CANONICS OF INVARIANCE: THE COUPLING OF MATTER WITH FIELDS

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7.1 ARTICULATION OF MATTER: FROM LOCAL GAUGE SYMMETRY TO GAUGE POTENTIALS

Physical geometry may be thought of as the outcome of a measurement procedure, where the interwoven notion of a group action discloses a particular act of measurement. The general conceptual attitude towards physical geometry emanating from *Felix Klein's* Erlangen program requires that the *geometric configuration* of states of a physical system and the *symmetry group* of transformations of those states should be considered *equivalent* through the *free* and *transitive action* of the symmetry group on the space of states.

Gauge-invariant field theories of material interaction are modeled in terms of *fiber bundle* geometries endowed with a *connectivity* structure, which integrate Klein's *group* theoretical conception of geometry with Riemann's infinitesimal *metrical* viewpoint. These physical models are based on the fundamental notion of a *connection* on a fiber bundle, which generalizes *Weyl's* viewpoint of a *purely infinitesimally* geometry, generated by means of an affine connection that can be made metric-compatible. The notion of a connection, in relation to a fiber bundle geometry, is indispensable for the effectuation of a *covariant derivative* operator, which acts on the local sections-states of the bundle. This operator expresses analytically in infinitesimal terms the process of *parallel translation* along paths on the base space of the bundle induced by this connection.

In this manner, the infinitesimal expression of the connection is interpreted as a local *gauge potential* corresponding to a physical field. It is expressed by means of a *differential* 1*-form*, which takes values in the *Sophus Lie algebra* of the symmetry group. In turn, the observable, and thus measurable, effects of the potential are expressed invariantly via a *tensor*, which is called the *curvature* of the connection. The latter is physically identified locally with the observable *strength* of the corresponding physical field. In the setting of gauge theories the symmetry group is modeled locally on the fibers of a *principal* fiber bundle, which is defined over spacetime. Concomitantly, the state space of a gauge theory is identified with the *group sheaf* of sections of this bundle, or equivalently, with the *vector sheaf* of sections of its associated *vector* bundle. We note that the above setting of a gauge field theory constitutes a concrete expression of the principle of *sheaf-theoretic localization*, referring to both, the physical state space, and the pertinent free and transitive group action.

The genesis of gauge theory can be traced back to *Hermann Weyl's* conception of the interrelation between some type of *material source* and the corresponding *field* governing a physical interaction. According to Weyl, analogously to the fact that charge may be conceived as an electromagnetic effect, mass may be conceived as a gravitational effect. In particular, mass is the flux of the gravitational field through a surface enclosing a particle in the same sense that charge is the flux of the electric field. Thus, in the same way that it is impossible to introduce charge without the electromagnetic field, it is also impossible to introduce a non-vanishing mass without the gravitational field. In this respect, Weyl's fundamental idea concerning the notion of a local gauge interaction pertaining to a type of matter source with a corresponding physical field stems from the requirement of *invariance*.

More concretely, the crucial realization is that the localization of an *internal* symmetry in the physical description pertaining to a material form always gives rise to a *corresponding* gauge potential of a field, in order that the physical description remains *invariant*. Therefore, physical geometry in the setting of a gauge theory as a result of an interaction *always* proceeds from the *localization* of a *global* symmetry pertaining to the description of a material form. The demand of invariance under this process of localization requires the instantiation of a gauge field potential *transferring* an interaction, which thus couples the material form with the gauge field. Therefore, the localization of symmetry in relation to a material form constitutes the *necessary* condition for expressing both, the notion of an interaction field via its gauge potentials, and the minimal coupling of these potentials with the matter sources, under the demand of invariance.

The principle of sheaf-theoretic localization, which forces the functorial transference of Grassmann's lineal extension framework from the level of vector states to the level of vector sheaves of states, by which we mean locally free modules of states over a commutative algebra sheaf of observables, gives the possibility of a precise description of the physical geometry derived as a result of the interaction between a physical field and some type of a matter source, according to the preceding. For this purpose, the *base space* of the sheaf-theoretic localization procedure, pertaining to both, the symmetries, and the physical states, is identified with the spacetime manifold. Nevertheless, its construal as a base space is only implicit, in the sense that its point structure is not explicit *ab initio*, but has to be *articulated* by physical means.

The basic idea is that the information regarding the *point* constitution of this space, which points are *occupied* by some material form, or the space is *penetrated* by some current, can be expressed by means of some global invariant characterizing the physical interaction field whose *sources* are precisely these material forms. In the same context, the notion of physical geometry is always induced by the localization of some free and transitive group action, which expresses a fundamental internal symmetry in the description pertaining to a material form, in the sense that it is tantamount to a conservation law. The culmination of this conceptual setting bears the name derived by the principle of *gauge invariance* leading to the modeling of physical interaction geometries in terms of *gauge theories*.

The localization of the internal symmetry of a material form simply means that it may *vary independently* from point to point of the base space, within the range of *ambiguity* determined by the corresponding symmetry group. In this sense, a *copy* of the same symmetry group is *replicated* from point to point of the base space, in such a way that, at each of these points, a symmetry element is subsumed in total *independence* from all the others at *any* other point. It is precisely due to this freedom of independent symmetry variation from point to point that a proper means of following this variation from the infinitesimal to the global is required. These means establish the standards of *congruence* according to an imposed rule of *parallelism*, which is to say a rule of parallel transport along temporally parameterized paths on the base space. The latter derives from the notion of a connection on the corresponding principal group sheaf of symmetry coefficients, or equivalently, by a connection on the associated vector sheaf of states, where the latter assumes the role of the state representation space of the symmetry group.

In a standard way, a connection always gives rise to a covariant derivative operator, i.e. to a covariant means of *differentiating* the *sections* of this sheaf. The physical interpretation is that a field can be *covariantly* detected in terms of its differentiating effects on the section-states of the vector sheaf, and thus it should be *locally* characterized in terms of its gauge potentials. Henceforth, gauge field potentials are instantiated as necessary elements proceeding from the localization of the internal symmetry group of a material form, and being minimally coupled with the latter, such that the physical description remains always *invariant* with respect to the algebra sheaf of observables.

In the setting of a gauge theory the invariance in the description of physical geometry is always meaningful with respect to a *commutative algebra sheaf* of observables. It is this requisite invariance that demands our focus once again on the *commutative shadow* of the *exterior* algebra of the vector sheaf of states. According to the qualification of geometric calculus in relation to the sheaf-theoretic localization of Grassmann's lineal extension method, the commutative shadow is of a *cohomological* origin.

More concretely, it is enunciated by means of the locally defined coboundary, or exterior derivative, operator of the observable algebra sheaf. It is precisely based on the idea of unfolding and separating the infinitesimal irreducible parts of the commutative shadow, as modules of differentials of different orders. Recalling that the notion of the coboundary operator is based on the articulation of a point as a bound of distinct temporal orders becoming dependent infinitesimally, i.e. in terms of commuting one-parameter infinitesimal flows, at this point, it is necessary to account for the modification of these flows by the effect of localization of the internal symmetry of a material form occupying the point itself. In the descriptive capacity of the commutative shadow, this type of modification needs to be expressed in an invariant manner locally around this point by means of the associated gauge field potentials, which become minimally coupled with the material form.

Therefore, the coboundary operator is *covariantly adapted* to the additional requirement of localizing matter symmetry, and thus extended to act on the representation vector sheaf of states of the internal symmetry group, as a covariant derivative operator stemming from a connection on this sheaf. As a result, and in terms of the commutative shadow, the articulation of a point where some matter form is pertinent can be accounted for by the failure of this modified coboundary operator to extend to a differential de Rham complex as in the former case. This *obstruction* to *extendibility* of the *modified coboundary* operator to the next higher order is interpreted physically as the *encoding trace* of the field strength, associated with the curvature of the employed connection on the vector sheaf of states. The curvature bears the transformation properties of a tensor, and thus behaves covariantly with respect to the commutative algebra sheaf of observables.

7.2 COVARIANCE: GAUGE TRANSFORMATIONS OF LOCAL POTENTIALS' STRENGTHS

A connection $\mathcal{D}_{\mathcal{E}} := \nabla_{\mathcal{E}}$ on the vector sheaf of states $\mathcal{E}$ is a $\mathbb{C}$*-linear sheaf* morphism:

$$\nabla_{\mathcal{E}} : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}, \quad (1)$$

referring to $\mathbb{C}$-vector space sheaves, such that the following Leibniz type of condition is satisfied:

$$\nabla_{\mathcal{E}}(a \cdot s) = a \cdot \nabla_{\mathcal{E}}(s) + s \otimes d^0(a) \quad (2)$$

At the next stage, and since we are interested in the *local* form of a connection, we will show that *every* connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf of states on X, can be decomposed locally as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega. \quad (3)$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms, called the *matrix potential* of $\nabla_{\mathcal{E}}$.

Moreover, under a change of local frame matrix $g = g_{\alpha\beta}$, we will demonstrate that the matrix potentials transform as follows:

$$\omega' = g^{-1}\omega g + g^{-1}d^{0}g. \quad (4)$$

If we consider a coordinatizing *basis of sections* of the vector sheaf $\mathcal{E}$ of rank-n, defined over an open cover U of X, denoted by:

$$e^{U} \equiv \{U; (e_{\alpha})_{1\leq\alpha\leq n}\} \quad (5)$$

called a *local sectional frame*, or equivalently, a *local gauge* of $\mathcal{E}$, then every continuous local section $s \in \mathcal{E}(U)$, where, $U \in \mathcal{U}$, can be expressed uniquely with respect to this local frame as the following *superposition*:

$$s = \sum_{\alpha=1}^{n} s_{\alpha} e_{\alpha}, \quad (6)$$

with coefficients s_{α} in $\mathcal{A}(U)$. The *action* of $\nabla_{\mathcal{E}}$ on these sections of $\mathcal{E}$ is expressed as follows:

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n}(s_{\alpha}\nabla_{\mathcal{E}}(e_{\alpha}) + e_{\alpha} \otimes d^{0}(s_{\alpha})). \quad (7)$$

$$\nabla_{\mathcal{E}}(e_{\alpha}) = \sum_{\alpha=1}^{n} e_{\alpha} \otimes \omega_{\alpha\beta}, 1 \leq \alpha, \beta \leq n, \quad (8)$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms. Consequently we have;

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n} e_{\alpha} \otimes (d^{0}(s_{\alpha}) + \sum_{\beta=1}^{n} s_{\beta}\omega_{\alpha\beta}) \equiv (d^{0} + \omega)(s). \quad (9)$$

Thus, *every connection* $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf on X, can be *decomposed locally* as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega. \quad (10)$$

In this context, $\nabla_{\mathcal{E}}$ is identified as a covariant derivative operator acting on the sections of the vector sheaf of states $\mathcal{E}$. This operator can be decomposed locally as a sum consisting of a *flat or integrable part* identical with d^0, and a generally *non-integrable* part ω, called the *local frame (gauge) matrix potential* of the connection.

The behavior of the local potential ω of $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the transformation law of local gauge potentials and is obtained as follows:

Let $e^U \equiv \{U; e_{\alpha=1\cdots n}\}$ and $h^V \equiv \{V; h_{\beta=1\cdots n}\}$ be two local frames of $\mathcal{E}$ over the open sets U and V of X, such that $U \bigcap V \neq \varnothing$. Let us denote by $g = g_{\alpha\beta}$ the following change of local frame matrix:

$$h_\beta = \sum_{\alpha=1}^{n} g_{\alpha\beta} e_\alpha. \quad (11)$$

Under such a local frame transformation $g_{\alpha\beta}$, we easily obtain that the local potential ω of $\nabla_{\mathcal{E}}$ transforms as follows in matrix form:

$$\omega' = g^{-1}\omega g + g^{-1} d^0 g. \quad (12)$$

The above is clearly a metaphora, which is expressed in terms of *conjugation* through the bridge g and its inverse:

$$\omega' = g^{-1}(d^0 + \omega) g. \quad (13)$$

Further, if we assume that the pair $(\mathcal{E}, \nabla_{\mathcal{E}})$ denotes a complex vector sheaf of states endowed with a *connection*, $\nabla_{\mathcal{E}}$, then $\nabla_{\mathcal{E}}$ induces a *sequence* of $\mathbb{C}$-linear morphisms:

$$\mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \to \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \quad (14)$$

or equivalently:

$$\mathcal{E} \to \Omega^1(\mathcal{E}) \to \ldots \to \Omega^n(\mathcal{E}) \to \ldots \quad (15)$$

where the morphism:

$$\nabla^n : \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{n+1}(\mathcal{A}) \otimes_{\mathbf{A}} \mathcal{E}, \quad (16)$$

is given by the formula:

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v), \quad (17)$$

for all $\omega \in \Omega^n(\mathcal{A})$, $v \in \mathcal{E}$. Immediately it follows that $\nabla^0 = \nabla_{\mathcal{E}}$.

The *composition* of $\mathbb{C}$-linear morphisms $\nabla^1 \circ \nabla^0$ is called the *curvature* of the connection $\nabla_{\mathcal{E}}$:

$$\nabla^1 \circ \nabla^0 := R_\nabla : \mathcal{E} \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} = \Omega^2(\mathcal{E}). \quad (18)$$

Consequently, we derive that the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$ on the vector sheaf of states $\mathcal{E}$ is an $\mathcal{A}$-*linear* sheaf morphism, that is, an $\mathcal{A}$-*covariant* morphism, or equivalently, an $\mathcal{A}$-*tensor*. The $\mathcal{A}$-covariant nature of the curvature R_∇ can significantly be contrasted with the connection $\nabla_{\mathcal{E}}$, which is *only* $\mathbb{C}$-covariant and *not* $\mathcal{A}$-covariant.

The *sequence* of $\mathbb{C}$-linear sheaf morphisms,

$$\mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \to \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \quad (19)$$

defines a *complex* of $\mathbb{C}$-vector space sheaves *if and only if* the following condition is satisfied:

$$R_\nabla = 0. \quad (20)$$

Thus, the curvature $\mathcal{A}$-covariant tensor R_∇ expresses the *obstacle*, or the *obstruction* for the above sequence to *qualify* as a complex. We say that the connection $\nabla_{\mathcal{E}}$ is an *integrable*, or equivalently, a *flat connection* if $R_\nabla = 0$. In this case, we refer to the above complex as the sheaf-theoretic *de Rham complex of the integrable connection* $\nabla_{\mathcal{E}}$ on the vector sheaf $\mathcal{E}$. Note that the universal $\mathbb{C}$-derivation d^0 on $\mathcal{A}$ always defines an integrable or flat connection.

A flat connection expresses a *maximally unobstructed* process of dynamical variation associated with the corresponding *field*. From a physical viewpoint, a flat connection sets up the standards of *congruence* under replication of the local internal symmetry group point by point. Thus, a non-vanishing curvature expresses covariantly the existence of a certain type of deviation from the maximally unobstructed form of this variation. Equivalently, curvature effects can be *cohomologically* identified as *obstructions* to deformation caused by the *matter* sources coupled to the field.

In case that, additionally, the representability principle over the field of the complex, or the real numbers is required at a point-event, the existence of *uniquely* defined *duals* is necessary. In this case, the physical field is identified with a *linear connection* on the $\mathcal{A}$-vector sheaf of states $\Xi = Hom(\Omega^1, \mathcal{A})$, which is cast *isomorphic* with Ω^1, by means of a bilinear form that plays the role of a *metric*:

$$g : \Omega^1 \simeq \Xi = \Omega^{1*}. \quad (21)$$

In this case, a physical observable geometry is considered with respect to a metric. Consequently, the physical field is properly expressed by *the pair* (Ξ, ∇_Ξ). The required *metric compatibility* of the connection is formulated as follows:

$$\nabla_{Hom_{\mathcal{A}}(\Xi,\Xi^*)}(g) = 0. \quad (22)$$

Taking into account the requirement of representability over the complex, or the real numbers, and thus considering the concomitant evaluation *trace* operator by means of the metric, we arrive at the *analogue* of *Albert Einstein's field equations*, which in the *absence* of matter sources with respect to $\mathcal{A}$, are expressed as follows:

$$\mathcal{R}(\nabla_\Xi)(\Xi) = 0. \quad (23)$$

where $\mathcal{R}(\nabla_\Xi)$ denotes the relevant *Ricci scalar curvature*. More precisely, we first define the *curvature endomorphism* $\mathfrak{R}_\nabla \in End(\Xi)$, called the *Ricci curvature operator*. Since the Ricci curvature $\mathfrak{R}_\nabla$ is locally matrix-valued, by taking its trace using the metric, that is, by considering its evaluation or contraction by means of the metric, we arrive at the definition of the Ricci scalar curvature $\mathcal{R}(\nabla_{\Xi})(\Xi)$ obeying the above equation.

Thus, the metric describing the *physical geometry*, as a *result* of field interactions, is dynamically determined as a *solution* of the above equation *in relation* to the metric compatible connection on the vector sheaf $Hom_{\mathcal{A}}(\Xi, \Xi^{*})$.

Finally, it is necessary to investigate the *local* form of the curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a locally free finite rank-n sheaf of modules (vector sheaf) $\mathcal{E}$ on X, defined by the following $\mathcal{A}$-linear morphism of sheaves:

$$R_{\nabla} := \nabla^{1} \circ \nabla^{0} : \mathcal{E} \to \Omega^{2}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} := \Omega^{2}(\mathcal{E}). \quad (24)$$

Due to its property of $\mathcal{A}$-covariance, a non-vanishing curvature represents in this context, the $\mathcal{A}$-covariant, and thus, *geometrically observable deviation* from the inertial form of variation corresponding to an integrable connection.

Further, since the curvature R_{∇} is an $\mathcal{A}$-linear morphism of sheaves of $\mathcal{A}$-modules, that is an $\mathcal{A}$-tensor, R_{∇} may be thought of as an element of $End(\mathcal{E}) \otimes_{\mathcal{A}} \Omega^{2}(\mathcal{A}) := \Omega^{2}(End(\mathcal{E}))$, as follows:

$$R_{\nabla} \in \Omega^{2}(End(\mathcal{E})) \quad (25)$$

Hence, the local form of the curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$, consists of local $n \times n$ matrices taking local 2-forms for entries. In particular, the local form of the curvature $R_{\nabla}|_{U}$, where U open in X, in terms of the local potentials ω is expressed by:

$$R_{\nabla}|_{U} = d^{1}\omega + \omega \wedge \omega, \quad (26)$$

as can easily be shown by substitution of the local potentials in the composition $\nabla^{1} \circ \nabla^{0}$. Furthermore, by application of the differential operator d^{2} on the above we obtain:

$$d^{2}R_{\nabla}|_{U} = R_{\nabla}|_{U} \wedge \omega - \omega \wedge R_{\nabla}|_{U}. \quad (27)$$

The behavior of the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the transformation law of the *gauge potentials' strength*. If we agree that $g = g_{\alpha\beta}$ denotes the change of local frame matrix that we considered in the discussion of the transformation law of the local gauge potentials previously, we derive the following local transformation law:

$$R_\nabla \overset{g}{\mapsto} R'_\nabla = g^{-1}(R_\nabla)g, \quad (28)$$

that is, the strength transforms covariantly by conjugation with respect to a local frame transformation.

Thus, we may summarize the preceding as follows:

i The local form of the curvature $R_\nabla |_U$, where U open in X, in terms of the local matrix potentials ω, is given by:

$$R_\nabla |_U = d^1\omega + \omega \wedge \omega \quad (29)$$

ii Under a change of local frame matrix $g = g_{\alpha\beta}$ the local form of the curvature transforms by conjugation with respect to *g*. Therefore, *g* and its inverse, are the bridges enunciating the metaphora of the field strength locally:

$$R_\nabla \overset{g}{\mapsto} R'_\nabla = g^{-1}(R_\nabla)g \quad (30)$$

We note that the above holds for *any* complex *vector* sheaf $\mathcal{E}$. Let us now specialize to the particular case of a *line* sheaf of states $\mathcal{L}$ endowed with a *connection*, denoted by the pair $(\mathcal{L}, \nabla)$. In this case, due to the isomorphism:

$$\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^* \cong Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \equiv End_{\mathcal{A}} \mathcal{L} \cong \mathcal{A}, \quad (31)$$

we obtain the following simplifications: the *local* form of a connection over an open set is *just* a *local* 1*-form* or a local potential, identified as a local continuous section of the sheaf Ω^1, whence the *local* form of the curvature of the connection over an open set is a *local* 2*-form*. The significant result obtained by the local transformation law in this case is that the curvature is actually a *local frame invariant*, i.e. it does not change under any local frame transformation.

$$R_\nabla \overset{g}{\mapsto} R_\nabla' = R_\nabla = d^1\omega \quad (32)$$

Thus, we obtain a global 2-form R_∇ defined over X, which is also a closed 2-form since:

$$dR_\nabla = 0. \quad (33)$$

Therefore, we conclude that for a *line* sheaf of states $\mathcal{L}$ equipped with a *connection* ∇, denoted by the pair $(\mathcal{L}, \nabla)$, the *curvature* R_∇ of the connection is a *global closed* 2-form.

7.3 INVARIANCE: GAUGE EQUIVALENT GEOMETRIC SPECTRA

The essence of a *geometric equivalence* problem in physical geometry refers to the determination of the conditions under which two *geometric spectra* of the same type are *equivalent* under an appropriate *group* of transformations. The most fruitful approach to geometric equivalence problems concerns the appropriate association of *invariants* with a type of geometric spectra, by which we mean attributes that *do not change* under an *isomorphism*. The idea is that invariants are capable of determining geometric spectra *uniquely* up to isomorphism.

In general, a *physical geometric spectral type* is expressed by an *equivalence* class of state spaces, each of which constitutes the geometric spectrum of a *commutative* algebra *sheaf* of observables. A state space incorporates the totality of *potential states* of a physical system, and is *vectorial* with respect to the algebra of observables of this system. In the case of a gauge theory, the notion of a state space arises from, and it is therefore equivalent, to the transitive group action space of an internal symmetry group, at least *locally*, in the sense that the symmetry group delineates locally the range of potential attributes that a physical geometric type can assume. Being a member of a geometric type, a physical matter entity can be in any of the permissible potential states locally.

This is the cornerstone of the local gauge freedom and constitutes a concrete physical manifestation of the *criterion* of geometric equivalence that a *local gauge group* furnishes in the case of gauge theories. The sheaf-theoretic formulation of gauge theories captures precisely the formation of physical geometric types under equivalence criteria constituted by the actions of local symmetry groups. In this state of affairs, a physical geometric spectral type is expressed by means of an *equivalence class* of *vector sheaves* endowed with a *connectivity* structure. The base manifold of a vector sheaf of states equipped with a connection is an integral part of the coupling between a gauge field and the matter sources occupying its points. For this reason, its point constitution is only implicitly posited, specified neither *ab initio* nor *a priori*. Rather, it is just the *base carrier* of the geometry by which matter is transformed, understood in this framework as a *structural quality* of the corresponding *physical field*.

The strategy to tackle the problem of equivalence of physical geometric spectra in the setting of gauge theories involves the determination of invariant *global characteristic classes* associated with vector sheaves of states endowed with a connection. A characteristic class is *represented* analytically in the pertinent geometric context in terms of an appropriate *differential form* stemming from the *connection*. For instance, in the case of a *line* sheaf, the global curvature differential form provides *the seed* for the articulation of the observable geometric spectrum, congredient with the field strength, via its *global de Rham cohomology class*.

We recall that the essential aspect of the de Rham cohomological analysis is precisely the *disclosure* of singular points on an implicitly assumed manifold, or more generally, a nice topological space, in terms of *invariant* quantities obtained by the *integration* of *differential forms*. In particular, closed differential forms are the natural integrands over cycles, meaning that they can be temporally integrated over closed chains encycling a singular point, such that the result of this integration procedure unveils a *residue* characterizing invariantly the *presence* of this singular point in terms of its *spectral periods*.

With this concept we reach the culmination of a key idea, an idea which addresses the specific problem of grasping the nature and essence of a *singular point* when some form of *matter* is in play. Since such a point *cannot* be bounded by infinitesimal flows converging on it, it can only be amenable to description by means of processes which *circulate* around it. These circulations are *not* equivalent, but they are in principle *distinguishable* in terms of some recognizable *global* attribute. Such an attribute obtained through integration is a spectral period, as above. Of especial significance is the resolving domain of spectral periods corresponding to the curvature form of the circulating connection. We stress the fact that cycles are not boundaries, since they generate homology classes encapsulating a *novel* type of temporal order with reference to singular points that is qualitatively different from boundaries. More concretely, the type of temporal order encapsulated by cycles is precisely characterized in terms of spectral periods via a global integration procedure, and not in terms of instants as in the former case.

In the context of the *equivalence* problem of *gauge geometric spectral types*, we consider two line sheaves of states which are equivalent due to an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$. Because of the *bijective* correspondence between line sheaves and coordinate 1-cocycles with respect to an open covering $\mathcal{U}$ of the base topological space X, which is considered to be paracompact, the set of *isomorphism classes* of line sheaves over X, denoted by $Iso(\mathcal{L})(X)$, is in bijective correspondence with the set of *cohomology classes* $H^1(X, \tilde{\mathcal{A}})$:

$$Iso(\mathcal{L})(X) \cong H^1(X, \tilde{\mathcal{A}}) \quad (1)$$

Moreover, the set of isomorphism classes of line sheaves of states over X, $Iso(\mathcal{L})(X)$, is an *Abelian group* with respect to the *tensor product* over the observable algebra sheaf $\mathcal{A}$. The tensor product of two line sheaves corresponds to the *product* of their respective coordinate 1-cocycles. Thus, $Iso(\mathcal{L})(X)$ is isomorphic with the Abelian group of cohomology classes $H^1(X, \tilde{\mathcal{A}})$. In this setting, $\mathcal{A}$ is considered as a soft sheaf, meaning that *every* section over some closed subset in X can be *extended* to a section over X.

Then, the process of *exponentiation*, in local sheaf-theoretic terminology, is expressed by the following *short exact sequence* of Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{exp} \tilde{\mathcal{A}} \to 1, \quad (2)$$

where $\mathbb{Z}$ is the *constant* abelian group sheaf of the *integers*, namely the sheaf of *locally constant* sections valued in the group of integers, such that:

$$Ker(e) = Im(\iota) \cong \mathbb{Z}. \quad (3)$$

Note that all the elements in the above short exact sequence are Abelian group sheaves *generalizing* the corresponding short exact sequences of constant Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{exp} \tilde{\mathbb{C}} \to 1, \quad (4)$$

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{exp(2\pi i)} \mathbb{U}(1) \to 1 \quad (5)$$

As an outcome, we obtain a *long exact sequence* in sheaf cohomology, which is reduced to a long exact sequence in Čech cohomology because of paracompactness of X:

$$\ldots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{A}) \to H^1(X,\tilde{\mathcal{A}}) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{A}) \to H^2(X,\tilde{\mathcal{A}}) \to \ldots \quad (6)$$

Furthermore, since $\mathcal{A}$ is a soft sheaf:

$$H^1(X,\mathcal{A}) = H^2(X,\mathcal{A}) = 0 \quad (7)$$

$$0 \to H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\delta_c} H^2(X,\mathbb{Z}) \to 0. \quad (8)$$

Thus, we obtain the following isomorphism of Abelian groups, called the *Shiing-Shen Chern isomorphism*:

$$\delta_c : H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\cong} H^2(X,\mathbb{Z}). \quad (9)$$

Finally, since $Iso(\mathcal{L})(X)$, is in *bijective* correspondence with the Abelian group of cohomology classes $H^1(X,\tilde{\mathcal{A}})$, we have:

$$Iso(\mathcal{L})(X) \cong H^2(X,\mathbb{Z}). \quad (10)$$

Therefore, the Chern isomorphism establishes that an *equivalence class* of line sheaves of states is in *bijective* correspondence with a *cohomology class* in the *integral* 2*-dimensional cohomology group* of X, called a *Chern class* of X. Taking into account that a line sheaf of states is actually the representation vector sheaf of states of the internal symmetry group sheaf of some corresponding matter form occupying a singularity, the Chern isomorphism reveals that this *matter* form is *encoded* in terms of a *two dimensional integral cohomology class* of X.

In other words, if we integrate a representative two dimensional cocycle of this class over a two dimensional cycle in X enclosing it, we obtain a characteristic *spectral period*, qualified in terms of a $\mathbb{Z}$-invariant. The pertinent problem, in relation to the equivalence problem of gauge geometric spectral types, is whether such an integer cohomology class can be *expressed* in terms of the cohomology class of a two-dimensional differential form, namely a de Rham cohomological invariant. The underlying reason is that the decoding of an integral cohomology class *in dynamical terms*, that is, in terms of the gauge potentials of the field, and by extension, in terms of the potentials' *strength*, or curvature, as a *flow* induced by this matter form, can be expressed only by means of an appropriate corresponding *differential* de Rham cohomology class.

7.5 THE OBSTACLE OF A MATTER SOURCE: HARMONICS OF SPECTRA AND QUANTIZATION

Initially, we observe that there exists a natural injection $\mathbb{Z} \infty \mathbb{C}$:

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}), \quad (11)$$

where any Čech cohomology class belonging to the image of the above map is called an *integral* cohomology class.

Next, we have to take into account the compatibility of the above short exact sequence of Abelian group sheaves with the short exact sequence of $\mathbb{C}$-vector sheaves,

$$0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{d^0} d^0\mathcal{A} \to 0, \quad (12)$$

arising as a fragment of the *de Rham resolution* of the constant sheaf $\mathbb{C}$, such that:

$$Ker(d^0) = Im(\varepsilon) \cong \mathbb{C}. \quad (13)$$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{exp} & \tilde{\mathcal{A}} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \frac{1}{2\pi i}\tilde{d}^0} \\ \mathcal{A} & \xrightarrow{d^0} & d^0\mathcal{A} \end{array}$$

Thus we obtain the following key relation:

$$2\pi i \cdot d^0 = \tilde{d}^0 \circ exp. \quad (14)$$

The dynamical, gauge field-theoretic, decoding of an integral cohomology class, or Chern class, of X requires to extend the notion of equivalence of two line sheaves to the corresponding one of two line sheaves *equipped* with a connection, which is induced *locally* by gauge potentials, $(\mathcal{L},\nabla)$ and $(\mathcal{L}',\nabla')$.

If we consider an isomorphism $h:\mathcal{L}\xrightarrow{\cong}\mathcal{L}'$ of line sheaves of states, we say that ∇ is *frame or gauge equivalent* under *metaphora* to ∇' if they are *conjugate connections* under the action of the isomorphism h:

$$\nabla' = h\cdot\nabla\cdot h^{-1}. \quad (15)$$

Thus, we may consider the set of equivalence classes on *pairs* of the form $(\mathcal{L},\nabla)$ under an *isomorphism* h as previously, denoted by $Iso(\mathcal{L},\nabla)$. It is necessary to investigate the relation between $Iso(\mathcal{L},\nabla)$ and the Abelian group $Iso(\mathcal{L})$. For this purpose, we need to make use of the local form of the pair $(\mathcal{L},\nabla)$.

We call a line sheaf $\mathcal{L}$ endowed with a connection ∇ a *differential line sheaf*, and we denote it by the pair $(\mathcal{L},\nabla)$. Then, The *local* form of a differential line sheaf is given by:

$$(\mathcal{L},\nabla)\leftrightarrow(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1) \quad (16)$$

Moreover, an arbitrary pair $(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1)$ determines a differential line sheaf if the *transformation* law of local gauge potentials is valid:

$$\delta^0(\omega_\alpha)=\tilde{d}^0(g_{\alpha\beta}) \quad (17)$$

where,

$$\tilde{d}^0(g_{\alpha\beta})=g_{\alpha\beta}^{-1}\cdot d^0 g_{\alpha\beta}, \quad (18)$$

and δ^0 denotes the 0-th coboundary operator $\delta^0 : C^0(\mathcal{U},\Omega^1)\to C^1(\mathcal{U},\Omega^1)$, such that:

$$\delta^0(\omega_\alpha)=\omega_\beta-\omega_\alpha. \quad (19)$$

In more detail, we note that a line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ associated with the covering $\mathcal{U}$. A connection ∇ is expressed bijectively in terms of a 0-cochain of gauge potential 1-forms, denoted by ω_α with respect to the covering $\mathcal{U}$ of X, that is $\omega_\alpha\in C^0(\mathcal{U},\Omega^1)$. Therefore, the local form of a differential line sheaf is the following:

$$(\mathcal{L},\nabla)\leftrightarrow(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1). \quad (20)$$

Conversely, an arbitrary pair $(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1)$ determines a differential line sheaf if the transformation law of local gauge potentials is satisfied by this pair, that is:

$$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}d^0 g_{\alpha\beta}, \quad (21)$$
$$\omega_\beta = \omega_\alpha + g_{\alpha\beta}^{-1}d^0 g_{\alpha\beta}. \quad (22)$$

Thus, given an open covering $\mathcal{U}=\{U_\alpha\}$, a 0-cochain (ω_α) valued in the sheaf Ω^1 determines the local form of a connection ∇ on the line sheaf $\mathcal{L}$, where the latter is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $(g_{\alpha\beta})$ valued in $\tilde{\mathcal{A}}$ with respect to $\mathcal{U}$, *if and only if* the corresponding local 1-forms ω_α of the 0-cochain with respect to $\mathcal{U}$ are pairwise inter-transformable, meaning *locally gauge-equivalent* on overlaps $U_{\alpha\beta}$ through the local gauge transition functions, namely the *local isomorphisms* $g_{\alpha\beta}\in\tilde{\mathcal{A}}(U_{\alpha\beta})$ according to the transformation law of local gauge potentials.

Next, we consider two line sheaves which are equivalent via an isomorphism $h:\mathcal{L}\xrightarrow{\cong}\mathcal{L}'$, such that their corresponding connections are *conjugate* under the action of h:

$$\nabla' = h\cdot\nabla\cdot h^{-1}. \quad (23)$$

Under these conditions the differential line sheaves $(\mathcal{L},\nabla)$ and $(\mathcal{L}',\nabla')$ are called *gauge or frame equivalent*. Thus, we may consider the set of gauge equivalence classes $[(\mathcal{L},\nabla)]$ of differential line sheaves as above, denoted by $Iso(\mathcal{L},\nabla)$. Then, it is easy to show that the set of gauge equivalence classes of differential line sheaves $Iso(\mathcal{L},\nabla)$, is indeed an *Abelian subgroup* of the abelian group $Iso(\mathcal{L})$.

If we consider the *local* form of the *tensor product* of two gauge equivalent differential line sheaves we have:

$$(\mathcal{L},\nabla)\otimes_{\mathcal{A}}(\mathcal{L}',\nabla')\leftrightarrow(g_{\alpha\beta}\cdot g'_{\alpha\beta},\omega_\alpha+\omega'_\alpha), \quad (24)$$

which satisfies the transformation law of local gauge potentials:

$$\tilde{d}^0(g_{\alpha\beta} \cdot g'_{\alpha\beta}) = \tilde{d}^0(g_{\alpha\beta}) + \tilde{d}^0(g'_{\alpha\beta}) = \delta^0(\omega_\alpha) + \delta^0(\omega'_\alpha) = \delta^0(\omega_\alpha + \omega'_\alpha). \quad (25)$$

Moreover, the *inverse* of a pair $(g_{\alpha\beta}, \omega_\alpha)$ is given by $(g^{-1}_{\alpha\beta}, -\omega_\alpha)$, whereas the *neutral* element in the group $Iso(\mathcal{L}, \nabla)$ is given by $(id_{\alpha\beta}, 0)$, which corresponds to the trivial standard differential line sheaf $(\mathcal{A}, d^0)$.

The most important consequence of the above characterization of gauge equivalent differential line sheaves in *local* terms with respect to an open covering $\mathcal{U}$, is that *all* gauge *equivalent geometric spectral types* are characterized by the *same curvature*. We denote the curvature of a gauge equivalence class of differential line sheaves by R. The differential form R is a *global* 2-form on X since it is *invariant* under a gauge transformation. Moreover, R is also a *closed* global 2-form on X, because of the fact that:

$$d \circ d\omega_\alpha = dR = 0. \quad (26)$$

Thus, the global 2-form R, which belongs to $Ker(d) : \Omega^2 \to \Omega^3$, called Ω^2_c, identified as a $\mathbb{C}$-vector sheaf *subspace* of Ω^2, determines a *global differential invariant* of gauge equivalent differential line sheaves. This is due to the fact that the global 2-form R determines a *2-dimensional complex-valued de Rham cohomology class* $[R]$. In turn, by virtue of the de Rham isomorphism, $[R]$ is *isomorphically* identified as a 2-dimensional complex Čech cohomology class in $H^2(X, \mathbb{C})$.

Most important, if we consider a differential line sheaf, the differential invariant de Rham cohomology class $[R]$ is *independent* of the *connection* used to *represent* R *locally*. Equivalently, a particular connection of a differential line sheaf provides *the means* to express this global differential invariant locally in terms of the corresponding *gauge potentials* of the field, whereas the latter is *independent* of the particular means used to represent it locally.

The fact that any two gauge equivalent differential sheaves have the *same* curvature, means that they are physically *indistinguishable*. Hence, the Abelian group $Iso(\mathcal{L}, \nabla)$ is spectrally *partitioned* into *orbits* over the image of $Iso(\mathcal{L}, \nabla)$ into Ω^2_c, where each orbit, or fiber of this partition, is labelled by a closed 2-form R of Ω^2_c:

$$Iso(\mathcal{L}, \nabla) = \sum_R Iso(\mathcal{L}, \nabla)_R. \quad (27)$$

Thus, the Abelian group of equivalence classes of differential line sheaves fibers over those elements of Ω_c^2, by which we mean that it fibers over those closed global 2-forms in Ω_c^2, which can be *identified* in terms of the *curvature* of the field.

Recapitulating the problem of gauge equivalent geometric spectra, the main issue is the *decoding* of a two-dimensional integral cohomology class, that is, the decoding of a *Chern class*, in *dynamical* terms expressed via the gauge field *potentials*. We bear in mind that a Chern class *encodes* the *singular* presence of a *matter* form via the Chern isomorphism.

This issue boils down to the idea of *decoding* a Chern class in terms of the curvature of the corresponding gauge field, which essentially amounts to decoding by means of the *flow* induced by the implicated singular matter form. According to this, the *Chern class* dynamical decoding takes place in an *invariant* manner through the *differential de Rham* cohomology class corresponding to the *curvature* of the field. In particular, since R is a global and closed 2-form on X it actually *determines* a 2-dimensional complex-valued de Rham cohomology class $[R]$, or equivalently a 2-dimensional complex Čech cohomology class in $H^2(X,\mathbb{C})$.

In the setting of gauge equivalent geometric spectra, the sheaf-theoretic formulation of the *Chern-Weil integrality theorem* states that the global closed 2-form is the curvature R of a differential line sheaf *if and only if* its 2-dimensional de Rham cohomology class is *integral*, more precisely, $[R] \in Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$.

This theorem establishes the *cohomological condition* for the consistency of the preceding encoding/decoding procedure, that is, the process of metaphora pertaining to gauge equivalent geometric spectra. More precisely, the singular presence of a matter form in gauge theory, encoded by means of a two-dimensional integral cohomology class according to the Chern isomorphism, is decoded dynamically by means of the two-dimensional cohomology class of the curvature of the gauge field, in such a manner that the latter is completely characterized intrinsically by the *integrality* cohomological condition.

It is precisely the above *integrality* condition that gives rise to *quantization* in the physical state of affairs. In other words, quantization is the condition that *resolves* the problem of *equivalence* of gauge geometric spectra. More precisely, a global closed 2-form is qualified as the curvature of a gauge field in its function to disclose the observable spectral periods of a singular matter form, *if and only if it is quantized*, meaning that the *integrality* condition is interpreted physically as a *quantization* condition, where the *intrinsic* characterization of the latter is purely of a *cohomological* nature.

The fact that the curvature cohomology class $[R]$ of any differential line sheaf in $H^2(X,\mathbb{C})$ is in the image of a cohomology class in the integral 2-dimensional cohomology group $H^2(X,\mathbb{Z})$ into $H^2(X,\mathbb{C})$ provides an intrinsic criterion for *recognition* of all those global closed 2-forms in Ω_c^2, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves, according to the *fibration*:

$$Iso(\mathcal{L},\nabla)=\sum_R Iso(\mathcal{L},\nabla)_R. \quad (28)$$

The proof of this theorem is based on the following commutative diagram in our set-up:

$$\begin{array}{ccc} H^1(X,\tilde{\mathcal{A}}) & \xrightarrow{\delta_c} & H^2(X,\mathbb{Z}) \\ \Big\downarrow \frac{1}{2\pi i}\tilde{d}^0 & & \Big\downarrow \iota^* \\ H^1(X,d^0\mathcal{A}) & \xrightarrow{\cong} & H^2(X,\mathbb{C}) \end{array}$$

Note that the image of a cohomology class of $H^2(X,\mathbb{Z})$ into an integral cohomology class of $H^2(X,\mathbb{C})$ corresponds to the cohomology class specified by the image of the 1-cocycle $g_{\alpha\beta}$ into $H^1(X,d^0\mathcal{A})$, viz. by the 1-cocycle $\frac{1}{2\pi i}\tilde{d}^0(g_{\alpha\beta})$. Moreover, due to the exactness of the exponential sheaf sequence of Abelian group sheaves:

$$0\to\mathbb{Z}\xrightarrow{\iota}\mathcal{A}\xrightarrow{exp}\tilde{\mathcal{A}}\to 1 \quad (29)$$
$$g_{\alpha\beta}=exp(w_{\alpha\beta}), \quad (30)$$

where $(w_{\alpha\beta})\in C^1(\mathcal{U},\mathcal{A})$, and by means of the Chern isomorphism:

$$\delta_c(w_{\alpha\beta})=(z_{\alpha\beta\gamma})\in Z^2(\mathcal{U},\mathbb{Z}). \quad (31)$$

Explicitly, we may consider $w_{\alpha\beta}=ln(g_{\alpha\beta})$, so that:

$$\delta_c(w_{\alpha\beta}) = (z_{\alpha\beta\gamma}) := \frac{1}{2\pi i}(ln(g_{\alpha\beta}) + ln(g_{\beta\gamma}) - ln(g_{\alpha\gamma})) \in Z^2(\mathcal{U},\mathbb{Z}). \quad (32)$$

The formulation of the *Chern-Weil* integrality theorem in *sheaf-theoretic* terms is the following:

$$[R] = 2\pi i \cdot [(z_{\alpha\beta\gamma})] \in H^2(X,\mathbb{C}), \quad (33)$$

$$[(g_{\alpha\beta})] = \frac{1}{2\pi i}[R] = [(z_{\alpha\beta\gamma})] \in H^2(X,\mathbb{Z}), \quad (34)$$

$$[R] \in Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})). \quad (35)$$

Thus, there exists an intrinsic *invariant* characterization of those global closed 2-forms in Ω_c^2, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves, denoted by $\Omega_{c,\mathbb{Z}}^2$. Concretely, a global closed 2-form is the *curvature* R of a differential line sheaf if and only if it is *quantized*, i.e. its 2-dimensional de Rham cohomology class is *integral*, meaning that $[R] \in Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$. Therefore the *unveiling* of a gauge field's observable *geometric spectrum* as a means to qualify the implicit presence of a *singular matter* form with which the field interacts with entails *quantization*.

To conclude, the quantization condition expresses the *harmonics* of gauge equivalent geometric spectra. The harmonics in their manifestation in terms of *integral spectral periods* are obtained dynamically by the integration of a global closed differential 2-form qualified as the curvature, which represents the *potentials' strength* of the corresponding *gauge field*. In particular, the integration of such a closed differential 2-form over a two dimensional cycle in X enclosing a material form with which the field is coupled to, gives rise to a characteristic integral spectral period. This can be grasped equivalently as a harmonic serving as the invariant of all gauge equivalent differential line sheaves sharing this form as their curvature.

7.6 THE OBSTACLE OF INERTIA: INTEGRABLE CONNECTIONS AND POLARIZATION

Čech cohomology theory is characterized by the fact that the group coefficient structure of the real or the complex numbers does not constrain the validity of Čech's theoretical framework in any way. Hence, *all* the steps of the Čech construction can be carried out for constant functions in any Abelian group, giving rise to Abelian cohomology groups with values in G for any abelian group of coefficients G. In particular, we may also consider a *locally variable* Abelian group of coefficients, i.e. an Abelian group sheaf of coefficients.

Since in the case of a *paracompact* topological space the calculation of cohomology with coefficients into a typical sheaf of coefficients is *equivalent* to the calculation of *Čech cohomology* theory with *values* in the corresponding *sheaf*, the calculation of cohomology with values in an Abelian group sheaf of locally constant functions is of major importance. The basic idea is that the natural argument of a cohomology theory is not just a global topological space or a topological manifold, which it is invoked *only implicitly* in our considerations, but a space *together* with an observable algebra *sheaf of coefficients*, in such a way that the point constitution of this space, including both standard and singular points, is *unveiled* via the *spectrum* of our geometric calculus, by which we mean, in terms of the pertinent *cohomological analysis*.

Prior to the modern formulation of sheaf theory, the significance of the notion of *homology with local coefficients* has been treated in relation to the fundamental group of a connected topological space X, an approach pioneered by *Norman Steenrod*.

For any point x of X we consider the *fundamental group* F_x of X based at x. If Γ is a curve from x to y, the class of curves from x to y *homotopic* to Γ is denoted by γ_{xy}. Its inverse is denoted by $\gamma_{xy}{}^{-1} := \gamma_{yx}$. In this way, the elements of F_x are denoted by α_x, β_x, and so on. Moreover, the product $\alpha_x \cdot \beta_x$ denotes the element of F_x obtained by first traversing a curve of the class α_x, and then of the class β_x. The class γ_{xy} determines an *isomorphism* of groups $F_x \rightarrow F_y$, denoted by the same symbol, and defined by *conjugation*, i.e. $\gamma_{xy}(\alpha_x) = \gamma_{yx} \cdot \alpha_x \cdot \gamma_{xy}$. In this context, the combination of two isomorphisms of the above form is also an isomorphism.

Then, we may define a *system of local groups* in X in terms of the following three conditions:

First, for each point x of X, there is given a group G_x;

Second, for each class of paths γ_{xy}, there is given a group isomorphism $G_x \rightarrow G_y$, denoted by the same symbol;

Third, the result of the isomorphism γ_{xy} followed by γ_{yz} is the isomorphism corresponding to the path $\gamma_{xy} \cdot \gamma_{yz}$. Note that the identity path from x to x is the identity transformation in the group G_x.

We note that a closed path o_x of F_x determines an *automorphism* of G_x. From the third property, it follows that F_x is a group of automorphism of G_x. Now, the *invariant subgroup* of F_x acting as the identity on G_x is denoted by $F^1{}_x$.

Then, we define a system of *local groups* $\{G_x\}$ in X to be simple if every $F^1{}_x = F_x$. Moreover, if this happens for one x, it will be true for all x in X. If $\{G_x\}$ is a simple system of groups, then the isomorphism γ_{xy} is independent of the path from x to y. Choosing a fixed point z as *origin*, each G_x is uniquely isomorphic to G_z. Thus, the local system consists of *one* G_z and as many *copies* of G_z as there are points $x \neq 0$. In this context, the collection $\{F_x\}$ is a simple system of local groups if and only if it is *Abelian*. Furthermore, the *abelianization* of the fundamental group gives rise to a simple system of local groups consisting of isomorphic copies of the same abelian group, identified as the *first homology group* of X.

From the perspective of sheaf theory, the main objective consists in the operative role of a cohomology theory with values in a locally constant group sheaf of coefficients, namely a group sheaf on X for which every point x in X has an open cover U such that the restriction of this sheaf to U is a *constant* sheaf. Local coefficients may be extended to Čech cycles by constructing a system of local groups in the *simplicial nerve* of a finite open covering of X, and then, demonstrating the *isomorphism* of cohomology -taking values in a system of local groups- as above, with Čech cohomology.

If we consider vector sheaves of rank 1, i.e. line sheaves, we recall that locally, for any point $x \in X$, an open cover U of X exists such that: $\mathcal{L}|_U \cong \mathcal{A}|_U$. Furthermore, if for any point $x \in X$ an open cover U of X exists such that $\Psi|_U \cong (\mathbb{C}|_U)$, meaning that it is isomorphic to the *constant* sheaf $\mathbb{C}$, then the locally free $\mathbb{C}$-module Ψ of finite rank 1 is a complex *linear local system* of rank 1; a *line local system*.

A very important observation is that a *constant* 1 *-cocycle* $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathbb{C}})$ can be interpreted as the *coordinate* 1-cocycle of a *particular type* of a *line sheaf* with respect to an open covering $\mathcal{U}$. Since the coordinate 1 -cocycle $(\xi_{\alpha\beta})$ is constant, the *bijectively* corresponding to it line sheaf is a complex linear local system of rank 1, or *a line local system*. The natural question arising in this context is from where do line local systems *descend from* and what is their role in the *spectrum unveiling* conducted through cohomological analysis.

A first observation regarding line local systems is that if the underlying space X is simply-connected, then they are actually constant. More generally, if X is s path-connected and paracompact base topological space, then the pullback of a line local system on X to the *universal covering* space of X becomes a *constant sheaf*. In relation to the physical framework of gauge theory, the pertinent issue refers to the means of instantiation of a line local system in dynamical terms, that is, in terms of a differential line sheaf. It turns out that a *bijective* correspondence pertains between *differential line sheaves* equipped with an *integrable connection* and *line local systems*. Given that an integrable connection subsumes *the covariantly constant*, by which we mean the inertial structure of a gauge field, we derive that the *inertial structure* is *encoded cohomologically* in terms of a corresponding *line local system* in its function as a *coefficient sheaf* for cohomology.

For this purpose, let us consider any differential line sheaf $(\mathcal{L}, \nabla)$, which lacks curvature and whose connection ∇ is thereby integrable. Then, the set of sections of $\mathcal{L}$, which reside in the *kernel* of the connection ∇, that is:

$$Ker(\nabla) := \{s \in \mathcal{L} : \nabla(s) = 0\} \quad (1)$$

forms a line local system. We call the sections of $Ker(\nabla)$ covariantly constant, or equivalently, inertial sections of $\mathcal{L}$ with respect to ∇. Inversely, given a line local system, denoted by Λ, we may define a differential line sheaf $(\mathcal{L}, \nabla)$ by the prescription $\mathcal{L} := \mathcal{A} \otimes_{\mathbb{C}} \Lambda$, and for every pair of local sections $a \in \mathcal{A}(U)$, $s \in \Lambda(U)$, $\breve{\nabla}(a \otimes s) := da \otimes s$. The above defined connection is *integrable*, and therefore we conclude that a *bijective* correspondence exists between differential line sheaves with an integrable connection, denoted by $(\mathcal{L}, \breve{\nabla})$ and line local systems. Most important, *every* line local system may be *identified* with the sheaf of *inertial sections* of an integrable differential line sheaf, meaning of a line sheaf $\mathcal{L}$ with respect to an integrable connection $\breve{\nabla}$ on $\mathcal{L}$, hence with $Ker(\breve{\nabla})_{\mathcal{L}}$.

Therefore, we derive the following equivalences:

$$Z^1(\mathcal{U}, \tilde{\mathbb{C}}) \nprec (\xi_{\alpha\beta}) \cong (\mathcal{L}, \breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \quad (2)$$

$$H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \quad (3)$$

Since the underlying topological space X is locally path-connected, then the above series of equivalences is refined as follows:

$$Hom(\pi_1(X),\tilde{\mathbb{C}}) \cong H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda], \quad (4)$$

where the first term denotes the set of *representations* of the fundamental group of the topological space X to $\tilde{\mathbb{C}}$.

All the previous considerations are immediately extended to the case of isomorphism classes of *Hermitian line sheaves*, that is, line sheaves equipped with a *Hermitian inner product* structure. More specifically, given a line sheaf $\mathcal{L}$ on X, an $\mathcal{A}$*-valued Hermitian inner product* on $\mathcal{L}$ is a skew-$\mathcal{A}$-bilinear sheaf morphism:

$$\mathcal{O} : \mathcal{L} \oplus \mathcal{L} \to \mathcal{A}, \quad (5)$$
$$\mathcal{O}(\alpha s, \beta t) = \alpha \cdot \bar{\beta} \cdot \mathcal{O}(s,t), \quad (6)$$

for any s, $t \in \mathcal{L}(U)$, α, $\beta \in \mathcal{A}(U)$, U open in X. Moreover, $\mathcal{O}(s,t)$ is skew-symmetric, viz. $\mathcal{O}(s,t) = \overline{\mathcal{O}(s,t)}$.

A line sheaf $\mathcal{L}$ on X, together with an $\mathcal{A}$-valued Hermitian inner product on $\mathcal{L}$ constitute a *Hermitian line sheaf*. A line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ associated with the open covering $\mathcal{U}$. A Čech coordinate 1-cocycle $g_{\alpha\beta}$ corresponding to a Hermitian line sheaf consists of local sections of $\mathcal{S}U(1,\mathcal{A})$, the *special unitary group sheaf* of $\mathcal{A}$ of order 1.

This is simply a coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$, such that the *unitarity* condition $|g_{\alpha\beta}| = 1$ is satisfied. Clearly, in the case that a coordinate 1-cocycle $g_{\alpha\beta}$ is constant, we have $g_{\alpha\beta}$ in $Z^1(\mathcal{U},U(1))$, or equivalently $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\mathbb{S}^1)$. Next, a connection ∇ on $\mathcal{L}$ is called *Hermitian* if it is compatible with $\mathcal{O}$:

$$d^0\mathcal{O}(s,t) = \mathcal{O}(\nabla s,t) + \mathcal{O}(s,\nabla t) \quad (7)$$

for any s, $t \in \mathcal{L}(U)$, U open in X. A *Hermitian differential line sheaf*, denoted by $(\mathcal{L}, \nabla, \circlearrowleft) := (\mathcal{L}, \nabla_{\circlearrowleft})$, is a Hermitian line sheaf equipped with a Hermitian connection.

The global 2-form R, which belongs to $Ker(d^2): \Omega^2 \rightarrow \Omega^3$, called Ω^2_c, identified as a $\mathbb{C}$-vector sheaf subspace of Ω^2, determines a global differential invariant of gauge equivalent Hermitian differential line sheaves, because the global 2-form R determines a 2-dimensional de Rham cohomology class $[R]$. The Hermitian connection of a Hermitian differential line sheaf provides *the means* to express this global differential invariant *locally*, whereas the latter is actually independent of the connection utilized to represent it locally.

From the curvature recognition integrality theorem, the abelian group $Iso(\mathcal{L}, \nabla)$ is partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each orbit is labelled by an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$, providing the differential invariant $[R]$ of this orbit in de Rham cohomology:

$$Iso(\mathcal{L}, \nabla) = \sum_{R \in \Omega^2_{c,\mathbb{Z}}} Iso(\mathcal{L}, \nabla)_R. \quad (8)$$

If we *restrict* the Abelian group $Iso(\mathcal{L}, \nabla)$ to gauge equivalent Hermitian differential line sheaves we obtain an Abelian subgroup of the former, denoted by $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$. It is clear that the latter Abelian group is also partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each orbit is labelled by an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$, where R is the curvature of the corresponding gauge equivalence class of Hermitian differential line sheaves.

We call each Hermitian differential line sheaf $(\mathcal{L}, \nabla_{\circlearrowleft})$, which belongs to an equivalence class, that is to an *orbit* $Iso(\mathcal{L}, \nabla_{\circlearrowleft})_R$ a *unitary* R*-ray*. Concomitantly, we call the orbit itself a *spectral* $[R]$*-beam*, which is characterized by the *integral differential invariant* $[R]$. Each spectral $[R]$-beam consists of gauge equivalent unitary R-rays, which are *indistinguishable* from the perspective of their common curvature integral differential invariant $[R]$, physically meaning that they are characterized dynamically by the *same* field strength.

A natural question arising in the context of gauge equivalent quantum unitary R-rays is *how* they are related to each other. In other words, although all gauge equivalent unitary R-rays *cannot* be distinguished from the *perspective* of their *curvature* differential *invariant*, is there any other *intrinsic* way that we can distinguish among them? It is precisely at this point that the significance of the *global inertial structure* of a spectral $[R]$-beam manifests in unveiling the constitution of gauge equivalent geometric spectra.

Henceforth, the only intrinsic and invariant way of distinguishing among gauge equivalent unitary R-rays is through their *global inertial structure*, which is induced by the action of a line local system of the form Λ. Equivalently, there exists a free group action of the abelian group $H^1(X,\mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L},\nabla_{\mathcal{O}})$, which is restricted to a free group action on each spectral $[R]$-beam.

First, we note that there exists a free group action of the Abelian group sheaf $\mathbb{S}^1 \infty \mathbb{C} \infty \mathcal{A}$ on the Abelian group sheaf $\tilde{\mathcal{A}}$ of invertible elements of $\mathcal{A}$, where $\mathbb{S}^1 \equiv \mathcal{U}(1) \equiv \mathcal{S}U(1,\mathbb{C})$:

$$\mathbb{S}^1 \times \tilde{\mathcal{A}}(U) \to \tilde{\mathcal{A}}(U), \quad (9)$$
$$(\xi, f) \mapsto \xi \cdot f. \quad (10)$$

with $\xi \in \mathbb{S}^1$ and $f \in \tilde{\mathcal{A}}(U)$ for any open U in X. This action is transferred naturally as a *free action* to the corresponding groups of coordinate 1-cocycles of the respective Abelian group sheaves:

$$Z^1(\mathcal{U},\mathbb{S}^1) \times Z^1(\mathcal{U},\tilde{\mathcal{A}}) \to Z^1(\mathcal{U},\tilde{\mathcal{A}}), \quad (11)$$
$$(\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}), \quad (12)$$

where $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U},\mathbb{S}^1)$, $(g_{\alpha\beta}) \in Z^1(\mathcal{U},\tilde{\mathcal{A}})$. This free action can be also *extended* to the corresponding *cohomology groups* still as a free action:

$$H^1(X,\mathbb{S}^1) \otimes H^1(X,\tilde{\mathcal{A}}) \to H^1(X,\tilde{\mathcal{A}}), \quad (13)$$
$$[(\xi_{\alpha\beta})] \cdot [(g_{\alpha\beta})] = [(\xi_{\alpha\beta} \cdot g_{\alpha\beta})], \quad (14)$$

where $[(\xi_{\alpha\beta})] \in H^1(X,\mathbb{S}^1)$, $[(g_{\alpha\beta})] \in H^1(X,\tilde{\mathcal{A}}) \cong Iso(\mathcal{L})$.

Next, we define a group action of $H^1(X,\mathbb{S}^1)$ on the Abelian group $Iso(\mathcal{L},\nabla)$ as follows: We consider $\xi \equiv [(\xi_{\alpha\beta})] \in H^1(X,\mathbb{S}^1)$, $[(\mathcal{L},\nabla)] \in Iso(\mathcal{L},\nabla)$, and we define the sought group action as follows:

$$\xi \cdot [(\mathcal{L},\nabla)] := [(\xi \cdot \mathcal{L},\nabla)] \equiv [(\mathcal{L}',\nabla)], \quad (15)$$

$$\mathcal{L}' = \xi \cdot \mathcal{L} \leftrightarrow (\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}). \quad (16)$$

We see easily that the pair $(\xi_{\alpha\beta} \cdot g_{\alpha\beta}, \omega_\alpha)$ satisfies the *transformation law* of local gauge potentials, i.e. $\delta(\omega_\alpha) = \tilde{d}^0(\xi_{\alpha\beta} \cdot g_{\alpha\beta})$. Given that $Ker(\tilde{d}^0) = \tilde{\mathbb{C}}$, as a consequence of the Poincaré Lemma, it follows directly that the above defined group action of $H^1(X, \mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L}, \nabla)$ is actually *free*, where $[(\xi_{\alpha\beta})] = 1 \in H^1(X, \mathbb{S}^1)$.

Consequently, the free group action of $H^1(X, \mathbb{S}^1)$ on $Iso(\mathcal{L}, \nabla)$ is *restricted* to a free group action on its *Abelian subgroup* of *unitary rays* $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$. Since the Abelian group $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$ is *partitioned* into spectral $[R]$-beams over $\Omega^2_{c,\mathbb{Z}}$ constituting its *spectrum*, we derive that the above free group action is finally transferred as a *free group action* of $H^1(X, \mathbb{S}^1)$ on *each spectral* $[R]$*-beam.*

We call a cohomology class in the abelian group $H^1(X, \mathbb{S}^1)$ a *polarization phase germ* of a spectral $[R]$-beam. Our terminology derives from the fact that a cohomology class in the Abelian group $H^1(X, \mathbb{S}^1) \cong H^1(X, U(1))$ is evaluated through a representative *cocycle* at a homology *cycle* $\gamma \in H_1(X)$ by means of the *integration pairing*:

$$H_1(X) \times H^1(X, U(1)) \to U(1) \quad (17)$$

to obtain a *global* observable *gauge-invariant phase factor* in the Abelian group $U(1)$. Thus, gauge equivalent unitary R-rays are *intrinsically* and *invariantly* distinguished by means of a polarization phase germ, which is identified as a cohomology class in the group $H^1(X, \mathbb{S}^1)$.

The significant observation in this context is that a polarization phase germ of a spectral $[R]$-beam is *always realized* via a *representation* of the *fundamental group* of the topological space X to $\mathbb{S}^1$. As an immediate consequence of the *Hurewicz isomorphism*:

$$Hom(\pi_1(X), \tilde{\mathbb{C}}) \cong H^1(X, \tilde{\mathbb{C}}), \quad (18)$$

if we restrict to the unitary case we obtain:

$$Hom(\pi_1(X), \mathbb{S}^1) \cong H^1(X, \mathbb{S}^1). \quad (19)$$

Thus, a polarization phase germ, expressed in terms of a cohomology class in $H^1(X,\mathbb{S}^1)$, is realized by a representation of the fundamental group of the topological space X to $\mathbb{S}^1$.

We have demonstrated thus far that the action of the group $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$ on each spectral $[R]$-beam is a *free* group action. The *encoding* of the global inertial structure in group-theoretic terms requires the investigation of the conditions qualifying this free action as a *transitive* one as well in the context of gauge equivalent spectra of unitary rays.

Given the validity of the Poincaré Lemma, we consider the following sequence of abelian group sheaves:

$$1\rightarrow\tilde{\mathbb{C}}\xrightarrow{\iota}\tilde{\mathcal{A}}\xrightarrow{\tilde{d}^0}\Omega^1\xrightarrow{d^1}d^1\Omega^1\rightarrow 0. \quad (20)$$

The closed 1-forms θ_α of Ω^1, satisfying $Im(\tilde{d}^0)=Ker(d^1)$ are called *logarithmically exact closed* 1*-forms*.

The significance of logarithmically exact closed 1-forms in relation to the global inertial structure lies on the fact that the above *sequence* of Abelian group sheaves is an *exact sequence* if *restricted* to logarithmically exact closed 1-forms. In this case, we obtain a 0-cochain (θ_α) of logarithmically exact closed 1-forms:

$$(\theta_\alpha)\in C^0(\mathcal{U},Ker(d^1))=(\theta_\alpha)\in C^0(\mathcal{U},Im(\tilde{d}^0))=\tilde{d}^0(C^0(\mathcal{U},\tilde{\mathcal{A}})). \quad (21)$$

Hence, for a 0-cochain (θ_α) of logarithmically exact closed 1-forms θ_α, a 0-cochain t_α in $\tilde{\mathcal{A}}$ exists, such that $\theta_\alpha=\tilde{d}^0(t_\alpha)$. This 0-cochain (θ_α) may be thought of as the *representative* of an *integrable connection* $\breve{\nabla}$ of a differential line sheaf, whose coordinate 1-cocycle with respect to an open covering $\mathcal{U}$ is given by $\zeta_{\alpha\beta}=t_\beta^{-1}t_\alpha$, i.e. it is a *coboundary*, which *satisfies* the transformation law of local gauge potentials $\delta^0(\theta_\alpha)=\tilde{d}^0(\zeta_{\alpha\beta})$.

Next, we consider a spectral $[R]$-beam, namely the spectral partition class, $Iso(\mathcal{L},\nabla_{\circlearrowleft})_R$, consisting of gauge equivalent unitary R-rays, which are indistinguishable from the perspective of their common differential invariant $[R]$. We also consider those closed 1-forms θ_α of Ω^1, which qualify as logarithmically exact. If we take a pair of equivalent unitary R -rays, denoted by $(\mathcal{L},\nabla_{\circlearrowleft})$, $(\mathcal{L}',\nabla'_{\circlearrowleft})$ correspondingly, we obtain:

$$R=(d\omega_{\alpha})=(d\omega'_{\alpha}), \quad (22)$$
$$d(\omega_{\alpha}-\omega'_{\alpha})=0. \quad (23)$$

We conclude that $(\omega_{\alpha}-\omega'_{\alpha})$ is of the form θ_{α} of Ω^1, hence it is a logarithmically exact closed 1-form. Based on this fact, we derive that the *free* group action of $H^1(X,\mathbb{S}^1)$ on a spectral $[R]$-beam is *also transitive* with respect to logarithmically exact closed 1-forms.
In conclusion, a spectral $[R]$ -beam becomes a $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$ -*affine space*, or equivalently, an affine space with *structure* group the *characters* of the fundamental group. The latter provides a complete characterization of the global inertial structure of a spectral $[R]$-beam in group-theoretic terms.

Each partition class of the spectrum, each orbit or fiber $Iso(\mathcal{L},\nabla_{\circlearrowleft})_R$, labelled by the curvature differential invariant $[R]$, which is to say each spectral $[R]$-beam is an affine space with structure group $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$. Thus, *any* two unitary R-rays *differ* by an element of $H^1(X,\mathbb{S}^1)$, and conversely any two unitary rays which differ by an element of $H^1(X,\mathbb{S}^1)$ are characterized by the *same* differential invariant $[R]$; in short, they constitute R-rays of the *same* spectral $[R]$-beam.

We conclude that, although all gauge equivalent unitary R-rays *cannot* be distinguished from the perspective of their common curvature differential invariant, nevertheless a free and transitive action of the group $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$ exists, characterizing the global inertial structure of a spectral $[R]$ -beam cohomologically in group-theoretic terms and corresponding *bijectively* to the respective line local system. Equivalently put, R-rays of the same spectral $[R]$-beam can be *distinguished inertially* via the characters of the fundamental group of X.

Inversely, from any one unitary R-ray of a spectral beam we can *generate* its *whole* orbit, identified with the *beam* itself, by means of the free and transitive action of the abelian group $H^1(X,\mathbb{S}^1)$ on the one depicted. Thus, whenever two unitary rays are characterized by the same differential invariant $[R]$, thereby belonging to the same orbit under the action of $H^1(X,\mathbb{S}^1)$ on $Iso(\mathcal{L},\nabla_{\circlearrowright})_R$, which is actually the *only* orbit due to transitivity of this action, identified as a spectral $[R]$-beam, then they differ by a *character* of the fundamental group of the carrier base topological space X.

7.7 INTERFERENCE: TOPOLOGICAL PHASE AND THE POLYDROMY OF A SPECTRAL BEAM

A polarization phase germ of a spectral $[R]$-beam, identified as a cohomology class in the Abelian group $H^1(X,\mathbb{S}^1)$, is realized by a representation of the fundamental group of the base connected topological space X to $\mathbb{S}^1$. We observe that the evaluation of a representative cocycle of $H^1(X,\mathbb{S}^1)$ at a homology cycle $\gamma \in H_1(X)$ by means of the integration pairing:

$$H_1(X)\times H^1(X,U(1))\rightarrow U(1) \quad (24)$$

gives a global observable gauge-invariant phase factor in the Abelian group $U(1)$. In this sense, gauge equivalent unitary spectral rays bearing the same strength are intrinsically distinguished by means of a polarization phase germ.

Consequently, the global polarization symmetry group of a spectral $[R]$-beam is *realized* in terms of a unitary line local system. Equivalently formulated, there exists a *bijective* correspondence between the polarization phase germs of a spectral $[R]$-beam and isomorphism classes of unitary line local systems, due to the following equivalences:

$$Z^1(\mathcal{U},\tilde{\mathbb{C}}) \nprec (\xi_{\alpha\beta}) \cong (\mathcal{L},\breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \quad (25)$$
$$H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \quad (26)$$

The above equivalences read as follows in the case of Hermitian integrable differential line sheaves, or integrable unitary rays:

$$Z^1(\mathcal{U},\mathbb{S}^1) \nprec (\xi_{\alpha\beta}) \cong (\mathcal{L},\breve{\nabla}_{\circlearrowright}) \cong Ker(\breve{\nabla}_{\circlearrowright})_{\mathcal{L}} \cong \Lambda_{\circlearrowright}, \quad (27)$$
$$H^1(X,\mathbb{S}^1) \cong [(\mathcal{L},\breve{\nabla}_{\circlearrowright})] \cong [Ker(\breve{\nabla}_{\circlearrowright})_{\mathcal{L}}] \cong [\Lambda_{\circlearrowright}]. \quad (28)$$

Let us examine in more detail the realization of the polarization symmetry group of a spectral $[R]$-beam at a point of the base space X. At each point of the topological space X, the polarization symmetry group of a spectral $[R]$-beam is *realized* thanks to the *monodromy group* of a unitary line local system at the depicted point, or equivalently by the group of monodromies of the corresponding integrable unitary ray whose covariantly constant sections form this unitary line local system. This is established as follows:

We already know that a polarization phase germ of a spectral $[R]$-beam is realized by a representation of the fundamental group of X to $\mathbb{S}^1$. Moreover, $\tilde{\mathbb{C}}$ is identified locally with the group of automorphisms of a line local system, $GL(1,\mathbb{C}) \cong \tilde{\mathbb{C}}$. Thus, in the unitary case, $\mathbb{S}^1$ is *identified locally* with the group of *automorphisms* of a unitary line local system $\Lambda_{\mathcal{O}}$, which may be thought of as the *locally constant* sheaf of the inertial, or covariantly constant sections of a corresponding Hermitian integrable line sheaf (*integrable unitary ray*) $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$.

We may fix a base point x_0 of X, and consider a path $\gamma:[0,1] \rightarrow X$, such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. Thus, if Λ is a line local system on X, then it is *pulled back* to $[0,1]$ as a *constant* sheaf, denoted by $\gamma^{\leftleftarrows}(\Lambda)$. Hence, we obtain that:

$$(\gamma^{\leftleftarrows}(\Lambda))_0 \cong (\gamma^{\leftleftarrows}(\Lambda))_1. \quad (29)$$

Henceforth, because of the *isomorphisms* $(\gamma^{\leftleftarrows}(\Lambda))_0 \cong \Lambda_{\gamma(0)} := \Lambda_{x_0}$ and $(\gamma^{\leftleftarrows}(\Lambda))_1 \cong \Lambda_{\gamma(1)} := \Lambda_{x_1}$, we have at our disposal a $\mathbb{C}$-vector space *isomorphism* $\Lambda_{x_0} \cong \Lambda_{x_1}$, which depends only on the *homotopy class* of γ.

Furthermore, this isomorphism may be thought of as being induced by the *parallel transport condition* of the corresponding integrable connection of the Hermitian integrable line sheaf $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$. If we consider a loop based at the point x_0 of X, we obtain an Abelian group homomorphism:

$$\mu: \pi_1(X, x_0) \rightarrow GL(\Lambda_{x_0}) \cong (GL(1,\mathbb{C}))_{x_0} \cong \tilde{\mathbb{C}}_{x_0}. \quad (30)$$

In the case of a unitary line local system, we correspondingly obtain the following Abelian group homomorphism:

$$\mu: \pi_1(X, x_0) \rightarrow \mathbb{S}^1_{x_0}. \quad (31)$$

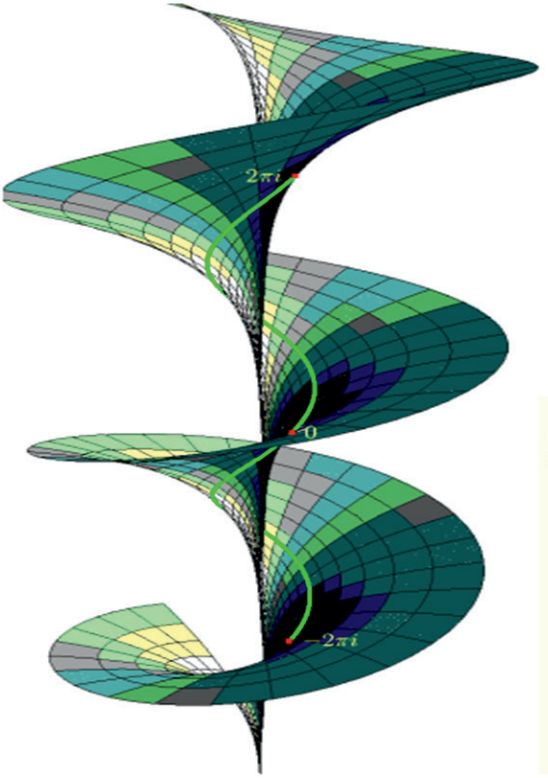
The *image* of μ in $\mathbb{S}^1_{x_0}$ is the *monodromy group* of the unitary local system $\Lambda_{\mathcal{O}}$. Thus, for each homotopy class of loops based at x_0, we obtain a topological integrable phase factor, identified with the monodromy of the unitary local system $\Lambda_{\mathcal{O}x_0}$. Equivalently, this is the same as the monodromy of the corresponding integrable quantum unitary ray $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$, derived through the parallel transport along a loop based at x_0 and belonging to a homotopy class in $\pi_1(X, x_0)$.

The notion of *monodromy* originates from complex function theory on Riemann surfaces, and more specifically, from the theory of linear differential equations on Riemann surfaces. In general, a solution to a linear differential equation in the setting of analytic or holomorphic dynamics, is characterized as *polydromic* if the *circulation* around a loop enclosing a *singularity* produces a *different value* in comparison to the *initial* one.

By contrast, it is called *monodromic* if this phenomenon does not arise. In this manner, the crystallized term *"monodromy"* constitutes a *misnomer* compared to the meaning enclosed in this Greek term, i.e. it should be properly called *"polydromy"*. This should also be in accordance with Riemann's distinction between single and multi-valued magnitudes and the concomitant process of analytic continuation of a local solution along different paths or *"dromoi"*. Unfortunately, the term "monodromy" is currently established as such, so we will adopt the present usage modulo the present clarification.

As an example, we may consider the differential equation $zf'(z) = 1$, which is *singular* at the *origin* of the complex plane. As a consequence, the local solution $f(z) = \log(z)$, if circulated around the origin, for each closed path or *"dromos"*, its value is modified by an *integer multiple* of $2\pi i$. For this reason, we need to invoke the *universal covering* space defined by the complex exponential $exp(z): \mathbb{C} \to \mathbb{C}^*$.

Accordingly, a polydromic solution to the given differential equation, that is, a determination of the *many-valued* function $\log(z)$ takes place only in terms of *local sections* of the complex exponential covering projection map $exp(z)$. Each section defined on $\mathbb{C} - \{ray\}$ constitutes an *inversion* of $exp(z)$, but only *locally*, and as such, it provides to a local determination of the logarithm. In particular, each section bears the form $\log(z) + 2\kappa i\pi$, where κ is an integer. The compatible gluing of all these local determinations over their non-trivial overlaps gives rise to the *Riemann surface* of the many-valued function $\log(z)$, represented below in helicoidal form:



A concrete observable manifestation of a global integrable phase factor in the context of *quantum mechanics* is provided by the effect discovered by *Yakir Aharonov and David Bohm*. This effect demonstrates the significance of the *local* electromagnetic gauge *potentials* of a spectral beam whose strength is null. More specifically, what constitutes the *Aharonov-Bohm effect* is an experimentally verified and observed *global relative phase factor* whose origin is *topological*, and thus expressed via an integrable connection pertaining to the description of the electromagnetic field.

The setting involves a very long solenoid restricting the magnetic field *flux* within its borders, in consequence rendering the region it occupies *inaccessible* to a charged particle. In this sense, the base topological space of localization that carries the field is multiply-connected, bearing the homotopical symmetry of a *circle*. The evolving states are transported by an *integrable connection* because the propagation takes place in the field strength-free region outside the solenoid. The observed global phase factor measures the *monodromy* of this integrable connection due to the topological obstacle imposed by the inaccessible region, which is enclosed by the *boundary* of the solenoid, where the *flux* of the magnetic field takes place.

The Aharonov-Bohm effect constitutes a perfect demonstration of the nature and significance of global observable integrable phase factors arising from *topological obstacles* in quantum theory. In particular, it demonstrates the following: First, the local gauge freedom of the phase of a quantum state; Second, the mutually implicative roles of the local and global levels in the quantum field theoretic description pertaining to a line sheaf of states endowed with an integrable connection. This consists of an extensive integration process of the contributions of all local gauge potentials to monodromies from the local to the global, and inversely, of a differential localization process of the global topological phase invariant, characterizing all gauge equivalent rays, in terms of the whole multiplicity of local gauge potentials. Taken all together, these gauge potentials express the permissible contextual variability of the connection with respect to this invariant.

Let us examine more specifically the details pertaining to an *integrable topological phase factor* from our viewpoint. We start from the observation that the Abelian group of quantum unitary rays $Iso(\mathcal{L},\nabla_{\mathcal{O}})$ constitutes a *central extension* of the Abelian group of integral global closed 2-forms R of $\Omega^2_{c,\mathbb{Z}}$ by the Abelian group of polarization phase germs $H^1(X,\mathbb{S}^1)$. This is due to the exactness of the following sequence of Abelian groups:

$$1 \to H^1(X,\mathbb{S}^1) \xrightarrow{\sigma} Iso(\mathcal{L},\nabla_{\mathcal{O}}) \xrightarrow{\kappa} \Omega^2_{c,\mathbb{Z}} \to 0. \quad (32)$$

In particular, if we consider spectral $[R]$-beams, each corresponding *partition spectral block* of *equipotent rays* $Iso(\mathcal{L},\nabla_{\mathcal{O}})_R$ is an *affine* space with structure *group* $H^1(X,\mathbb{S}^1)$. The integrality condition in this context is tantamount to *Paul Dirac's quantization condition*, meaning cohomologically that $(2\pi i)^{-1}[R]$ is a 2-dimensional *integral* cohomology class of X. Moreover, a zero curvature spectral $[R]$-beam $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$ is isomorphic to the Abelian group of polarization phase germs $H^1(X,\mathbb{S}^1)$, according to the preceding.

We have that $Iso(\mathcal{L},\nabla_{\mathcal{O}})_R = \kappa^{-1}(R)$, where $R \in \Omega^2_{c,\mathbb{Z}}$. Therefore, due to the fact that $H^1(X,\mathbb{S}^1) \infty\ Iso(\mathcal{L},\nabla_{\mathcal{O}})$ we obtain:

$$\kappa^{-1}(0) = Iso(\mathcal{L},\nabla_{\mathcal{O}})_0 \cong H^1(X,\mathbb{S}^1). \quad (33)$$

Consequently, a spectral $[R]$-beam is an $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$-*torsor*, by which we mean an affine space with respect to *logarithmically exact closed* 1-*forms*.

The Aharonov-Bohm global phase factor refers to the realization of a zero curvature spectral $[R]$-beam $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$. Equivalently, it is the experimentally realized *global* gauge-invariant phase characteristic of a gauge equivalence class of integrable Hermitian differential line sheaves, or integrable quantum unitary rays. Thus, we derive the following:

$$Iso(\mathcal{L},\nabla_{\mathcal{O}})_0 \cong H^1(X,\mathbb{S}^1) \cong Hom(\pi_1(X),\mathbb{S}^1). \quad (34)$$

In particular, for each point x_0 of the base topological space X, we have:

$$\mu : \pi_1(X,x_0) \to \mathbb{S}^1{}_{x_0} \cong U(1). \quad (35)$$

Therefore, the *global phase factor* of the Aharonov-Bohm type is realized as the global *monodromy* group element $\mu(\gamma) \in U(1)$ for *each homotopy class* of loops γ based at x_0. Note that in the experimental setting the base topological space X is homotopically *contractible* to the circle, and hence, its second integer cohomology is trivial.

These features establish a gauge equivalence class of integrable Hermitian differential line sheaves, or equivalently a gauge equivalence class of line local systems on the *circle*, which form a zero curvature spectral beam. The global gauge-invariant phase factors by which this beam is realized is the monodromy group associated with it, which is identified with the image of the fundamental group of the circle; the integers $\mathbb{Z}$ into $U(1)$. The monodromy depends *only* on the *integer winding number* and is observed as a *shift* in the interference pattern of the beam. Physically, the integer winding numbers obtained topologically, descend from the *harmonics* of the spectrum of the beam, whereas *interference* constitutes its observable geometric manifestation.

We emphasize again that an Aharonov-Bohm type of phase for non-simply connected base space X, due to the presence of obstructions such as those in our experimental case, refers to the global gauge-invariant observable factor pertaining to a zero curvature beam, i.e. to a gauge equivalence class of zero curvature quantum unitary rays. From a *cohomological* perspective, the analysis of these types of phases proceeds as follows:

We notice that a zero curvature beam is actually *isomorphic* to the *Abelian group* $H^1(X,U(1))$. Hence, it can be grasped as follows:

$$[\Psi(\theta_\alpha,-)] = exp(\frac{ie}{\hbar c}\int_{[-]}\theta_\alpha), \quad (36)$$

where we have inserted the corresponding physical units, and we consider θ_α real-valued in the Lie algebra of $U(1)$.

We conclude that a *representative* of the *class* identified with a zero curvature beam $[\Psi(\theta_\alpha,-)]$ is an element of $H^1(X,U(1))$, which is *evaluated* at a homology cycle $\gamma \in H_1(X)$ by means of the *pairing* of groups:

$$H_1(X)\times H^1(X,U(1)) \to U(1), \quad (37)$$

$$(\gamma,[\Psi(\theta_\alpha,-)]) \mapsto [\Psi(\theta_\alpha,\gamma)] = exp(\frac{ie}{\hbar c}\int_\gamma \theta_\alpha), \quad (38)$$

where $[\Psi(\theta_\alpha,\gamma)]$ is identified as a global Aharonov-Bohm gauge-invariant topological phase factor of the beam in $U(1)$.

For reasons pertaining to the significance of the partition spectrum under investigation, it is instructive to point out that if we consider any unitary ray of this beam, then we only obtain a real-valued phase, defined by:

$$\Psi(\theta_\alpha,\gamma) = \frac{e}{\hbar c}\int_\gamma \theta_\alpha. \quad (39)$$

Due to the isomorphism of groups $\mathbb{R}/\mathbb{Z} \cong U(1)$, we have to take the quotient of the set of all $\Psi((\theta_\alpha)_i,\gamma)$ for all unitary rays by the equivalence relation: $\Psi((\theta_\alpha)_1,\gamma) \sim \Psi((\theta_\alpha)_2,\gamma)$ if $((\theta_\alpha)_1 - (\theta_\alpha)_2) \in \mathbb{Z}$. In physical terms, this means that the interference phase patterns of quantum unitary rays differing by an integer *cannot* be distinguished experimentally, and thus the physically meaningful global gauge-invariant information is only the topological phase factors of the form:

$$[\Psi(\theta_\alpha,\gamma)] = exp(\frac{ie}{\hbar c}\int_\gamma \theta_\alpha), \quad (40)$$

referring to the global realization of the beam, hence to the global realization of the *whole gauge equivalence class* of quantum unitary rays.

7.8 ANHOLONOMY: GEOMETRIC PHASE AND THE MEMORY OF A SPECTRAL BEAM

If the energy operator, the Hamiltonian of a quantum system, is functionally dependent on an underlying set of *control variables*, then a quantum state becomes *localized* parametrically on the base space constituted by these variables. The dynamical evolution of a quantum state is therefore driven by the *implicit temporal* dependence of the Hamiltonian through the control variables. Under the assumption that the set of these variables forms a smooth manifold, then the time dependence is depicted by means of *differentiable paths* on this space.

In the approach pioneered by *Michael Berry*, the *adiabatic cyclic* evolution of a quantum state if of a fundamental significance. The cyclic evolution signifies the *periodicity* property of a quantum state with respect to the control variables, whence the adiabatic hypothesis is equivalent to the specification of a *connection*, a parallel transport condition on the evolution of normalized state vectors, which in general is considered to be path-dependent. Due to this *path-dependency*, a quantum state upon completion of cyclic path acquires a *global non-integrable* geometric phase factor, called the *anholonomy* of the transport.

The fibers of the induced spectral line bundle stand for the eigenspaces of the energy operator. Thus, the adiabatic transportation rule amounts to a non-integrable connection, according to which, an eigenstate of the Hamiltonian is required to remain in the eigenspace of the same instantaneous eigenvalue during the adiabatic evolution. In turn, the non-integrable connection gives rise to a covariant derivative operator on the sections of the corresponding Hermitian line sheaf, constituted by the eigenstates of the Hamiltonian. The non-dynamical, by which we mean non-Hamiltonian dependent, global phase assembled during a cyclic evolution along a closed path on the base space is thought of as the *memory* of the evolution, since it *encodes* the global geometric features of the space of control variables in the algebraic, and more specifically, *group*-theoretic structure of the anholonomy of the connection.

The observable global phase factor is called *geometric* because it depends *solely* on the geometry of the base space *pathway* along which the quantum state is transported. If the eigenvalues of the Hamiltonian are degenerate or close to each other, then the adiabatic transportation constraint is not realistic and is substituted by another appropriate connection depending on the particular context. In this case, the gauge freedom of a state vector, localized at a fiber over an eigenspace of the Hamiltonian, is not an one-dimensional complex phase any more, but rather is an n-dimensional complex matrix of phases, called a *non-Abelian complex phase*. It turns out that even the adiabatic transportation rule is *not* necessary for the experimental detection of a global phase factor. This has been demonstrated convincingly through the *intrinsic line bundle* formulation of the complex Hilbert space of states over the *complex projective* Hilbert space. In this formulation, analogously to the preceding approach, the *one-dimensional projection* operators play the role of control variables. This line bundle, or equivalently, the line sheaf of its sections is endowed with a natural connection obtained by differentiation of the Hermitian inner product of the normalized sections, which are the quantum state vectors, and the adiabatic hypothesis is *not* involved at all. Then, the global phase factor is identified in terms of the global *anholonomy* of this connection with respect to a closed path on the complex projective space.

From the general perspective of our cohomological analysis pertaining to spectral beams, all observable geometric phase factors are actually *generated* by the *curvature* of a spectral $[R]$-beam. In particular, the *memory* of a spectral beam is the *global encoding* of the fact that $(2\pi i)^{-1}[R]$ is a 2-dimensional *integral* cohomology class of the base topological space of variables X. As already established, global observable topological phase factors can be completely understood in terms of the monodromies of zero curvature beams. Again, it is instructive to emphasize that an observable anholonomy pertains to a partition class of equipotent unitary R-rays; to a *spectral* $[R]$*-beam*, rather than to an individual unitary R-ray.

A global geometric phase factor arises cohomologically as follows: For any real valued form ϕ of degree k we define:

$$\hat{\phi}(\eta) := \Psi(\phi,\eta) = (\int_{\eta}\phi) + \mathbb{Z}, \quad (1)$$

where η is a k*-chain* of X and $\hat{\phi}$ is a k*-cochain* of X with values in $\mathbb{R}/\mathbb{Z}$. Next we consider the homomorphism of groups:

$$\Xi : Z_k(X) \to \mathbb{R}/\mathbb{Z}, \quad (2)$$

such that, a $k+1$-form τ exists, which satisfies $\Xi \circ \partial = \hat{\tau}$, or equivalently:

$$\Xi(\partial\zeta) = (\int_{\zeta}\tau) + \mathbb{Z} = \hat{\tau}(\zeta) \in \mathbb{R}/\mathbb{Z}, \quad (3)$$

for any smooth map $\zeta : \Delta_{k+1} \to X$. We note that Δ_n stands for the standard n-dimensional *simplex*, and the space of n-chains is generated by Δ_n.

In this framework, the following relation emerges:

$$d(\hat{\tau}) = d\tau = \Xi \circ \partial^2 = 0, \quad (4)$$

and therefore, τ is a *closed* $k+1$-form.

Next, we consider a unitary R-ray $(\mathcal{L}, \nabla_{\mathcal{O}})$ and apply the *quantization* condition, according to which, R is an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$. Similarly to the definition of gauge potentials in the case of Hermitian differential line sheaves, we consider R as a purely *imaginary closed* 2-form, such that $R = i \cdot \Theta$. Thus, according to the above, and since Θ is an *integral global* real-valued form of degree 2, we may instantiate the following group homorphism, which we call the *anholonomy homomorphism*:

$$\mathbb{H} : Z_1(X) \to \mathbb{R}/\mathbb{Z}, \quad (5)$$

$$\mathbb{H}(\partial S) = (\int_S \Theta) + \mathbb{Z} = \hat{\Theta}(S) \in \mathbb{R}/\mathbb{Z}, \quad (6)$$

for any smooth map $S : \Delta_2 \to X$. Equivalently, we have:

$$\mathbb{H} \circ \partial = \hat{\Theta}. \quad (7)$$

We observe that for a *fixed* curvature R of $\Omega^2_{c,\mathbb{Z}}$, the *same* anholonomy homorphism $\mathbb{H}$ is defined for any other unitary R-ray. Thus, it provides a characterization of the whole *gauge equivalence class* of unitary R-rays *classified* by the differential invariant $[R]$. Equivalently, it provides a characterization of a spectral $[R]$-beam, and therefore, we obtain an *anholonomy cohomology class* in $H^1(X, U(1))$, as follows:

$$Hol(\partial S) = exp(i \int_S \Theta). \quad (8)$$

From the above, we form the conclusion that taking into account the *Chern homomorphism*:

$$\delta_c : H^1(X, U(1)) \to H^2(X, \mathbb{Z}), \quad (9)$$

and the natural homomorphism:

$$\imath : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}), \quad (10)$$

the anholonomy cohomology class in $H^1(X, U(1))$ of a spectral beam for *fixed* Θ, is in the *inverse image* of the Chern characteristic class:

$$c_1 = -\frac{1}{2\pi i}[\Theta] \in H^2(X, \mathbb{Z}), \quad (11)$$

under δ_c, or equivalently, it is located in the inverse image of the cohomology class $[\Theta]$ in $H^2(X, \mathbb{R})$, such that:

$$(\imath \circ \delta_c)(Hol) = [\Theta]. \quad (12)$$

Thus, we conclude that an $\mathbb{R}/\mathbb{Z} \cong U(1)$-observable *anholonomy*, which formalizes the notion of a *non-integrable geometric phase factor*, is a global observable gauge-invariant characteristic of a spectral beam, qualified as the *memory* pertaining to the *whole* gauge equivalence class of quantum unitary rays having the *same curvature*, such that the above cohomological relation is satisfied.