcal{B}$ under the adoption of *colimits of diagrams* of Boolean structure probes, that is, of spectral invariants to be *adjoined* on a quantum event algebra.

An object $\mathbf{P}$ of $Sets^{\mathcal{B}^{op}}$ is thought of as a *right action* of the category $\mathcal{B}$ on a set of observables, which is *partitioned* into a variety of Boolean *spectral kinds* parameterized by the Boolean event algebras B in $\mathcal{B}$. Such an action $\mathbf{P}$ is equivalent to the specification of a diagram in $\mathcal{B}$, simply considered as a $\mathcal{B}$-variable set, called a *presheaf of sets* on $\mathcal{B}$. For each probe B of $\mathcal{B}$, $\mathbf{P}$(B) is a set, and for each arrow $f:C\rightarrow B$, $\mathbf{P}(f):\mathbf{P}(B)\rightarrow\mathbf{P}(C)$ is a set-theoretic function such that if $p\in\mathbf{P}(B)$, the value $\mathbf{P}(f)(p)$ for an arrow $f:C\rightarrow B$ in $\mathcal{B}$ is called the *restriction* of p along f and is denoted by $\mathbf{P}(f)(p)=p\cdot f$.

Each Boolean probe B in $\mathcal{B}$ gives rise to a *contravariant representable* Hom-functor $\mathbf{y}^{B}:=\mathbf{y}[B]:=Hom_{\mathcal{B}}(-,B)$. This functor defines a $\mathcal{B}$-variable set on $\mathcal{B}$, represented by B. The functor $\mathbf{y}$ is a *full* and *faithful* functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$, i.e.:

$$\mathbf{y}:\mathcal{B}\rightarrow Sets^{\mathcal{B}^{op}},$$

giving rise to the *Yoneda embedding* $\mathcal{B}\infty\ Sets^{\mathcal{B}^{op}}$.

The category of *presheaves* of sets on Boolean probes, denoted by $Sets^{\mathcal{B}^{op}}$, has objects all *functors* $\mathbf{P}:\mathcal{B}^{op}\rightarrow Sets$, and morphisms all *natural transformations* between such functors, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$, meaning all the arrows are inverted. In the setting of the functor category $Sets^{\mathcal{B}^{op}}$ it becomes possible to *realize* a quantum event algebra L in $\mathcal{L}$ in terms of a *distinguished* presheaf of Boolean event algebras, which models the *spectral capacity* of the latter to act as locally invariant logical probing frames of a quantum event algebra. These Boolean frames of an L in $\mathcal{L}$ are objectified as L-targeting morphisms in $\mathcal{L}$:

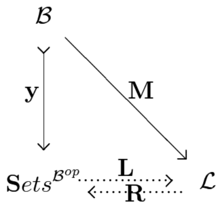
$$\psi_{B}:\mathbf{M}(B)\rightarrow L,$$

being *interrelated* by the operation of *restriction*. Explicitly, this means that for each Boolean *homomorphism* $f:C\rightarrow B$, if $\psi_{B}:\mathbf{M}(B)\rightarrow L$ is a

Boolean frame of L, the corresponding Boolean frame $\psi_C : \mathbf{M}(C) \to L$ is given by the restriction or pullback of ψ_B along f, denoted by $\psi_B \cdot f = \psi_C$. Thus, we obtain a contravariant presheaf functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$, called the *functor of Boolean frames* of L. Since the physical interpretation of the functor $\mathbf{R}(L)(-)$ refers to the functorial realization of a quantum event algebra L in $\mathcal{L}$ in terms of Boolean probes B in $\mathcal{B}$, we think of $\mathbf{R}_L(-)$ as the variable *Boolean spectral functor* of L through the Boolean frames adjoined to it as *local invariants.*

Due to the categorical *Yoneda Lemma*, an *injective* correspondence obtains between *elements* of the set $\mathbf{P}(B)$ and *natural transformations* in $Sets^{\mathcal{B}^{op}}$ from $\mathbf{y}[B]$ to $\mathbf{P}$ and this correspondence is natural in both $\mathbf{P}$ and B, for *every* presheaf of sets $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ and probe B in $\mathcal{B}$. The functor category of presheaves of sets on Boolean probes $Sets^{\mathcal{B}^{op}}$ is a complete and cocomplete category. Thus, the Yoneda embedding $\mathbf{y} : \mathcal{B} \to Sets^{\mathcal{B}^{op}}$ constitutes the free completion of $\mathcal{B}$ under colimits of diagrams of Boolean probes.

The significance of this boils down to the fact that, if we consider a Boolean shaping functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ there can be *precisely one unique*, up to isomorphism, *colimit-preserving functor* $\widehat{\mathbf{M}} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$, such that the following diagram commutes:



Consequently, *every* morphism from a Boolean probe B in $\mathcal{B}$ to a quantum event algebra L in $\mathcal{L}$ *factors uniquely* through the functor category $Sets^{\mathcal{B}^{op}}$ and the specification of the colimit-preserving functor $\widehat{\mathbf{M}} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$ is instrumental for understanding how the underlying structure of L in $\mathcal{L}$ *emerges* through communication, emerges in other words, through the *canonics* of adjoining Boolean probes as local spectral invariants to it.

More precisely, the functor $\widehat{\mathbf{M}}$ plays the role of a *left adjoint* $\mathbf{L}$, and thus serves as a *colimit-preserving* functor, from $Sets^{\mathcal{B}^{op}}$ to $\mathcal{L}$. Equivalently, the functor $\widehat{\mathbf{M}} := \mathbf{L}$ is the *left adjoint* of the categorical *adjunction* between the categories $Sets^{\mathcal{B}^{op}}$ and $\mathcal{L}$, where the *right adjoint* $\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}}$, is interpreted as the *realization functor* of $\mathcal{L}$ in terms of variable Boolean probing frames.

More specifically, the variable Boolean probes-induced *realization functor* of $\mathcal{L}$ in $Sets^{\mathcal{B}^{op}}$ is defined as follows:

$$\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}},$$

such that the contravariant presheaf functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ in the image of $\mathbf{R}$ into in $Sets^{\mathcal{B}^{op}}$, for a fixed L in $\mathcal{L}$, is the presheaf functor of Boolean frames of L.

We conclude that the problem of specification of a quantum event algebra L in $\mathcal{L}$ by means of *diagrams of Boolean probes* B has a *universal solution*, which is provided by the *left adjoint* functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$ to the *realization* functor $\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}}$. Equivalently, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit *inductive synthesis* of a quantum event algebra L in $\mathcal{L}$ by means of appropriate diagrams of Boolean probes in a functorial manner. Therefore, a quantum event algebra is essentially *generated*, and indirectly completely *specified*, by the canonics of *adjoining* locally invariant *Boolean spectral frames* thereto for the qualification and measurement of events, in short, through quantum-classical communication.

Technically speaking, a categorical adjunction pertains between the categories $Sets^{\mathcal{B}^{op}}$ and $\mathcal{L}$, called the *Boolean frames–quantum* adjunction. It is characterized by a pair of adjoint functors $\mathbf{L} - \mathbf{R}$ as follows:

$$\mathbf{L} : Sets^{\mathcal{B}^{op}} \rightleftarrows \mathcal{L} : \mathbf{R}$$

Thus, we obtain a *bijection*, which is *natural* in both $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ and L in $\mathcal{L}$:

$$Hom_{\mathbf{Sets}^{\mathcal{B}^{op}}}(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

abbreviated as follows:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

The "*category of elements*" of a presheaf plays an essential role in the canonics of this adjunction. We bear in mind that every presheaf functor $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ gives rise to a category, called the *category of elements* of $\mathbf{P}$ and denoted by $\int(\mathbf{P}, \mathcal{B})$.
The objects of this category are all pairs (B, p), and the morphisms $(B', p') \to (B, p)$ are those morphisms $u : B' \to B$ of the underlying category of probes $\mathcal{B}$, satisfying the condition that $p \cdot u = p'$, namely that the restriction, or the pullback of p along u is p'.
If we project onto the second coordinate of $\int(\mathbf{P}, \mathcal{B})$, we obtain a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \to \mathcal{B}$. Therefore, every presheaf functor $\mathbf{P}$ induces a *split discrete* and *uniform* fibration, where $\mathcal{B}$ is the *base* category of the fibration.
The fibers are categories in which the only arrows are identity arrows. If B is a Boolean probe in $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}(B)$, although its elements are written as pairs so as to form a disjoint union.

$$\begin{array}{ccc} \int(\mathbf{P}, \mathcal{B}) & & \\ \Big\downarrow {\scriptstyle \int_{\mathbf{P}}} & & \\ \mathcal{B} & \xrightarrow{\mathbf{P}} & \mathbf{Sets} \end{array}$$

A *natural transformation* τ between the presheaves $\mathbf{P}$ and $\mathbf{R}(L)$ on the category of Boolean probes $\mathcal{B}$, $\tau : \mathbf{P} \to \mathbf{R}(L)$ is equivalent to a *family* of compatible mappings between sets:

$$\tau_B : \mathbf{P}(B) \to Hom_{\mathcal{L}}(\mathbf{M}(B), L),$$

indexed by the probes B of $\mathcal{B}$. Each such mapping τ_B is identical with the following mapping:

$$\tau_B : (B, p) \to Hom_{\mathcal{L}}(\mathbf{M} \circ \int_{\mathbf{P}}(B, p), L).$$

Therefore, a natural transformation τ between the presheaves $\mathbf{P}$ and $\mathbf{R}(L)$ can be equivalently represented as a family of arrows of $\mathcal{L}$ targeting L, which is being indexed by the objects (B, p) of the *category of elements* of the presheaf $\mathbf{P}$, namely

$$\{\tau_B(p) : \mathbf{M}(B) \to L\}_{(B,p)}.$$

These arrows $\tau_B(p)$ considered *jointly* give rise to a *cocone* from the functor $\mathbf{M} \circ \int_{\mathbf{P}}$ to L. The categorical definition of a colimit determines that each such cocone emerges by the composition of the colimiting cocone with a *unique* arrow from the *colimit* $\mathbf{LP}$ to L. Equivalently, a bijection exists, which is natural in $\mathbf{P}$ and L:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L).$$

Hence, the Boolean probes-induced *realization* functor of $\mathcal{L}$, realized for each L in $\mathcal{L}$ by the presheaf of Boolean probing frames $\mathbf{R}(L) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ in $Sets^{\mathcal{B}^{op}}$, has a *left adjoint* functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf of sets $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ as the *colimit* $\mathbf{L}(\mathbf{P})$.

The pair of *adjoint* functors $\mathbf{L} - \mathbf{R}$ formalizes category-theoretically the *functorial* process of *encoding* and *decoding* information between diagrams of Boolean probes B and quantum event algebras L through the action of Boolean probing frames $\psi_B : \mathbf{M}(B) \to L$.

The existence of an adjunction between two categories always gives rise to a family of universal morphisms, called *unit* and *counit* of the adjunction, one for each object in the first category and one for each object in the second. Furthermore, every adjunction gives rise to an *adjoint equivalence* of certain subcategories of the initial functorially correlated categories. It is precisely this category-theoretic fact which determines the necessary and sufficient conditions for the *isomorphic* representation of a quantum event algebra L in $\mathcal{L}$ by means of suitably qualified functors, forming *sheaves* of Boolean probing frames.

For any presheaf $\mathbf{P}$ in the functor category $Sets^{\mathcal{B}^{op}}$, the *unit* of the adjunction is defined as follows:

$$\delta_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbf{RLP}.$$

On the other side, for each quantum event algebra L in $\mathcal{L}$ the *counit* is defined as follows:

$$Ú_L : \mathbf{LR}(L) \rightarrow L.$$

We conclude that the problem of establishing a *functorial representation* of a quantum event algebra in terms of Boolean logical probes has a *universal solution* in terms of quantum-classical natural communication, which is provided by the left adjoint functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$ to the Boolean realization functor $\mathbf{R} : \mathcal{L} \rightarrow Sets^{\mathcal{B}^{op}}$. In other words, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit articulation of quantum event algebras by means of suitably qualified diagrams of Boolean probing frames based on partial or local congruences between the Boolean and quantum kinds of event structure.

The *counit* natural transformation $Ú_L$ defines the spectral enunciation of a quantum event algebra L in $\mathcal{L}$ through *metaphora* by means of the *colimiting–interconnection* of Boolean probing frames of L, whose domains are *partially congruent* Boolean event algebras to L.

In more detail, the left adjoint functor of the Boolean frames–quantum adjunction, $\mathbf{L} : Sets^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$, is defined for each presheaf $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ as the colimit $\mathbf{L}(\mathbf{P})$. The functorial enunciation of a quantum event algebra L in $\mathcal{L}$ by means of the counit natural transformation requires an explicit calculation of the colimit $\mathbf{LR}(L)$ of the presheaf functor of Boolean probing frames of a quantum event algebra L. The corresponding category of elements $\int(\mathbf{R}(L), \mathcal{B})$ has objects all pairs (B, ψ_B), where B is a Boolean event algebra and $\psi_B : \mathbf{M}(B) \rightarrow L$ is a Boolean frame of L defined over B. The morphisms of $\int(\mathbf{R}(L), \mathcal{B})$, denoted by $(B', \psi_{B'}) \rightarrow (B, \psi_B)$, are those Boolean event algebra homorphisms

$u: B' \to B$ of the base category $\mathcal{B}$ for which $\psi_B \cdot u = \psi_{B'}$, that is, the restriction or pullback of the Boolean frame ψ_B along u is $\psi_{B'}$.

The pertinent calculation of the colimit $\mathbf{LR}(L)$ is simplified by the observation that we have an *underlying colimit-preserving* faithful functor from the category $\mathcal{L}$ to the category $Sets$. Thus, the sought colimit can be calculated by means of *set-valued equivalence classes*, or partition blocks in $Sets$, under the constraint that the derived set of equivalence classes are able to carry the *structure* of a quantum event algebra:

$$\mathbf{L}(\mathbf{R}(L)) = \mathbf{L}_{\mathbf{M}}(\mathbf{R}(L)) = Colim\{\int(\mathbf{R}(L), \mathcal{B}) \to \mathcal{B} \to \mathcal{L} \to \mathbf{Sets}\}.$$

The *indexing* category corresponding to the functor $\mathbf{R}(L)$ is the category of its elements $\mathcal{I} \equiv \int(\mathbf{R}(L), \mathcal{B})$, whence the functor $[\mathbf{M} \circ \int_{\mathbf{R}(L)}]$ defines the *diagram* $\mathcal{I} \to \mathcal{L}$ over which the colimit should be calculated. Since a colimit-preserving functor from the category $\mathcal{L}$ to $Sets$ exists, the sought *colimit* is equivalent to the definition of the *tensor product* $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ of the set valued functors:

$$\mathbf{R}(L): \mathcal{B}^{op} \to Sets, \qquad \mathbf{M}: \mathcal{B} \to Sets,$$

where the *contravariant* functor $\mathbf{R}(L)$ is considered a *right* $\mathcal{B}$*-module* and the *covariant* functor $\mathbf{M}$ a *left* $\mathcal{B}$*-module*, in analogia with the *algebraic* definition of the tensor product of a right $\mathcal{B}$-module with a left $\mathcal{B}$-module over a *ring* of coefficients $\mathcal{B}$. The above defines the *functorial* tensor product decomposition of the colimit in the category of elements of $\mathbf{R}(L)$ induced by the Boolean shaping functor $\mathbf{M}: \mathcal{B} \to \mathcal{L}$ of $\mathcal{L}$.

Therefore, for a Boolean probing frame $\psi_B \in \mathbf{R}(L)(B)$, $v: B' \to B$ and $q' \in \mathbf{M}(B')$ the *elements* of the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(\psi_B, q)$. This element can be written as:

$$\chi(\psi_B, q) = \psi_B \otimes q, \quad \psi_B \in \mathbf{R}(L)(B), q \in \mathbf{M}(B),$$

such that:

$$\psi_B \cdot v \otimes q' = \psi_B \otimes v(q'), \quad \psi_B \in \mathbf{R}(L)(B), q' \in \mathbf{M}(B'), v: B' \to B$$

We conclude that the set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ is actually the *quotient* of the set

$$\sum_{B}\mathbf{R}(L)(B)\times\mathbf{M}(B)$$

by the smallest *equivalence* relation generated by the above equations, whence the elements $\psi_B\otimes q$ of the quotient set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ are the *equivalence classes* of this relation. Since ψ_B denotes a probing Boolean frame of L and q denotes a projection operator $q\in\mathbf{M}(B)$, we conclude that the quotient set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ is a set of equivalence classes, or partition blocks of *pointed Boolean frames*, called *Boolean germs*.

Most important, this set can be naturally *endowed* with a *quantum* event algebraic structure by defining the *orthocomplementation* operator according to the assignment $(\psi_B\otimes q)^*:=\psi_B\otimes q^*$, and the *unit* element according to $\mathbf{1}:=\psi_B\otimes 1$. Notice that two equivalence classes in the quotient set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ can be *ordered* if and only if they have a *common refinement*. Consequently, the partial order structure is defined by the assignment,

$$\left(\psi_B\otimes q\right)\preceq\left(\psi_C\otimes r\right),$$

if and only if,

$$d_1\preceq d_2,$$

where we have made the following identifications,

$$(\psi_B\otimes q)=(\psi_D\otimes d_1)$$
$$(\psi_C\otimes r)=(\psi_D\otimes d_2),$$

with d_1, d_2 $\in$ $\mathbf{M}(D)$, according to the *pullback* diagram,

$$
\begin{array}{ccc}
\mathbf{M}(D) & \xrightarrow{\ \beta\ } & \mathbf{M}(B) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\
\mathbf{M}(C) & \xrightarrow{\ \lambda\ } & L
\end{array}
$$

such that $\beta(d_1) = q$, $\gamma(d_2) = r$, and $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$, $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$ denote the pullback of $\alpha : \mathbf{M}(B) \to L$ along $\lambda : \mathbf{M}(C) \to L$ in the category of quantum event algebras. Thus, the ordering relation between any two equivalence classes of pointed Boolean frames in the colimiting set $\mathbf{LR}(L) = \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the existence of pullback compatibility conditions between the corresponding Boolean frames.

We conclude that the *spectral constitution* of a quantum event algebra L in $\mathcal{L}$ through the Boolean frames–quantum *adjunction* is based on the *action* of the *endofunctor* $\mathbf{G}$ on $\mathcal{L}$, defined by:

$$
\begin{gathered}
\mathbf{G} := \mathbf{LR} : \mathcal{L} \to Sets^{\mathcal{B}^{op}} \to \mathcal{L} \\
L \to \mathbf{R}(L) \to \mathbf{LR}(L) \to L,
\end{gathered}
$$

which acts as the global spectral constitution endofunctor of a quantum categorical event structure $\mathcal{L}$ via Boolean probing frames.

In particular, for each quantum event algebra L in $\mathcal{L}$ the counit universal morphism of the Boolean frames-quantum adjunction evaluated at L is expressed in terms of equivalence classes, or partition blocks of pointed Boolean frames, that is, in terms of Boolean germs:

$$
\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \to L.
$$

Thus, the counit ϵ_L fits into the following diagram:

$$
\begin{array}{ccc}
\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} & & \\
\uparrow{\scriptstyle \psi_B \otimes (-)} & \searrow{\scriptstyle \epsilon_L} & \\
\mathbf{M}(B) & \xrightarrow{\ \psi_B\ } & L
\end{array}
$$

Accordingly, for every Boolean frame $\psi_B : \mathbf{M}(B) \to L$ the *projection* operator $q \in \mathbf{M}(B)$ is mapped to an *event* in L only through its *factorization* via the adjoined *Boolean germ* $\psi_B \otimes q$, i.e. through the partition spectral block of pointed Boolean frames it belongs to, according to:

$$\acute{\mathrm{U}}_L([\psi_B \otimes q]) = \psi_B(q), \qquad q \in \mathbf{M}(B).$$

This epitomizes the means of *communication* between the *Boolean* and the *quantum* structural kinds of event structure by means of germinal partial structural *congruence* through the canonics of the established adjunction. The crucial idea here is that the information encoded in the quantum structural kind can be accessed, decoded, qualified, and quantified, only indirectly in terms of *modulation* by Boolean germs, or partition spectral blocks of equivalent pointed Boolean frames.

In the same vein of ideas, it is revealing to examine the conditions that force the *counit* natural transformations of the *identity functor* in the category of quantum event algebras $\mathcal{L}$ into an *isomorphism*. The counit isomorphism expresses the property of *invariance* of a quantum event algebra under the two-step procedure of *encoding* in terms of appropriate families of Boolean event algebras through probing frames in $\mathbf{R}(L)$, and then *decoding* back by means of the action of the left adjoint on the former, denoted by $\mathbf{LR}(L)$.

Note that that if the *counit* evaluated at L is an *isomorphism*, then L can be considered as a *fixed* point of the corresponding global spectral constitution *endofunctor* of L through the action of Boolean probing frames. In general, the counit natural transformation $\acute{\mathrm{U}}_L$ is a natural isomorphism, if and only if the *right adjoint* functor of the Boolean frames–quantum adjunction is *full* and *faithful*, or equivalently, if and only if the *cocone* from the functor $\mathbf{M} \circ \int_{\mathbf{R}(L)}$ to L is *universal* for each L in $\mathcal{L}$. In the latter case, the functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ is characterized as a *dense* Boolean shaping functor.

It is worth specifying in more detail the necessary and sufficient conditions which force the counit $\acute{\mathrm{U}}_L$ to be an isomorphism. These conditions amount to the notion of *sheaf-theoretic localization* of L through the probing frames $\psi_B : \mathbf{M}(B) \to L$.

If the counit natural transformation $\acute{\mathrm{U}}_L$ at L is *restricted* to an isomorphism:

$$\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \cong L$$

then the quantum structural information of L is completely *decoded* in terms of its *adjoined* Boolean *germs*, that is, through spectral equivalence classes of pointed Boolean frames. This will rest on the satisfaction of appropriate conditions by restricted families of Boolean frames in $\mathbf{R}(L)$, distinguished qualitatively by their function as local Boolean *covers* of L.

The requirements qualifying Boolean frames as local Boolean covers of L are the following: First, they should constitute a minimal *generating* class of Boolean frames instantiating a *sieve*, i.e. a subfunctor of the functor of Boolean frames $\mathbf{R}(L)$ of L. Second, they should jointly form an *epimorphic family covering* L entirely on their overlaps. Third, they should be *compatible* under refinement, or more generally *pullback* operations in L. Fourth, they should be *transitive* such that subcovers of covers of L can be qualified as covers themselves.

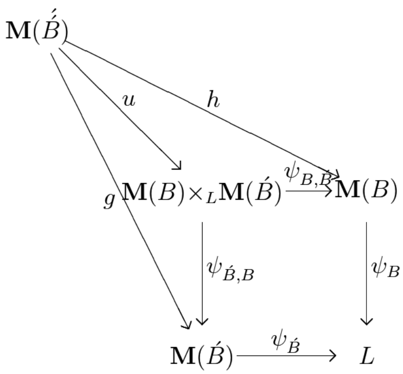
A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a *subfunctor* $\mathbf{T}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L, i.e. $\mathbf{T} \infty\ \mathbf{R}(L)$. For each Boolean algebra B in $\mathcal{B}$, a subfunctor $\mathbf{T} \infty\ \mathbf{R}(L)$ is equivalent to a *right ideal*, or equivalently a *spectral sieve* of quantum homomorphisms $\mathbf{T} \triangleright \mathbf{R}(L)$, defined by the requirement that, for each B in $\mathcal{B}$, the set of elements of $\mathbf{T}(B) \subseteq [\mathbf{R}(L)](B)$ is a set of Boolean frames $\psi_B : \mathbf{M}(B) \to L$ of $\mathbf{R}(L)(B)$, called *Boolean covers* of L, satisfying the following property:

If $[\psi_B : \mathbf{M}(B) \to L] \in \mathbf{T}(B)$, i.e. it is a Boolean cover of L, and $\mathbf{M}(v) : \mathbf{M}(B') \to \mathbf{M}(B)$ in $\mathcal{L}$ for $v : B' \to B$ in $\mathcal{B}$, then [$\psi_B \circ \mathbf{M}(v) : \mathbf{M}(B') \to \mathcal{L}] \in \mathbf{T}(B)$, i.e. it is also a Boolean cover of L}.

A *family* of Boolean covers $\psi_B : \mathbf{M}(B) \to L$, B in $\mathcal{B}$, is the *generator* of a *spectral sieve* of Boolean coverings $\mathbf{T}$, if and only if, this sieve is the *smallest* among all containing that family. The spectral sieves of Boolean coverings for an L in $\mathcal{L}$ constitute a *partially ordered* set under inclusion of subobjects. The minimal sieve is the empty one, namely $\mathbf{T}(B) = \varnothing$ for all B in $\mathcal{B}$, whereas the maximal sieve is the set of all probing Boolean frames of L for all B in $\mathcal{B}$, considered as Boolean covers.

We recall that the ordering relation between any two equivalence classes of pointed Boolean frames in the colimiting set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the property of *pullback* compatibility between the corresponding Boolean frames. Therefore, if we consider a functor of Boolean coverings $\mathbf{T}$ for a quantum event algebra L, we require that the generating family of Boolean covers they belong to is compatible under pullbacks.

The *pullback* of the Boolean covers $\psi_B : \mathbf{M}(B) \to L$ and $\psi_{B'} : \mathbf{M}(B') \to L$, where B and B' are Boolean event algebras in $\mathcal{B}$, with common codomain a *quantum* event algebra L, consists of the Boolean cover $\mathbf{M}(B) \times_L \mathbf{M}(B')$, together with the two *projections* $\psi_{BB'}$ and $\psi_{B'B}$, as shown in the diagram:



If the Boolean probing frames ψ_B and $\psi_{B'}$ are *injective*, then their pullback is given by their *intersection*. Next, we define the pairwise *gluing isomorphism* of the Boolean probing frames ψ_B and $\psi_{B'}$, as follows:

$$\Omega_{B,B'} : \psi_{B'B}(\mathbf{M}(B) \times_L \mathbf{M}(B')) \to \psi_{BB'}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$$
$$\Omega_{B,B'} = \psi_{BB'} \circ \psi_{B'B}{}^{-1}$$

From the previous definition, we derive the following *cocycle conditions*:

$$\Omega_{B,B} = id_B,$$
$$\Omega_{B,B'} \circ \Omega_{B',B''} = \Omega_{B,B''},$$
$$\Omega_{B,B'} = \Omega^{-1}{}_{B',B}$$

where, in the first condition id_B denotes the identity of $\mathbf{M}(B)$, in the second condition $\psi_B \times_L \psi_{B'} \times_L \psi_{B''} \neq 0$, and in the third condition $\psi_B \times_L \psi_{B'} \neq 0$.

Thus, the gluing isomorphism between any two Boolean frames of a spectral sieve $\mathbf{T}(L)$ assures that $\psi_{B'B}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$ and $\psi_{BB'}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$ probe L on their common refinement in a *compatible* way. This provides the sought-after *criterion* for the indirect *isomorphic* representation of a quantum event algebras in terms of a *spectral sieve* $\mathbf{T}(L)$ adjoined to it, under the proviso that the family of all Boolean covers $\psi_B : \mathbf{M}(B) \to L$, for variable B in $\mathcal{B}$, generating this spectral sieve jointly form an epimorphic family covering L completely:

$$T_L : \sum_{(B_j, \psi_j : \mathbf{M}(B_j) \to L)} \mathbf{M}(B_j) \twoheadrightarrow L,$$

where T_L is an *epimorphism* in $\mathcal{L}$ with codomain a quantum event algebra L.

A sieve *adjoined* to a quantum event algebra L is a Boolean *localizing* spectral sieve of L, or equivalently a functor of *Boolean localizations* of L, if and only if it is *closed* with respect to an *epimorphic* family of Boolean covers of L and the above *cocycle* conditions are satisfied. The conceptual significance of a Boolean localizing spectral sieve of L lies in the fact that the functor of Boolean probing frames $\mathbf{R}(L)$ becomes a *structure sheaf* of local Boolean frames when restricted to it. Then, for a dense epimorphic generating family of Boolean covers in a Boolean localization functor $\mathbf{T}$ of L, the counit of the Boolean frames-quantum adjunction is restricted to a quantum algebraic isomorphism, that is at once structure-preserving, injective and surjective.

In turn, the right adjoint functor of the adjunction restricted to a Boolean localization functor is full and faithful. This argument is formalized more precisely in *topos*-theoretic terminology by means of the *subcanonical* Grothendieck topology consisting of epimorphic families of covers on the base category of Boolean event algebras. Consequently, $\mathcal{L}$ becomes a *reflection* of the topos of variable local Boolean frames $Sets^{\mathcal{B}^{op}}$, and the structure of a quantum event algebra L in $\mathcal{L}$ is *preserved* by the action of a family of Boolean frames if and only if this family forms a *Boolean localization functor* of L. In this case, *any* compatible family of

Boolean frames ψ_j in the localized structure sheaf has a *unique amalgamation*, in the sense that there exists a unique colimiting Boolean frame $\psi : \mathbf{T}(L) \otimes_{\mathcal{B}} \mathbf{M} \to L$, such that the restriction of ψ along u_j gives ψ_j, that is $\psi \cdot u_j = \psi_j$.