CANONICS OF FUNCTORIAL RELATIONS: SIEVES AND REPRESENTABILITY

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9.1 FUNCTORIAL COORDINATIZATION: CATEGORIES OF RINGS OF OBSERVABLES

The first fundamental notion, which is necessary for the intelligibility of the modeling process of any natural system's behavior is representation. *Representation* is expressed categorically as a canonical functorial relation for formulating the bidirectional process of categorical structural correspondence between *natural systems* and *formal symbolic systems*. The concept of a natural system refers to a deliberate abstraction that schematizes the external world in terms of a dyadic relationship between the former and its environment.

Hence, it is suitable to adopt a *flexible*, non-rigid notion of what a natural system actually is which, eventually, allows a *variability* in the determination of the separating *boundary* between what constitutes a natural system and what its environment. The essential aspect of this relation amounts to representing a natural system, conceived in the above sense, by means of an appropriate formal system, capable of being effectively used for providing predictions about the behavior of the former. The notion of a formal system is formulated in the algebraic terms of *rings* capturing the structure of attributes of natural systems, *encoded* by means of generalized number-like quantities, called *observables*, where, the notion of observable signifies a physical attribute that, in principle, can be measured.

The basic hypothesis underlying the *bidirectional* process of representation of natural systems by mathematical formal systems, is that the behavior of the former can be adequately understood by establishing an appropriate *functorial congruence* between the structures of phenomena corresponding to the behavior of natural systems and some suitable algebraic structures of generalized number-like quantities. The latter are enriched with the semantics of observable attributes of natural systems in various measurement situations. Hence, the essential aspect of representation amounts to a structure preserving process of *coordinatization* or *arithmetization* of attributes in terms of observables. The strategy of arithmetization will prove to be successful, if and only if, a formal system becomes capable of providing predictions about the behavior of the natural system which represents.

Of course, as a condition required to substantiate that possibility, observables must amount to more than mere amorphous collections of coordinates; they must also be amenable to collective manipulations under the fundamental arithmetic operations of addition and multiplication, such that, the collection remains *closed* under these operations, preserving the algebraic *morphe* of such a *structure* of observables. Most significantly, the form of coordinatization of phenomena in the above sense, should not depend structurally on any particular natural system depicted in the external world. This means, that the form of the arithmetization process, effectuated in terms of an appropriate associatively closed universe (category) of algebraic structures of observables, should be constructed *covariantly* with respect to a corresponding associatively closed universe (category) of natural systems abstracted from the external world.

Categorically speaking, we mean that *arithmetization* should be properly expressed as a *functorial* process. More specifically, arithmetization is defined as a *functor* from the abstract category of natural systems to a concrete category of formal systems, identified algebraically, with structures of observables closed under the operations of addition and multiplication (rings). Note that, the notion of an observable denotes a physical attribute that, in principle, can be measured. The quantities admissible as measured results should belong to a number field, or more generally, to a ring of scalar quantities to be interpreted as generalized numbers in some context of observation. The latter defines the *valuation* codomain of observables in terms of measurement *scales*, conforming to the algebraic specification of, *at least*, the *ring*-theoretic structure. Note also that, the designation of both, the observables and the measurement scales, should be *structurally* on an *equal* footing, meaning that they should both be algebraic structures (rings) in a *category* of objects of the same kind (category of rings).

In this sense, the process of measurement is precisely formalized by means of a *surjective* morphism of rings, within the corresponding category of formal systems. It is important to note that, from this perspective, an abstract ring of number-like elements (scalars) simultaneously incorporates a *dual role* within the categorical environment where it operates. More concretely, if it stands on the *domain* of a surjective morphism of rings it is interpreted as a closed algebraic structure of observables *measured* by means of a closed algebraic structure of the same form consisting of measurement scales or numbers in the codomain of this morphism, within that category.

Equally, if it stands on the *codomain* of the rings morphism, it is interpreted as an algebraic structure of number-like values, which *measure* the observables belonging in the domain-ring. The essence of this *duality* of roles amounts to a *symmetrization* of the notions of *observer* and *observed* in an algebraic categorical framework of reasoning based primarily on the concept of structure preserving transformation between objects.

From this perspective, the *same* algebraic object can serve simultaneously, as *both* an observer and as an observed, *depending on* the *context* of interpretation of morphisms in the algebraic category emulating formal systems. In more detail, the positioning of a ring of number-like elements in the *codomain* of a surjective morphism acquires the semantics of an *observer*, that coordinatizes the observables of the system it measures in terms of distinguishable measurement scales. On the other hand, the positioning of that ring in the *domain* of a surjective morphism in the algebraic category, acquires the semantics of an *observed*, measured by means of evaluation of its observable attributes at the measurement scales of the codomain-ring.

In this sense, we are able to capture the natural duality between the symmetric functional roles of observer-observed in a categorical context of interpretation of surjective algebraic morphisms of rings. According to this relational categorical framework, the description of attributes related to the behavior of a natural system should be *covariant* with respect to *all* rings of measurement scales implemented as observational means, or equivalently, covariant with respect to the base ring (of observer's measurement scales) *change*.

We conclude that, arithmetization or coordinatization constitutes a directed functorial process $\mathbf{E}: \mathcal{S} \to \mathcal{F}$ of *encoding* attributes related to an abstract category of natural systems $\mathcal{S}$, in terms of a concrete category of formal systems $\mathcal{F}$, constructed algebraically as a category of rings of scalar number-like quantities (coordinates), interpreted as observables of natural systems, which can be measured by means of surjective morphisms (evaluations) in the latter algebraic category. We say that the modeling of the abstract category of natural systems by the concrete category of formal systems is proper, if an *inverse* functorial process $\mathbf{D}: \mathcal{F} \to \mathcal{S}$ of *decoding* exists, which can be used for making predictions about the behavior of natural systems, such that, the pair of functorial processes $(\mathbf{E}, \mathbf{D})$ constitutes a *categorical adjunction*:

$$\mathbf{E}: \mathcal{S} \rightleftarrows \mathcal{F}: \mathbf{D}$$

Thus, the bidirectional functorial process of representation of natural systems by formal systems is defined accurately as a proper categorical relation of *metaphora*, according to the previous adjunction. Finally, the encoding arithmetization functor $\mathbf{E}: \mathcal{S} \to \mathcal{F}$ is said to define an *equivalence* between the categories of natural systems and formal systems respectively, if a decoding functor $\mathbf{D}: \mathcal{F} \to \mathcal{S}$ exists, such that the composite functor $\mathbf{DE}$ is isomorphic to $\mathbf{Id}_S$ and the composite functor $\mathbf{ED}$ is isomorphic to $\mathbf{Id}_F$.

9.2 REPRESENTABLE FUNCTOR: MEASUREMENT AND COMMUNICATION

Let us now concentrate our attention on the category of formal systems $\mathcal{F}$, in an attempt to describe explicitly its representational functionality. The category $\mathcal{F}$ is defined as a category of unital rings of observables that coordinatize structurally the attributes of natural systems. The observables are measured by means of *evaluation* morphisms in appropriately specified rings of measurement *scales* in the same category $\mathcal{F}$. A ring in $\mathcal{F}$ is *commutative* if and only if the multiplication is commutative.

We will restrict ourselves intentionally to the case of commutative rings of scalars. This is due to the fact that the hypothesis of commutativity of the observable attributes of a natural system can be satisfied at least *locally* within the categorical environment for all natural systems, even if, globally their rings of observables are only *partially* commutative, or even *non-commutative*.

For the sake of completeness, we should recall that, a ring is a division ring, if and only if, every non-zero element has a multiplicative inverse, whereas, in case that it is also commutative, it is called a *field*. Moreover, a commutative ring without non-trivial divisors of zero is called an *integral domain*. According to our previous comments, the morphisms in $\mathcal{F}$ are considered to be the additive and multiplicative identity-preserving homomorphisms of commutative rings. The surjective morphisms in $\mathcal{F}$ formalize the structure preserving process of measurement of observables (contained in the domain of a rings-morphism) with respect to observers' measurement scales (contained in the codomain respectively), according to the preceding physical interpretation.

The essential aspect of casting the measurement process in a categorical form, as above, is twofold: Firstly, the categorical framework provides the means to state precisely a *criterion* of complete determination of a natural system's behavior, formulated in terms of its coordinatization ring of observables, by way of measurements in variable rings of measurement scales, according to the Yoneda-Grothendieck philosophy. Secondly, the same framework, seen from a *dual* categorical perspective, permits the *geometric* encoding of the information acquired by evaluating a physical attribute at a measurement scale, in terms of the notion of *state*. Thus, it becomes possible to conceptually and technically unify, through duality transformations in the category of formal systems, the *algebraic information* acquisition by measuring observables at scales, together with, the *geometric representation* of that information in terms of state-spaces.

The covariant description of the measurement process of natural systems in relation to the above two-fold interpretational schema renders a functorial formulation of that process. In turn, this is to be achieved using the machinery of *representable* functors. More precisely, we consider the category of formal systems restricted to surjective morphisms of unital commutative rings of observables, and let $\mathcal{A}$ be a ring of observables of a corresponding natural system. Then, the functor *represented* by $\mathcal{A}$ is the *covariant* functor $\mathbf{y}_{\mathcal{A}}:\mathcal{F}\to\mathbf{Sets}$, defined as follows:

i For all rings $\mathcal{X}$ in $\mathcal{F}$, $\mathbf{y}_{\mathcal{A}}(\mathcal{X}):=Hom_{\mathcal{F}}(\mathcal{A},\mathcal{X})$.

ii For all rings-morphisms $f:\mathcal{X}\to\mathcal{Y}$ in $\mathcal{F}$,

$$\mathbf{y}_{\mathcal{A}}(f):Hom_{\mathcal{F}}(\mathcal{A},\mathcal{X})\to Hom_{\mathcal{F}}(\mathcal{A},\mathcal{Y})$$

is defined as post-composition with f, viz., $\mathbf{y}_{\mathcal{A}}(f)(g):=f\circ g$.

The covariant representable functor $\mathbf{y}_{\mathcal{A}}:\mathcal{F}\to\mathbf{Sets}$, can be thought of as constructing an image of $\mathcal{F}$ in $Sets$ covariantly, or equivalently, as a covariant translation of the information induced by measurement procedures in the category $\mathcal{F}$ into that of $Sets$. The important thing to notice is that, if we set up some measurement procedure of the observables of a natural system, represented formally as a surjective morphism of rings $f:\mathcal{A}\to\mathcal{B}$, then we get an *induced* morphism of representable functors (*natural transformation*) $\mathbf{y}_f:\mathbf{y}_{\mathcal{B}}\to\mathbf{y}_{\mathcal{A}}$ by pre-composing with f.

Subsequently, the implementation of the Yoneda lemma in the current setting, gives that, if $\eta:\mathbf{y}_{\mathcal{B}}\to\mathbf{y}_{\mathcal{A}}$ is *any* natural transformation of covariant representable functors, then, there is a measurement procedure, a morphism of rings $f:\mathcal{A}\to\mathcal{B}$, such that, $\eta=\mathbf{y}_f$. This implies, in particular, that if two rings of observables represent *isomorphic functors*, then the corresponding formal systems themselves are *isomorphic*.

The importance of this fact, constituting the cornerstone of the Yoneda-Grothendieck philosophy, is that a formal system conceived within the associatively closed structure of a category of objects and structure-preserving morphisms of the same form $\mathcal{F}$, encoded algebraically as a ring of observables of a natural system, can be *classified* completely up to *unique* isomorphism, by analyzing the set-valued functor that it *represents*. More precisely, the information encoded in the ring of observables of a natural system, can be *recovered* completely by means of *all* measurement procedures applied to that ring, incorporated in the definition of the corresponding representable functor as ring-morphisms from that ring of observables towards variable rings of measurement scales of observers within the same category.

Consequently, this realization means that the behavior of a corresponding natural system within $\mathcal{S}$, can be reproduced in its entirety, by studying the totality of *variable* observational perspectives imposed upon it within $\mathcal{S}$, or equivalently, the totality of its interactions with all other natural systems within $\mathcal{S}$. Note that, the notion of variable observational perspective on a natural system in $\mathcal{S}$, is operationally realized by the *instantiation* of measurement procedures of the coordinatization ring of observables $\mathcal{A}$ of that natural system with respect to variable rings of measurement scales in $\mathcal{F}$, and consequently, modeled by means of the corresponding *covariant representable* functor $\mathbf{y}_{\mathcal{A}} : \mathcal{F} \to \mathbf{Sets}$. In the same vein of ideas, we may say conclusively that, the abstraction related to the notion of a natural system, within a closed categorical environment $\mathcal{S}$, is completely understood by the system of all *relational* referential viewpoints on it (variable observational perspectives) instantiated within $\mathcal{S}$, encoded as variable measurement procedures of its coordinatization ring of observables with respect to variable rings of measurement scales within the modeling category of formal systems $\mathcal{F}$. This is finally translated covariantly in $Sets$ via the corresponding representable functor of that ring of observables.

9.3 FUNCTORIAL SPECTRUM: SIEVES AND MULTI-LAYERED RESOLUTION

At a further stage of development, the operational role of the covariant representable functor of a ring of observables, classifying it completely up to unique isomorphism, is equivalent to the functorial process of translating geometrically the information collected by all evaluations of that ring at all rings of measurement scales, in sum, the information collected by the totality of variable measurement procedures as above. In this manner, the *geometric encoding* of the information related to the behavior of a natural system is being generated *functorially* through the functioning of the covariant representable functor of the ring of observables arithmetizing the former.

The key concept that explicates the geometric representation of natural systems in appropriate functorial terms is the notion of state. Subsequently, the geometric representation of natural systems in terms of state-spaces is generated functorially. Generally speaking, a *state* of a ring of observables $\mathcal{A}$ *over* a ring of measurement scales $\mathcal{B}$ (called a $\mathcal{B}$-state of $\mathcal{A}$) signifies the geometric encoding of the information acquired by evaluating the physical attributes contained in the former ring at the measurement scales of the latter. Thus, any $\mathcal{B}$-state of $\mathcal{A}$ is a geometric representation of a morphism of rings $f : \mathcal{A} \to \mathcal{B}$ in the intended semantic interpretation.

Consequently, the *set* of *all* morphisms of rings $f : \mathcal{A} \to \mathcal{B}$, where the domain ring is considered as a ring of observables and the codomain ring as a ring of measurement scales, should be tautosemous with the set of all $\mathcal{B}$-states of $\mathcal{A}$. This set is called the $\mathcal{B}$-*spectrum* of the unital ring of observables $\mathcal{A}$, where the ring $\mathcal{B}$ is called the *coordinatizing frame* of that state. The geometric semantics of this connotation denotes the set of elements which can be $\mathcal{B}$-observed by a measurement procedure on $\mathcal{A}$. Eventually, that set of elements, constituting the $\mathcal{B}$-spectrum of $\mathcal{A}$, are properly identified with the $\mathcal{B}$-*coordinatized points* of a geometric state-space that can be observed by means of the ring $\mathcal{A}$. Obviously, the above conception of a geometric state-space of a natural system is *functorial,* since it admits a *covariant* description with respect to variation of the evaluation ring of measurement scales $\mathcal{B}$ in the category $\mathcal{F}$. This fact effectively means that, the geometric state space of a natural system is identified with the covariant representable functor $\mathbf{y}_{\mathcal{A}} : \mathcal{F} \to \mathbf{Sets}$ of the coordinatization ring of observables of that natural system.

For this reason, $\mathbf{y}_{\mathcal{A}}$ is called the *functorial spectrum* of $\mathcal{A}$, denoted by $\mathbf{S}pec_{\mathcal{A}}$, which gives rise to a *spectral sieve* of $\mathcal{A}$. It describes functorially the *multi-layered* geometric state-space related to the behavior of a natural system under *variable* observational perspectives, where its evaluation at a layer $\mathcal{B}$, constituting the $\mathcal{B}$-spectrum of the ring of observables $\mathcal{A}$, gives the set of all $\mathcal{B}$-states of $\mathcal{A}$. In this vein of ideas, each ring of measurement scales, where the evaluation of a ring of observables takes place, by the effectuation of a corresponding measurement procedure, is the *spectral carrier* of a specific geometric *layer of spatiality*, corresponding to the ontological observational perspective of *point-schematization* dictated by the nature of measurement scales contained in that ring.

9.4 RELATIVIZATION: MODULES AND REPRESENTABILITY OF COMPOSITION

A natural question arising in this categorical setting is the following: Is it possible to express the notion of a *module* of a commutative unital ring of observables $\mathcal{A}$ in the category $\mathcal{F}$, *intrinsically* with respect to the information contained in the category $\mathcal{F}$? This can be accomplished by using the method of *categorical relativization*, which is based on the passage to the *slice* category $\mathcal{F} / \mathcal{A}$. More concretely, the basic problem has to do with the possibility of representing the information contained in an $\mathcal{A}$-module, where $\mathcal{A}$ is a commutative unital ring of observables in $\mathcal{F}$, by means of a suitable object of the relativization of $\mathcal{F}$ with respect to $\mathcal{A}$, namely with an object of the slice, or comma category $\mathcal{F}/\mathcal{A}$.

For this purpose, we define the *split extension* of the commutative ring $\mathcal{A}$ by an $\mathcal{A}$-module M, denoted by $\mathcal{A}\bigoplus M$, as follows: The underlying set of $\mathcal{A}\bigoplus M$ is the cartesian product $\mathcal{A}\times M$, where the group and ring theoretic operations are defined respectively as;

$$(a,m)+(b,n):=(a+b,m+n)$$
$$(a,m)\bullet(b,n):=(ab,a\cdot n+b\cdot m)$$

Note that the identity element of $\mathcal{A}\bigoplus M$ is $(1_{\mathcal{A}},0_M)$, and also that, the split extension $\mathcal{A}\bigoplus M$ contains an *ideal* $0_{\mathcal{A}}\times M:=\langle M\rangle$, that corresponds *naturally* to the $\mathcal{A}$-module M. Thus, given a commutative ring $\mathcal{A}$ in $\mathcal{F}$, the information of an $\mathcal{A}$-module M, consists of an object $\langle M\rangle$ (*ideal* in $\mathcal{A}\bigoplus M$), together with a *split short exact sequence* in $\mathcal{F}$;

$$\left\langle M\right\rangle \rightarrowtail \mathcal{A}\oplus M \twoheadrightarrow \mathcal{A}$$

We infer that the ideal $\langle M\rangle$ is identified with the *kernel* of the epimorphism $\mathcal{A}\oplus M\twoheadrightarrow\mathcal{A}$:

$$\left\langle M\right\rangle = Ker(\mathcal{A}\oplus M\twoheadrightarrow\mathcal{A})$$

From now on we focus our attention to the comma category $\mathcal{F}/\mathcal{A}$, noticing that $id_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}$ is the *terminal* object in this category. If we consider the split extension of the commutative ring of observables $\mathcal{A}$, by an $\mathcal{A}$-module M, that is $\mathcal{A}\bigoplus M$, then the morphism:

$$\lambda:\mathcal{A}\bigoplus M\to\mathcal{A}$$
$$(a,m)\mapsto a$$

is obviously an object of $\mathcal{F}/\mathcal{A}$. Moreover, it easy to show that it is actually an *Abelian group* object in the comma category $\mathcal{F}/\mathcal{A}$. This equivalently means that for every object ξ in $\mathcal{F}/\mathcal{A}$ the set of morphisms $Hom_{\mathcal{F}/\mathcal{A}}(\xi,\lambda)$ is an Abelian group in $\mathbf{Sets}$. Moreover, the arrow $\gamma:\kappa\to\lambda$ is a morphism of Abelian groups in $\mathcal{F}/\mathcal{A}$, if and only if for every ξ in $\mathcal{F}/\mathcal{A}$ the morphism:

$$\hat{\gamma}_{\xi}:Hom_{\mathcal{F}/\mathcal{A}}(\xi,\kappa)\to Hom_{\mathcal{F}/\mathcal{A}}(\xi,\lambda)$$

is a morphism of Abelian groups in $\mathbf{Sets}$. We denote the *category of Abelian group objects* in $\mathcal{F}/\mathcal{A}$ by the suggestive symbol $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}$. Based on our previous remarks, it is straightforward to show that the category of Abelian group objects in $\mathcal{F}/\mathcal{A}$ is *equivalent* with the *category* of $\mathcal{A}$*-modules*:

$$[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}\cong\mathcal{M}^{(\mathcal{A})}$$

Thus, we have managed to characterize $\mathcal{A}$-modules intrinsically as Abelian group objects in the *relativization* of the category of commutative unital rings of observables $\mathcal{F}$ with respect to $\mathcal{A}$, and moreover, we have concretely identified them as kernels of split extensions of $\mathcal{A}$.

Let us now consider two $\mathcal{A}$-modules M, N. The *tensor product* of M and N over $\mathcal{A}$, denoted by $M\bigotimes_{\mathcal{A}}N$ is the *unique* object in $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}\cong\mathcal{M}^{(\mathcal{A})}$, which satisfies the following *universal* property: There exists a *bilinear* morphism $\tau:M\times N\to M\bigotimes_{\mathcal{A}}N$ such that given *any* $\mathcal{A}$*-module* S and *any bilinear* morphism $f:M\times N\to S$ there exists a *unique* $\mathcal{A}$-modules (*linear*) morphism $g:M\bigotimes_{\mathcal{A}}N\to S$ such that the bilinear morphism admits the *factorization* $f=g\circ\tau$. Consequently, factorization via the tensor product of two $\mathcal{A}$-modules M, N, that is, $M\bigotimes_{\mathcal{A}}N$, should be understood as the *universal* way to *linearize* any bilinear morphism form $M\times N$ to any other $\mathcal{A}$-module S. Note that, since the tensor product construction is defined by a universal mapping property it is *unique* up to unique isomorphism, as a consequence of Yoneda's lemma.

The tensor product of two $\mathcal{A}$-modules M, N, viz. $M\bigotimes_{\mathcal{A}}N$, is *generated* by elements of the form $m\otimes n$. This means that every element of $M\bigotimes_{\mathcal{A}}N$ is of the form $\sum_{i=1}^{k}m_i\otimes n_i$, where $m_i\in M$, $n_i\in N$ for $1\le i\le k$. Moreover, if M, N are *free* $\mathcal{A}$-modules of *finite* rank m,

n correspondingly, then their tensor product over $\mathcal{A}$, $M \bigotimes_{\mathcal{A}} N$ is a *free* $\mathcal{A}$-module of finite rank mn.

The tensor product in the category $\mathcal{M}^{(\mathcal{A})}$ is an *associative*, *commutative* and *unital* operation, that is:

i If M, N, P are $\mathcal{A}$-modules, then:

$$(M \bigotimes_{\mathcal{A}} N) \bigotimes_{\mathcal{A}} P = M \bigotimes_{\mathcal{A}} (N \bigotimes_{\mathcal{A}} P)$$

ii If M, N are $\mathcal{A}$-modules, then:

$$M \bigotimes_{\mathcal{A}} N = N \bigotimes_{\mathcal{A}} M$$

iii The commutative ring $\mathcal{A}$ is a *unit* for the tensor product, that is, for any $\mathcal{A}$-modules M:

$$\mathcal{A} \bigotimes_{\mathcal{A}} M = M \bigotimes_{\mathcal{A}} \mathcal{A} = M$$

Most importantly, the tensor product is a *functorial* operation, meaning that if M is an $\mathcal{A}$-module, then:

$$M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$$

is a *functor*, defined on objects as:

$$(M \bigotimes_{\mathcal{A}} -)(N) = M \bigotimes_{\mathcal{A}} N$$

and on morphisms $f : N \to L$ in $\mathcal{M}^{(\mathcal{A})}$ as follows:

$$(M \bigotimes_{\mathcal{A}} -)(f) = 1 \otimes f : M \bigotimes_{\mathcal{A}} N \to M \bigotimes_{\mathcal{A}} L$$

It is also defined on generators by:

$$(1 \otimes f)(m \otimes n) = m \otimes f(n)$$

The functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ is *right exact*. More precisely, if we consider a *short exact sequence* of $\mathcal{A}$-modules:

$$0 \to N_1 \to N \to N_2 \to 0$$

then, *tensoring* by M over $\mathcal{A}$, we get the *right exact sequence* of $\mathcal{A}$-modules:

$$M \bigotimes_{\mathcal{A}} N_1 \to M \bigotimes_{\mathcal{A}} N \to M \bigotimes_{\mathcal{A}} N_2 \to 0$$

Note that, in general, the above sequence of $\mathcal{A}$-modules is *not left exact* as well. In case that the tensor product functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ is *exact* (meaning that it is left exact as well) then the $\mathcal{A}$-module M is called a *flat* module.

The conceptual meaning of the tensor product functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ and consequently of the tensor product operation on the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$ arises if we consider the *covariant representable functor* in the category $[\mathcal{F} / \mathcal{A}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{A})}$ valued in the same category, that is the covariant functor:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, -) : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$$

represented by the $\mathcal{A}$-module M, defined as follows:

i For all $\mathcal{A}$-modules N in $\mathcal{M}^{(\mathcal{A})}$, the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor maps N to the $\mathcal{A}$-module $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, N)$.

ii For all $\mathcal{A}$-modules $f : N \to P$ in $\mathcal{M}^{(\mathcal{A})}$,

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(f) : \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, N) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, P)$$

is defined as post-composition with f, viz., $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(f)(g) := f \circ g$.

The advantage of considering the covariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, that is, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, -)$, is that, by virtue of the categorical equivalence $[\mathcal{F} / \mathcal{A}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{A})}$, we *retain* the physical interpretation of the $\mathbf{Hom}$-functor (enriched in the category of $\mathcal{A}$-modules) in terms of measurement procedures relativized with respect to a fixed ring of observables $\mathcal{A}$, according to the Yoneda-Grothendieck philosophy explained previously.

The covariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, viz. $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ is a *left exact* functor. More precisely, if we consider a short exact sequence of $\mathcal{A}$-modules:

$$0 \to N_1 \to N \to N_2 \to 0$$

then, applying the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor, we get the *left exact sequence* of $\mathcal{A}$-modules:

$$0 \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N_1) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N_2)$$

Note that, in general, the above sequence of $\mathcal{A}$-modules is *not* right exact as well. In case that the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor is exact (meaning that it is right exact as well) then the $\mathcal{A}$-module M is called a *projective* module. Analogously we may consider the contravariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, that is, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(,M)$. If the latter functor is *exact* then the $\mathcal{A}$-module M is correspondingly called an *injective* module. Another immediate observation is that:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(\mathcal{A},N) = N$$

The above properties of the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$, and the corresponding properties of the tensor product functor $(M \bigotimes_{\mathcal{A}} -)$ imply that they stand for *inverse functorial processes* in the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$. This reveals the presence of an *adjunction*, that is, $(M \bigotimes_{\mathcal{A}} -)$ and $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ are *adjoint functors*, described by the *natural isomorphism*:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M \bigotimes_{\mathcal{A}} N, P) \cong \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N,P))$$

for any $\mathcal{A}$-modules M, N and P, where the tensor product functor is the *left adjoint* and the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor is the *right adjoint* of the $\left[\otimes \rightleftarrows \mathbf{Hom}\right]_{\mathcal{M}(\mathcal{A})}$ adjunction.

Consequently, the physical meaning of the tensor product operation in the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})} \cong [\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}$ is obtained as follows: We consider the interpretation of the covariant $\mathbf{Hom}$-functors $\mathbf{Z}_M := \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ and $\mathbf{Z}_N := \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N,-)$ (enriched in the category of $\mathcal{A}$-modules) in terms of corresponding measurement procedures on the rings $\mathcal{A} \bigoplus M$ and $\mathcal{A} \bigoplus N$ relativized with respect to the fixed ring of observables $\mathcal{A}$.

Then, the *composition* of the covariant $\mathbf{Hom}$-functors $\mathbf{Z}_M$ and $\mathbf{Z}_N$, by which we mean the composition of the corresponding measurement procedures with respect to the fixed ring $\mathcal{A}$, becomes *representable* in the category of $\mathcal{A}$-modules by the tensor product *entanglement* of M and N, that is we have the *natural isomorphism*:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M \bigotimes_{\mathcal{A}} N, P) \cong \mathbf{Z}_M \circ \mathbf{Z}_N(P) = \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N, P))$$

9.5 COMPARISON OF THE FUNCTORIAL WITH THE CLASSICAL REPRESENTABILITY

At this stage, a brief discussion of the formal model of a natural system idealized by classical theories will serve to throw light on the connection with the categorical generalization implied by the previous analysis. The basic postulate of classical theories stipulates in advance that the form of observation be globally expressible by *real number representability*, and subsequently, observables are modeled by *real-valued* functions corresponding to measuring devices calibrated to register real numbers.

At a further stage of development of this idea, two further assumptions are imposed on the structure of observables: the first of them specifies the *algebraic* nature of the set of all observables used for the description of a natural system, by assuming the structure of a commutative unital ring, which is, a *commutative unital algebra* $\mathcal{A}$ over the *real* numbers. The second assumption restricts the content of the set of real-valued functions corresponding to physical observables to those that admit a mathematical characterization as *measurable*, *continuous* or *smooth*.

Thus, depending on the means of description of a physical system, observables are modeled by $\mathbb{R}$-algebras of measurable, continuous or smooth functions corresponding to suitably specifiable measurement environments in each case. Usually the assumption of *smoothness* is postulated because it is desirable to consider *derivatives* of observables and effectively set-up a *dynamical* framework of description in terms of *differential equations*. Moreover, since we have initially assumed that real-number representability constitutes the prefixed form of observation in terms of the readings of measuring devices, the set of all $\mathbb{R}$ -algebra unital morphisms $\mathcal{A} \to \mathbb{R}$, assigning to each observable in $\mathcal{A}$, the reading of a measuring device in $\mathbb{R}$, encapsulates all the *states-related* information collected about a system in measurement situations in terms of algebras of real-valued observables.

Mathematically, the set of all $\mathbb{R}$ -algebra morphisms $\mathcal{A} \to \mathbb{R}$ is identified as the $\mathbb{R}$ *-spectrum* of the unital commutative algebra of observables $\mathcal{A}$. The physical semantics of this connotation denotes the set that can be $\mathbb{R}$ *-observed* by means of this algebra. It is well known that, in case $\mathcal{A}$ stands for a smooth algebra of real-valued observables, $\mathbb{R}$ -algebra morphisms $\mathcal{A} \to \mathbb{R}$ can be legitimately identified (thanks to the *Gelfand duality*) with the $\mathbb{R}$ *-coordinatized points* of a space, which can be *observed* by means of $\mathcal{A}$. These are the points of a compact real differential *manifold* that, in turn, denote the states of the observed system. From this perspective, geometric state spaces in classical theories are compact real differential manifolds N consisting of sets of points being $\mathbb{R}$ -observed by means of unital $\mathbb{R}$ -algebras of smooth real-valued functions, denoted by $\mathcal{C}^{\infty}(N)$.

If we attempt a comparison of the functorial with the classical case, outlined above, we notice that, according to the generalized definition of a geometric state introduced previously, each state of a ring of observables $\mathcal{A}$ may have a *different coordinatizing frame*, depending upon the ring of scales employed for measurement. Thus, the *new* notion of a geometric state-space, is a *multi-valued* one, in the sense that its generalized points may be coordinatized by means of *different scales*, namely scales belonging to different rings. Thus, in contradistinction to the classical conception, the notion of generalized state introduced, *rejects* the *absolute representability* principle of the classical theory over the coordinatizing field of real numbers, rather to allow, in this sense, the *geometric* representation of states, in terms of *generalized* points of a *multi-valued* geometric state space, as well as, the *evaluation* of observable attributes at those points. This is achieved by the *relativization* of *representability* over a multitude of measurement scales, belonging to different coordinatizing rings, giving eventually rise to the above multi-valued geometric state space.

9.6 THE GEOMETRIC QUALIFICATION OF THE SPECTRUM FUNCTOR

We have already concluded that, in the context of the functorial approach to modeling a natural system's behavior, the geometric state space of a natural system is identified with the *covariant representable functor* $\mathbf{y}_A : \mathcal{F} \to \mathbf{Sets}$ of the coordinatization ring of observables of that natural system. For this reason, the functor $\mathbf{y}_A$ is called the *spectrum* of $\mathcal{A}$, and denoted as, $Spec_A$.

It is essential to examine the geometric semantics of the Spectrum functor in some detail. If we consider the opposite of the category of rings of observables, that is, the category with the same objects but with arrows *reversed* $\mathcal{F}^{op}$, each object in the context of this category can be thought of as the *locus of states* of a ring of observables, or else it carries the connotation of *space*. The crucial observation is that, any such space is determined, up to canonical isomorphism, if we know *all* morphisms into this locus from *any* other locus in that category. For instance, the set of morphisms from the one-point state locus to the locus A in the categorial context of $\mathcal{F}^{op}$ determines the set of point-states of the locus A. The philosophy behind this approach amounts to treating *any* morphism in $\mathcal{F}^{op}$ with the locus A for target as a *generalized point state* of A.

Let us consider the category of loci of states $\mathcal{F}^{op}$, and let A be an object in this category. Then, the functor *represented* by A is the *contravariant* functor $\mathbf{y}^A : \mathcal{F}^{op} \to \mathbf{Sets}$, defined as follows:

i For all loci of states B in $\mathcal{F}^{op}$, $\mathbf{y}^A(B) := Hom_{\mathcal{F}^{op}}(B, A)$.

ii For all loci-morphisms $f : C \to B$ in $\mathcal{F}^{op}$,

$$\mathbf{y}^A(f) : Hom_{\mathcal{F}^{op}}(B, A) \to Hom_{\mathcal{F}^{op}}(C, A)$$

is defined as pre-composition with f, viz., $\mathbf{y}^A(f)(g) := g \circ f$.

The functor $\mathbf{y}^A : \mathcal{F}^{op} \to \mathbf{Sets}$ being represented by the locus of states A, is called the *functor of generalized point states* of A. Moreover, the information contained in the locus of states A is classified completely by its functor of generalized point states $\mathbf{y}^A$. Hence, the functor $\mathbf{y}^A$ gives a *geometric* form to the abstract extension of the spatial locus of states A in the environment of $\mathcal{F}^{op}$. From the above, the direct conclusion is that the *Spectrum functor* can be specified *equivalently* in a dual manner: it can be specified by means of the *contravariant* representable functor from the category of loci of states to the category of sets, or equivalently, via the *covariant* representable functor from the category of rings of observables to the category of sets. Thus, it admits a well-defined operational determination in terms of measurement procedures referring to a coordinatization ring of observables, according to our preceding remarks.

9.7 QUANTUM SYSTEMS: FUNCTORIAL GEOMETRIC SPECTRA

At this stage there is every need to distinguish between classical and quantum systems. A category of classical natural systems, according to the above general description, admits a representation in terms of a category of formal systems (commutative, unital $\mathbb{R}$ -algebras of observable attributes) whose qualitative features are well understood and can be simultaneously determined with precision by means of valuations to the field of real numbers.

In this sense, a category of systems is characterized as a category of *quantum* systems in relation to the global complexity of the corresponding representing *formal* systems, with respect to those used to model the behavior of *classical* systems. This aspect of global complexity is, according to quantum theory, due to the *inability* of *simultaneous precise* measurement of all the attributes of the corresponding natural system within the same *local* measurement *context*.

More concretely, *Heisenberg's uncertainty* principle sets the *limits* of simultaneous precise measurement of *incompatible* observables by means of valuations in the field of real numbers, like position and momentum, within the same measurement context. On the other hand, *maximal families* of compatible observables can be *simultaneously* measured *precisely* within some appropriate measurement context, but these families stand in a *complementary* relation to each other with respect to the property of *incompatibility* of their observables. Thus, the behavior of a quantum system can be *approximated* in terms of maximal compatible families of simultaneously measurable observables within an appropriate local commutative measurement context, considered together with the notion of complementarity between such families.

In this frame of reasoning, the functorial conception of natural systems proves to be particularly relevant for the representation of quantum systems. More specifically, in striking *contrast* to the *global* classical conception, the notion of *generalized state* of a natural system, *rejects* the *absolute* global and simultaneous representability of all observables of the classical theory over the coordinatizing field of real numbers, rather it allows, in this sense, the geometric representation of states, in terms of *generalized* points of a *multi-valued* geometric state space, as well as, the *evaluation* of observable attributes at those points. This is achieved by a process of *local relativization* of physical *representability* with respect to a *multitude* of measurement scales, belonging to *different local commutative* coordinatizing rings, leading eventually to the above multi-valued geometric state space.

In relation to quantum theory, if we further assume that *local* commutative rings of measurement scales, corresponding to maximal families of compatible observables, can *co-finally* be translated simultaneously into the field of real numbers by means of suitable event-registering measuring devices, then the representation of quantum systems can be made possible by means of *diagrams* of simultaneously determinable *commutative* and *unital local* $\mathbb{R}$ -algebras of compatible observables.

This essentially means that the transition from the classical to the quantum representation of natural systems, is effectuated by the *replacement* of a *category* of commutative unital $\mathbb{R}$ -algebras of observables evaluated globally and simultaneously to the field of real numbers, by a *category of appropriate diagrams* of local commutative unital $\mathbb{R}$ -algebras of compatible observables, amenable to a local simultaneous translation into the field of real numbers. Most importantly, such a semantic transition becomes possible only via the *functorial* representational framework of natural systems, which entails the relativization of physical representability of observable attributes of natural systems with respect to varying local commutative coordinatizing frames of scale coefficients.

The force of this present line of argument will be missed without a precise indication of the sense in which the qualifications of global and local nature are used here. We have seen that in the environment of the functorial representation of natural systems, a representing formal system stands for a structure specified concretely as a coordinatizing unital ring of observables of some corresponding natural system. Thus, the *distinction* between classical and quantum systems should also be reflected in the appropriate *qualification* of their corresponding *rings* of observables.

The crucial distinguishing requirement with respect to the coordinatizing rings of observables has to do with the property of *global* commutativity. In this line of thought, a classical system is being globally represented by a commutative ring ($\mathbb{R}$ -algebra) of observable attributes, whereas, a quantum one is represented by a ring of observables attributes which is only *partially commutative*, and thus, *globally non-commutative*. The conceptual underpinning of this distinction, referring to the property of commutativity of observable coordinates, has to do with the fact that, a globally non-commutative or partially commutative ring of observables *determines* an underlying *diagram* of *commutative* rings. Then, each commutative ring can be *locally* identified with a ring of commutative scalar coordinates. Thus, there exists the possibility that the information contained in a ring of a quantum system's observables may be approximated or recovered by a *sheaf-theoretic pasting* construction referring to *diagrams* of commutative subrings, identified *locally* with *commutative* rings of compatible observables.

The implications of these considerations are the following: Firstly, we claim that observational *complexity* is a property of a system's behavior that is conceived *topologically* as a global attribute, admitting an *algebraic* description in terms (globally non-commutative but locally commutative) of its *observables*. Secondly, the behavior of a quantum system can be modeled in terms of well-defined families of local commutative algebras of compatible observables, such that the globally complex aspect of its behavior is due to the *non-trivial interlocking* of its local manifesting *icons* in commutative localization measurement environments. Thirdly, a suitable representational framework of quantum systems, taking into account the functorial interpretation of the measurement processes of natural systems, should *not* be a category of non-commutative rings of observables, but a category of appropriate *diagrams* of local commutative unital $\mathbb{R}$ -algebras of compatible observables. This means that the behavior of a quantum system in relation to the measurement processes is not captured by a rigid non-commutative $\mathbb{R}$ -algebra of observables, but by diagrams of local commutative unital $\mathbb{R}$ -algebras of compatible observables. In turn, this is equivalent to the preconditional requirement for *local variation* of the *coordinatizing frame-ring* of coefficients to enable the capture of quantum phenomena, together with the requirement of *amalgamation* of locally compatible information into diagrams of such a form. Fourthly, according to the above, physical *representability* of observables of quantum systems should be *relativized* or *localized* with respect to local commutative reference frames of measurement, since *simultaneous global* determination of their attributes into the field of real numbers is *impossible*. We shall see later that the mathematical transcription of these ideas requires the explicit adoption of a *topos-theoretic* model of the *physical continuum* for the description of quantum phenomena, realized as a category of *(pre)-sheaves* over a base *localizing* category of commutative measurement contexts.

By taking into account the previous distinction, we represent a category of quantum systems by means of a cocomplete category of formal systems $\mathcal{Z}$, such that: Its objects (called *quantum objects*), Z , are quantum information structures, identified as partially commutative unital rings ($\mathbb{R}$ -algebras) of observables, whereas, its *arrows* are the structure-preserving morphisms between them.

The basic idea is that the behavior of a quantum system can be comprehended in terms of a *local to global* contravariant functorial construction referring to its *Spectrum functor*. This can be realized for each quantum structure Z in $\mathcal{Z}$, as an *interlocking* family of incoming morphisms from the loci of intentionally depicted commutative measurement structures, which, characterize the behavior of simple, sufficiently understood *local* systems. From this perspective, we consider a locally small category $\mathcal{Y}$, whose objects, Y, are intentionally selected commutative unital rings ($\mathbb{R}$ -algebras) of scalar coordinates, called *partial* or *local information carriers*, whereas its *arrows* are structure-preserving maps of these carriers. Their role is inextricably connected with the philosophy illustrating their attachment to a *quantum* information structure as *localization* devices, or information *filters* or even as modes of perception.

The epistemological purpose of their introduction is, eventually, the construction of a *covering* system of a quantum information structure with respect to commutative domains of measurement. The notion of a covering system signifies an intentional structured decomposition of a globally non-commutative information structure in terms of partial or local commutative carriers, such that the functioning of the former can be *approximated*, or completely *recovered*, by the *interconnecting* machinery governing the organization of the covering system. Evidently, each local or partial information carrier, includes the amount of information related to a *filtering* process, objectified by a specified context, or a localization environment, and thus, it represents the abstractions associated with the intentional aspect of its use. A further claim, necessary for the development of the proposed functorial model, has to do with the technical requirement that the category of quantum information structures must meet a condition, phrased in category- theoretic language, as *cocompleteness*. This condition means that the category of quantum information structures has arbitrary *small colimits*. The existence of colimits expresses the basic intuition that a quantum object may be conceived as arising from the structured *interconnection* of partially or locally defined information carriers in a specified covering system.

We recall that the formal system corresponding to a quantum system is completely determined by its Spectrum functor in the corresponding representing categorical environment $\mathcal{Z}$. From the preceding discussion we have concluded that the Spectrum functor can be specified equivalently, either by, the contravariant representable functor from the category of partially commutative loci of states to the category of sets, or by means of, the covariant representable functor from the category of partially commutative rings of observables to the category of sets.

The crucial thing to remember is that any locus of states of a quantum system is determined, up to canonical isomorphism, if we know all morphisms into this locus from any other locus in the same category.

For our purposes we consider the description of a locus Z in terms of *all* possible morphisms from all other objects of $\mathcal{Z}^{op}$ as *redundant*. For this reason, we may restrict the generalized point states of Z to all those morphisms in $\mathcal{Z}^{op}$ having as domains loci corresponding to commutative subrings of the globally non-commutative ring of observables of a quantum system. Variation of generalized point states over all loci of the subcategory of $\mathcal{Z}^{op}$, consisting of commutative measurement loci, identified with $\mathcal{Y}^{op}$, produces the *Spectrum functor* of Z restricted to the subcategory of commutative loci.

The Spectrum functor of Z, specified as above, stands as an object in the category of *presheaves* (variable sets) $Sets^{\mathcal{Y}^{op}}$, representing a quantum object in the variable environment of the *topos* of presheaves over the category of its commutative subobjects. This methodology proves to be successful by the establishment of an *isomorphic* representation of Z in terms of its generalized point states $Y_i \to Z$, considered as morphisms in the same category, and further, amalgamated by sheaf-theoretic means.