UNDECIDABILITY-UNCERTAINTY: INDIRECT LOGICAL CYCLES AND COMPUTATION

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4.1 INDIRECT LOGICAL *METAPHORA*: GÖDEL'S FIRST INCOMPLETENESS THEOREM

The conceptual essence of *Kurt Gödel's* first incompleteness theorem may be summarized in the assertion that if a formal system containing arithmetic, meaning any arithmetic structure endowed with the operations of addition and multiplication, is consistent, then it contains undecidable propositions, namely statements whose truth or falsity cannot be expressed within the language of this formal system. According to Gödel, the reason for the existence of undecidable propositions in a formal system containing arithmetic is that a complete epistemological description of a language A cannot be given in the same language A, because the concept of *"truth"* of sentences of A cannot be defined within A. Thus, the "truth" of the propositions of a language cannot be expressed in the same language, while *provability*, which is an arithmetic relation can. In a nutshell, true $\neq$ provable.

Gödel's proof of the first incompleteness theorem is based on the explicit construction of an arithmetical formula that asserts its own non-provability, and thus, it is undecidable within the language of its formal system. From our viewpoint, the particular interest in Gödel's proof stems from three factors:

1 According to a remark of Gödel himself, there exists an analogia between his undecidable proposition within a formal system containing arithmetic and classical semantic paradoxes, like the Liar paradox. The analogia is based on the existence of direct *self-referentiality*;

2 The correct solution of these semantic paradoxes derives from the method of proof that Gödel devised in order to evade the logical obstacle of direct strong self-referentiality within the language of his formal system. Concisely put, this method of proof involved an argument requiring a stratification into two *hypostatic* levels, one of which is called the mathematical level and the other the *metamathematical* level. In other words, the circumvention of self-referentiality required a metaphora into another level of hypostasis, such that the direct obstruction is avoided by dint of ascending to another level and then descending back. In this way, *indirect* self-reference, leads to a well-defined legitimate statement and not to a paradox.

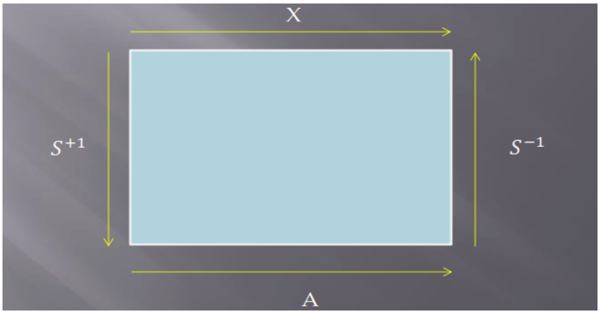
3 The process of ascending from the mathematical to the metamathematical level and then descending back, or the other way round symmetrically, effecting indirect self-reference, and thus eventually, producing a legitimate statement asserting its own unprovability, required a metaphora, i.e. the instantiation of

encoding/decoding bridges for translating between these two levels. This is precisely the role of "Gödel's numbering" or "Gödel's ordering" idea.

From our viewpoint, "Gödel's numbering" is actually *"Gödel's gnomon"*, utilized as a means to indicate or label propositions at both the mathematical and the metamathematical level, in such a way that a certain type of *homeotic equivalence* can be established between these two levels. In other words, the role of "Gödel's gnomon" is to logically conjugate the intractable problem of direct, strong self-referentiality at one level of hypostasis by a definite tractable process at the other level of hypostasis, where the latter is qualified in terms of Cantor's diagonalization method, as we shall show below.

Henceforth, the key to understanding Gödel's argument from our view, consists in delineating the stratification of the argument into levels and identifying the "gnomon" which induces an appropriate "homeotic criterion" permitting the "metaphora by logical conjugation", or else, descending and ascending between these levels. Gödel's argument requires a stratification into two levels: the mathematical level involves general propositions about numbers and the metamathematical level involves general propositions *about* general propositions about numbers.

Gödel's argument refers to a true proposition at the metamathematical level, whose truth is established by "logical conjugation" through the mathematical level. It is clear that this argument involves an indirect self-reference, which is legitimate since it is arrived at by descent to and re-ascent from the mathematical level, as we have stressed previously. Gödel's gnomon is a gnomon of numbering or ordering and it is utilized to establish encoding and decoding reciprocal translation bridges between these two levels. In terms of Gödel's theorem, the possibility of establishing a true proposition at the metamathematical level, is proved by descending to the mathematical level, such that a particular argument can be formulated by means of an infinite closure operator, qualified in terms of *"Georg Cantor's diagonalization"*, which is then transferred back to the metamathematical level by means of ascending the inverse bridge to prove the theorem.



Since the alphabet of arithmetic is countable, it is possible to instantiate a fixed schema of numbering or ordering, which assigns a unique positive integer to every legitimate arithmetic formula. The same schema can be extended to order finite strings of arithmetical formulas. Of course, many such appropriate schemas of ordering or numbering exist, but the essential idea is that by fixing any one of them the function of ordering or numbering can be carried out. For example, we may fix the ordering gnomon provided by the natural numbers' sequence, such that every arithmetic formula and every finite string of arithmetic formulas is assigned a unique number in this sequence, called its Gödel number. It follows directly that in the way described the ordering structure of the natural numbers may be adjoined to the structure of an arithmetic. In particular, the proof of an arithmetic formula K constitutes a finite string ending with K itself, and thus proofs are naturally assigned Gödel numbers in the ordering.

Gödel starts his argument by considering the proposition $p(x,y)$ at the metamathematical level stating the following:

"$p(x,y)$: x is the Gödel number of an arithmetic formula whose proof has Gödel number y".

Then, still at the metamathematical level, he considers the associated proposition,

$$\forall y \neg p(x,y),$$

which reads as follows:

"$\forall y \neg p(x,y)$: No number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is x".

The last proposition simply means that the x-th formula in our ordering schema is not provable.

The crucial idea in the last proposition boils down to the fact that the variable x is a free variable. Then, the natural question to ask is the following: Is the proposition

$$\forall y \neg p(x, y)$$

at the metamathematical level Gödel-numberable itself? Equivalently stated, does Gödel's gnomon apply to this proposition? This is the crux of the matter because, as we already know, a gnomon is effective if it induces a "homeotic criterion" to the structure it is adjoined to, that permits the descent and ascent between the metamathematical and the mathematical level.

Clearly, such a criterion is feasible in the present case, only if Gödel's gnomon actually assigns a unique number to the proposition $\forall y \neg p(x, y)$, where x is a free variable.

We realize from this reasoning why the major part of Gödel's paper is devoted to showing that the aforementioned proposition is indeed Gödel-numberable. Let us denote the Gödel number of the metamathematical level proposition

$$\forall y \neg p(x, y), \text{ where } x \text{ is free,}$$

by the number ξ at the mathematical level. The criterion can now be implemented using Gödel's gnomon by applying Cantor's diagonalization process at the mathematical level in order to achieve closure. This simply amounts to substituting the free variable x in the proposition $\forall y \neg p(x, y)$ by the definite number ξ to obtain now at the mathematical level the proposition

$$\forall y \neg p(\xi, y),$$

which means that the concrete ξ-th formula in our ordering schema is not provable.

The role of Gödel's gnomon is enunciated as follows: If we apply this gnomon the metamathematical level proposition

$$\forall y \neg p(x, y), \text{ where } x \text{ is free,}$$

is precisely mirrored at the number ξ at the mathematical level. This means that the above metamathematical level proposition is symmetrical, and thus equivalent, to a certain arithmetic formula at the mathematical level whose sequential number is ξ modulo the gnomon employed.

Henceforth, the metamathematical level proposition $\forall y \neg p(x,y)$, where x is free, is symmetrical modulo the gnomon, and thus homeotically identical, with the ξ-th arithmetic formula in the ordering induced by the gnomon at the mathematical level.

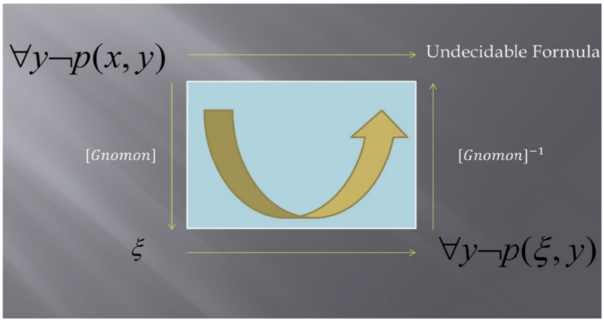
It is important to notice that the process of Cantorian diagonalization at the mathematical level involves a reflexive action, since we feed this fixed ordering number ξ as an argument in the place of the free variable x of $\forall y \neg p(x,y)$.

In this manner, we obtain a legitimate proposition at the mathematical level

$$\forall y \neg p(\xi,y),$$

which states that the concrete ξ-th formula in our ordering schema is not provable, since no number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is ξ.

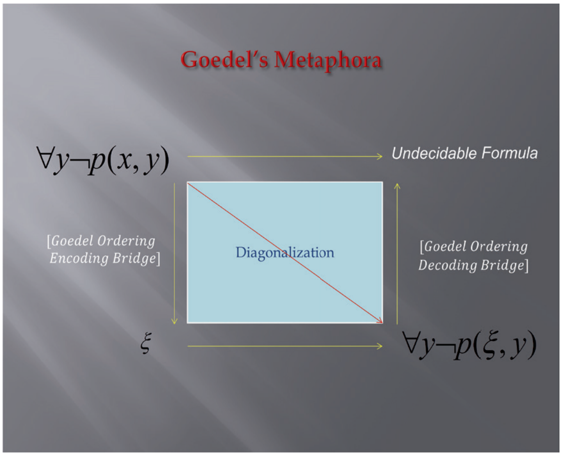
Finally, using the homeotic criterion established by Gödel's gnomon in reverse, we ascend back to the metamathematical level, where we finally obtain a proposition that ascertains its own unprovability. It is precisely this proposition that expresses Gödel's incompleteness theorem itself, since this proposition is undecidable given the consistency of our arithmetic.



Gödel's gnomon and the previously described metaphora by means of logical conjugation between the metamathematical and mathematical levels is operative with respect to the whole structure of an arithmetical formal system, that is with respect to both the additive and multiplicative structure of an arithmetic system. In case that only the additive structure is considered, Gödel's gnomon does *not* induce a homeotic criterion between the metamathematical and mathematical levels, and it can be shown that the incompleteness theorem is not valid.

Epigrammatically, Gödel's gnomon effects an indirect self-reference at the metamathematical level by means of the descent to and re-ascent from the mathematical level. The indirect self-reference is conducted through the mathematical level by utilizing the infinite closure operation of Cantor's diagonalization process. In other words, employing Gödel's gnomon renders indirect self-reference feasible by conjugating the initial intractable problem to an infinite closure operation, and thus circumventing it appropriately.

The far-reaching consequence of the above is that Turing's argument, according to which, the halting problem inherent to a universal Turing machine is undecidable, should be viewed as the computational variant of Gödel's first incompleteness theorem. The reason is that Turing's argument can be also considered as a logical conjugation argument of the same form, meaning that indirect self-reference at the level of a universal Turing machine is feasible by conjugating it to the infinite closure operation of Cantor's diagonalization. "Turing's gnomon" is similarly a gnomon of ordering or numbering programs by means of the natural numbers' sequence.



4.2 INDIRECT FORCING IN LOGIC: GENERIC FILTERS AND CONTEXTUALIZED TRUTH

Since the formulation of Gödel's first incompleteness theorem, it is well known that for any comprehensible list of set theoretic axioms, there will be statements neither provable nor unprovable from those axioms. What is really interesting, however, is how many of the most natural questions about sets are not decidable by the standard axioms of Zermelo-Fraenkel-Choice (ZFC) set theory, and how many ways of deciding these questions there are available.

In this state of problematics, *Paul Cohen* managed to establish the independence of the Axiom of Choice (AC) from Zermelo-Fraenkel (ZF) and the independence of the *Continuum Hypothesis* (CH) from ZFC. This became possible by developing a novel technique, called forcing, that constitutes a far-reaching generalization of the logical notion of implication, which he used for extending a standard model of set theory. Cohen's extension method constitutes another significant case of the paradigmatic schema of analysis we developed in the previous Section in relation to Gödel's first incompleteness theorem. This schema involves a process of indirect self-reference through extension to a new logical level of hypostasis by a metaphora providing bidirectional bridges for translating between the initial standard model of set theory and some novel model of set theory internally distinguishable from the former one.

Epigramatically, Cohen discovered the precise means of operationally extending a standard model of set theory to some other admissible one bearing specific properties without altering the ordinals. The central technical innovation was based on the notion of a *forcing condition*, through which satisfaction for the extension could be approached in the ground standard model.

In general terms the method of forcing consists in the instantiation of a novel model of set theory from a standard model by the adjunction of certain sets with particular properties via "forcing conditions" encoding information about those sets. Intuitively, some conditions are stronger than others, and this serves as a criterion for partially ordering them within the ground standard model. In this state of affairs, proving general results about how the elements of a partially ordered set P force certain conditions to hold allows one to prove statements referring to the novel constructed models, without looking closely at the forcing conditions themselves anew in each particular case. It is this generality that lends the method its efficiency and universality.

Let us start with a ground model V of set theory (countable and transitive) and consider a specific partially ordered set (poset) P in V. The set P is to be interpreted as a partial order of forcing conditions, ordered by means of their strength. In this way, a nonempty poset $(P,<)$ in induces a notion of forcing, whose elements will play the role of "forcing conditions". In general, if p, and $p<q$, we say that p is stronger than q. That is, p represents a stronger condition than q. It is customary for the poset to have a largest element, denoted 1, such that all elements of P are stronger than 1. The elements p and q

are compatible if there exists an element r in $(P,<)$ stronger than both p and q, otherwise they are incompatible.

It is important to note that the poset and its ordering must be elements of V. However, the idea is to utilize the poset $(P,<)$ as a partial order of forcing conditions, so as to construct and eventually adjoin to V certain sets that are not already in V. In particular, by adjoining a "generic set" G to the ground model V, a new model $V[G]$ can be consistently formed by extension.

A dense subset of P is defined as a set D such that for all $p \in P$, there is $q \in D$ stronger than p. A filter on a poset P is a nonempty subset F of P such that: (a) if $p \in F$ and $p \leq q$, then $q \in F$ and (b) if p, $q \in F$ there is $r \in F$ extending both p and q. If this filter intersects every dense set D in the ground model V, then it is called a "generic filter".

Considering that the subset $G \subseteq P$ is a *"generic filter"*, meaning that G contains members from every *dense* subset of P in V, one proceeds to build the forcing extension of the ground standard model V by G, denoted by $V[G]$, such that $V \subseteq V[G]$ by requiring closure under all elementary set-theoretic operations.

Consequently, the forcing extension has adjoined the *"ideal"* object G to V, in much the same way that one might build an algebraic extension of a ring by means of an ideal. In particular, every object in $V[G]$ has a name in V and is constructed algebraically from its name and the generic filter G. Remarkably, the forcing extension $V[G]$ is always a model of ZFC. Nevertheless, the crucial point is that it can exhibit different set-theoretic properties in a way that can be precisely controlled by the choice of the poset of forcing conditions P.

More precisely, one may proceed by defining the forcing relation on a proposition ϕ, denoted by p ë ϕ, which holds whenever every generic filter G containing the forcing condition p implies $V[G]$ ë ϕ. For this purpose, it is important to consider the complete Boolean algebra B, which functions as the completion of the selected poset of forcing conditions P. We say that a condition $p \in P$ forces ϕ if and only if p is less or equal to the "Boolean value" of ϕ in B.

It is worth recalling that in a standard model of set theory, all propositions are evaluated as "true" or "false", meaning that they are strictly evaluated with respect to the two-valued Boolean algebra $\{0,1\}$. In comparison, referring to a non-standard model, propositions can take values on any element of the Boolean algebra under consideration. The concrete meaning of this logical maneuver will be examined subsequently. Presently, if we take for granted the notion referring to the *"Boolean value"* of ϕ in B, the notion of forcing implicates that if the proposition ϕ is being forced by p, then it is going to be also forced by any condition that is stronger than p.

The fundamentally significant elements of the method of forcing, pioneered by Cohen, are the following:

1. The forcing extension of the ground standard model V by G, namely $V[G]$, satisfies the axioms of ZFC set theory.
2. Every proposition ϕ that holds in $V[G]$, is forced by some condition p in P.
3. The forcing relation p ë ϕ is definable in the ground model for fixed ϕ.

Of course, the crux of the matter, is the delineation of an appropriate generic filter that actually accomplishes the required extension. In the case of a countable and transitive ground model V, for any chosen *partial order* P of elements from V, to be interpreted as forcing conditions, there exist only countably many dense subsets of P, which may be enumerated externally as D_0, D_1, D_2, and so on. Then, we may pick in order any condition $p_0 \in D_0$, then $p_1 \in D_1$ below p_0, and so on. With this procedure, and using the method of diagonalization, we can construct a descending sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$, such that $p_n \in D_n$. Then, the filter G generated by this sequence is generic, and therefore, suitable for the construction of the forcing extension $V[G]$.

The restriction that V stands for a countable and transitive model ground model of set theory can be effectively lifted. In this case, it is not possible to delineate a generic filter as described above, and consequently, the method of "Boolean values" seems to provide the most general approach, to the semantics to which we now turn our attention.

Cohen's main innovation lies in the distillation of the notion of forcing by means of a chosen partial order in the ground model, and the positing of a generic filter in this partial order containing elements not already grasped in the ground model. This innovation made it possible to secure suitable properties of a novel set, which emerged through the extension of a standard model by a generic filter, without having distinguished all of the members *ab initio*. According to this conception, the generic set G will not be determined completely, but in spite of this, properties of G will be completely determined on the basis of very incomplete information about G. This is phenomenally contradictory, because how could one decide whether a statement about G is true, before we have determined G itself? The seeming contradiction stems from the standard conception of truth in terms of evaluations of a priori distinguishable elements with respect to the two-valued Boolean algebra $\{0,1\}$. The method of forcing requires a re-conceptualization of the notion of truth in a novel way. Actually this novelty re-enforces the ancient conception of truth as "aletheia". In this manner, truth emerges temporally, in the precise sense of being unveiled, through a process of percolation through the generic filter.

The main idea is that in the ground model V, the devised set of forcing conditions P, where each condition is the carrier of partial information toward an eventual generic filter G, is precisely ordered according to the potential amount of information. In the case of a countable ground model V, a complete sequence of stronger and stronger conditions p_0, p_1, p_2, $\ldots$ is applied, so that every proposition, or its negation, is forced by some member of this sequence. Therefore, it is owing to this sequence that a generic object G is eventually manifested bearing the appropriate properties.

In the general case, a new meaning is elucidated in relation to the notion of truth, which may be described as follows: Working inside the ground model V, we consider the set of elementary conditions P which forces a given set either to lie in the generic set G, or not in G. Because of the fact that forcing is defined in the ground model V, we can examine all the possibilities of assigning sets of elementary conditions P, which force the members of A to lie in an arbitrary B. This set of elementary conditions is the "truth value" of the statement. As we mentioned above, the notion of truth in this context bears the meaning of "aletheia", in the sense that the "truth value" is identified with the set of conditions that unveil the statement. Conclusively, in a standard model of set theory a subset of X is determined by a two-valued function on $\{0,1\}$ applied to the members of X. In a non-standard model, a subset is determined by a function taking its values in the subset of the elementary conditions. Since these values are all in the ground model V, quantification is possible over all possible truth values.

The precise connection with the idea of evaluating propositions in the complete Boolean algebra B obtained by the completion of the poset of forcing conditions P emerges in the following manner: The subsets of the set of elementary conditions P, which determine the truth or falsity of each statement, are thought of as elements of a

Boolean algebra. Then, a Boolean-valued model of set theory arises, in the sense that the truth values are identified with the elements of the concomitant Boolean algebra, which is different from the bivalent $\{0,1\}$. The notion of forcing is expressed by saying that the set P forces a statement S, if no extension of P forces the negation of S".

In the above setting, the essential idea is that the conceptual maneuver of evaluating statements by means of the Boolean algebraic completion B of the chosen set of forcing conditions P in the ground model V, provides the means to embrace the obstacle; namely, that the elements of the generic set G are not specified *ab initio*. This gives rise to a non-standard Boolean-valued model of set theory, denoted by V^B. In other words, a Boolean-valued model, besides the *crystallized* elements, contains elements that are "partially or locally distinguishable", where the extent of their distinguishability in the *percolation* process of *unveiling* is provided by the "truth value" they are assigned in B, thought of as a domain of truth values, and not in the bivalent domain $\{0,1\}$. This logical maneuver in the specification of the Boolean-valued set V^B allows us to think of it as a set of fluid potential members in the process of eventual crystallization.

In this state of affairs the role of the generic set G is precisely to determine which elements of V^B will eventually be crystallized giving rise to a novel standard model of set theory satisfying the axioms of ZFC. Henceforth, the generic set G plays the role of Cohen's gnomon for extending a standard ground model of set theory to a novel standard model obeying ZFC, which is nonetheless internally distinguishable from the former one. This is accomplished by means of indirect self-reference through the level of Boolean-valued models of set theory as follows:

1 The encoding bridge is from the level of standard ZFC models of set theory to the level of non-standard Boolean-valued models obtained by completion of any chosen partial order of forcing conditions in the ground standard model we are starting with;

2 At the level of a Boolean-valued model V^B, an appropriate equivalence relation is formulated by utilizing Cohen's gnomon, i.e. a generic set G in B. This establishes a homeotic criterion of symmetry for the members of the Boolean-valued set V^B in the sense of a common logical measure provided by G giving rise to a *partition spectrum*. In particular, the generic set G is qualified as a *generic ultrafilter* in B. The ultrafilter characterization means that:

a $1 \in G$,

b 0 not $\in G$,

c if $\kappa, \lambda \in G$, then $\kappa \wedge \lambda \in G$,

d if $\kappa \in G$, and $\kappa \leq \lambda$, then $\lambda \in G$,

e For all $\xi \in B$, either $\xi \in G$ or $\neg\xi \in G$.

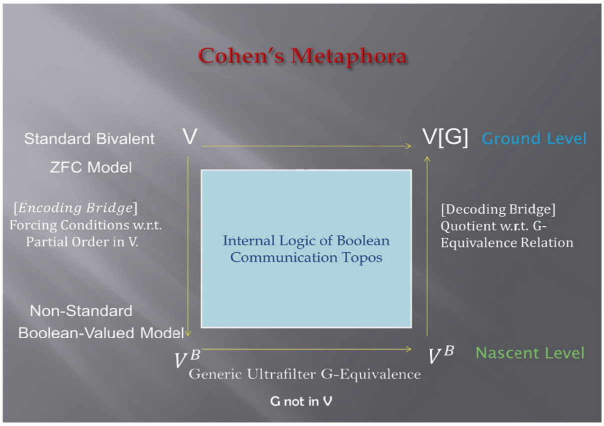
Then, the common logical measure implied by a generic ultrafilter G, i.e. the symmetry, or homeotic criterion, for elements of the Boolean-valued model V^B with respect to the gnomon G, is formulated in terms of the following equivalence relation:

$$\text{For all } x, y \in V^B,$$
$$x \sim_G y \text{ iff } \|x = y\|^B = \xi \in G,$$

where, $\|x = y\|^B$ denotes the *"Boolean value"* characterizing the extent to which $x = y$ for all $x, y \in V^B$.

Since, the generic set G (not required to be in V) is an ultrafilter in B, the equivalence classes of elements of V^B, i.e. the blocks of the corresponding partition spectrum, with respect to the above symmetry criterion provided simply by means of membership in G or not, that is, in a bivalent manner, are the candidates for forming the elements of the new standard model of set theory, constituted or crystallized by the forcing extension of the ground standard model V by G, i.e. $V[G]$.

3 The decoding bridge is from the level of Boolean-valued models of set-theory back to the level of standard ZFC models of set theory. The standard ZFC model obtained by imposition of the above equivalence relation on elements of V^B with respect to Cohen's gnomon G is the quotient set of equivalence classes V^B / G, which is bivalent. The ingenuity of Cohen's proof rests on the requirement of generic status for his gnomon, in the sense that the quotient set V^B / G is actually a standard ZFC model, if the utilized ultrafilter G is a generic one. Actually, if G is a generic ultrafilter then $V^B / G := V[G]$ is the smallest standard ZFC model containing both V and G.



Finally, it is noteworthy that Cohen's method of forcing has been adapted in category-theoretic language. The relevant context is the topos of *sheaves* over a partially ordered set bearing the semantics of Cohen's forcing conditions. In this manner, it turns out that a non-standard Boolean-valued model of set theory can be equivalently represented as a sheaf over the considered "complete Boolean algebra" completion. Consequently, the internal logic of this topos of sheaves can be adequately depicted by means of the above diagram of metaphora between the indicated levels, from the viewpoint of natural communication.

4.3 ALGORITHMIC COMPLEXITY AND THE HALTING PROBABILITY

In the context of *algorithmic* or program-size *complexity* theory, *Gregory Chaitin* came to focus on the implications of Gödel's first incompleteness theorem, which finally lead to a refinement of the former in a computational context. The algorithmic complexity of a string is essentially defined by the length of the shortest program that generates this string and then halts. In this way, a finite string is characterized as random if its complexity is equal approximately to its length. There are strings with arbitrarily large algorithmic complexity and the problem of program-size complexity is undecidable. In this context, Chaitin's incompleteness theorem states that given a consistent arithmetic, there exists a number C depending upon that arithmetic, such that any proposition of the form "the program-size complexity of the string s is greater than C" is not provable. Thus, since there are true instances of such propositions, it follows that there are propositions of the above form which remain undecidable within the context of the given arithmetic.

Chaitin's argument constitutes a refinement of Gödel's first incompleteness theorem because it involves a metaphora extension in depth. First, Chaitin's gnomon is based on counting the number of bits in a program, whence the homeotic criterion is applied for *self-delimiting* programs, defined as strings having the property that one can tell where they end. Second, Chaitin's program-size counting gnomon is modified probabilistically, by a deeper stage logical conjugation at the measure-theoretic level involving the probability $P(x)$ that a program will give a number x at the higher level, while preserving the same homeotic criterion as applied to self-delimiting programs.

This is called the *algorithmic probability* of x, and a summation of probabilities over all possible outputs x yields the halting probability $\Omega = \sum_x P(x)$, where Ω is interpreted as a random infinite sequence of bits. In particular, the halting probability Ω is a random real number. The most intuitive conception of randomness is tied to the notion of absence of predictability. In other words, if one knows the first n-bits of a random sequence it is not possible to predict the next $n+1$-bit. Here, the central objects of our attention are elements of the continuum $\{0,1\}^N := 2^N$.

Elements of 2^N may be viewed either as infinite sequences of bits (infinite strings) or as sets of natural numbers, which can be identified with their characteristic functions. We denote the set of finite binary strings as $2^{[N]}$. The set $2^{[N]}$ can be canonically identified with N, so that subsets of N may be thought of as sets of strings. We also denote the length of a finite string σ by $|\sigma|$. Using finite binary strings, we may define a topology on 2^N as follows: First, we define the extension of a finite string σ by the *clopen* set $E(\sigma) = \{x \in 2^N : \sigma = [x]_{|\sigma|}\}$, where $[x]$ denotes the operation of restriction. Second, we consider clopen sets of the form $E(\sigma)$, where σ is a finite binary string, as the base of a topology on 2^N, where each $E(\sigma)$ is a basic clopen set, to be thought of as an interval in the continuum. In particular, we may identify 2^N with the interval of real numbers $[0,1]$ by associating each real number with its usual binary representation. If we regard μ as the *Lebesgue measure* on $[0,1]$, then we have that $\mu(E(\sigma)) = 2^{-|\sigma|}$. Now, we expect that non-random sequences form a set of measure zero. Intuitively, using the above defined topology, we require that the extensions of longer and longer initial segments σ of a string $x \in 2^N$ become arbitrarily small. In this manner, *random* sequences are defined from a complementary viewpoint measure-theoretically on the basis of the fact that non-random sequences should form sets of measure zero.

Next, if we recall the intuitive conception of randomness as related with the absence of predictability, we may require that there is no algorithm α which can ever compute, and thus uniformly measure, $[x]_{|\sigma|}$ from any sorter string. Here an algorithm is considered as a function $\alpha : 2^{[N]} \to \{0,1\}$.

The above idea constitutes, in effect, a complexity measure based on program-size. The notion of program-size complexity introduced by Chaitin to this effect, regards σ as a self-delimiting program, i.e. as a program delimited by an end-marker. Clearly, no extension $\circlearrowright$ of a self-delimiting program can be a self-delimiting program, since the end-marker will not be in the right place. If $\psi : 2^{[N]} \to 2^{[N]}$ is a partial recursive function with prefix-free domain, which means computable by a self-delimiting reference *universal Alan Turing machine*, the Chaitin complexity of σ, or the algorithmic information content of σ is defined by $I(\sigma) = min\{|\tau| : \psi(\tau) = \sigma\}$. This is the length of the shortest program τ of the self-delimiting universal Turing machine that outputs σ. Then, we define an infinite sequence $x \in 2^N$ to be random if all its extensions have high Chaitin complexity, capturing in this way the above intuitive conception of randomness.

More precisely, an infinite sequence x is random if and only if there exists a constant k, such that $(\forall n)[I([x]_n) \geq (n-k)]$. The infinite sequences that satisfy this condition form a set of measure one, and thus random sequences form a set of measure one. This result is in good compatibility with the measure theoretic characterization of non-random sequences as sets of measure zero derived in the previous paragraph. In this sense, the characterization of random sequences according to Chaitin or program-size complexity is in agreement with the measure-theoretic characterization completing the logical conjugation.

Chaitin's incompleteness theorem constitutes not only a refinement of Gödel's first incompleteness theorem due to the deeper stage logical conjugation at the measure-theoretic level, or equivalently via the program-size complexity level, but it also contains the seeds of two powerful generalizations: The first comes from an even deeper level conjugation via the level of generic sets and Cohen's forcing conditions, based on an effective analogical relation between the notions of *random* sets and *generic* sets. The second comes from an interpretation of the constant involved in the definition of random sequences in terms of an uncertainty relation between two logically conjugate domains in the spirit of *Werner Heisenberg's uncertainty principle* in quantum mechanics.

4.4 BOOLEAN-VALUED SETS: FROM RANDOMNESS TO GENERICITY

Regarding the first of the previously stated issues, the deeper stage metaphora takes place through the hypostatic level of Boolean-valued sets. The concomitant logical conjugation utilizes the effective analogical relation between random sets and generic sets. Both of them can be formulated as Boolean-valued models of set theory, or equivalently as variable sets, called sheaves, over a Boolean algebra. In the first case, the Boolean algebra is identified with the Borel algebra of clopen sets, which comprehends both closed and open sets, defined above, modulo the sets of measure zero (non-random sequences). In the second case, it is identified with the Boolean algebra of Cohen forcing conditions. In this manner, the proposed deeper level logical conjugation is based on the consideration of random sequences as Cohen forcing conditions with respect to a Boolean measure algebra, in the context of a Boolean-valued model of set theory containing a consistent arithmetic. Intuitively stated, the sets in this Boolean-valued model, or equivalently the sheaves over the Boolean measure algebra, are to be thought of as sets, whose elements are not evaluated to the two-valued Boolean algebra 2, but are evaluated on the clopen sets of the Boolean measure algebra.

The basic idea can be put as follows: Let us think that we start from a standard model of set theory, which we agree to call constant sets. The elements of constant sets are characterized by valuations in the two-valued Boolean algebra 2. Then we adjoin a multiplicative encoding bridge from the level of constant sets to the level of variable sets, which in this case are the sets varying over a Boolean algebra. From *Marshall Harvey Stone's representation theorem* for Boolean algebras, the *spectral* representation of a Boolean algebra is a totally *disconnected* and *compact* Hausdorff space, called the Stone space. Then, we are able to think of the pertinent variation in terms of measurable functions over this space.

If we arrest the variation at a point of this space, i.e. at a principal ultrafilter of the associated Boolean algebra, then we force a homeotic criterion of identity, by the stipulation that two functions are equivalent if their measurable values agree at this point. Thus, after having identified the partition spectrum, representing the equivalence classes induced by this criterion, we can ascend back to the level of constant sets. In other words, the quotient set obtained is a standard set at the initial level. If we arrest the variation at an ideal point of the Stone space instead, i.e. at a non-principal ultrafilter of the Boolean algebra, then a new possibility arises. More concretely, if we apply the same homeotic criterion for ideal points, we obtain a new quotient set at the level of constant sets, which is an extension of the constant set we started with, called a *Boolean ultrapower* of this set. The Boolean ultrapower is a new constant set, which is internally indistinguishable from the initial set we started with.

In this light, Cohen's forcing employing the gnomon of generic filters is a refinement of the method of evaluation at ideal points aiming toward the construction of new constant sets internally distinguishable from the set we started with. Instead of ideal points, one considers a partially ordered set P of forcing conditions. Arresting the variation with respect to these forcing conditions, one obtains a generic distinguishable extension of the initial set at the level of constant sets, such that a proposition is true in the generic extension if and only if it is forced by some generic forcing condition in P. Note that the generic set of forcing conditions is not contained in the initial constant set, and thus Cohen's forcing requires logical conjugation through the deeper level of variable sets. Moreover, Cohen's method of forcing via some generic set is equivalent to forcing with respect to a Boolean algebra, which in the present case is identified with a Boolean measure algebra. This is why, the notion of random sets involved in applying Chaitin's gnomon may be interpreted by logical conjugation via the notion of generic sets underpinning Cohen's gnomon.

4.5 QUANTUM UNCERTAINTY AND COMPLEMENTARITY

In quantum mechanics, Heisenberg's uncertainty relation involves a limit or bound, which is defined in terms of *Max Planck's constant*, pertaining to the simultaneous determination of two *conjugate observables*, for example, the position and momentum of a quantum system.

We note that observables in quantum mechanics are defined as self-adjoint operators, bearing thus a spectral resolution in terms of *projection operators*. In this way, each observable is associated with a complete Boolean algebra of projection operators obtained by its spectral decomposition. If two observables commute, then they can be resolved by means of a common Boolean algebra of projectors. In other words, a *commutative* algebra of observables is logically characterized by means of the Boolean algebra of projectors (idempotent elements of the commutative algebra), which simultaneously resolve all the observables belonging in this algebra. The non-commutativity of observables like the position and the momentum of a quantum system, quantified by means of Heisenberg's uncertainty principle, signifies the fact that there does *not* exist a universal Boolean algebras of projectors resolving all the observables in quantum mechanics.

Thus, the internal logic of a quantum system is not a Boolean logic of projection operators, but a globally non-Boolean *amalgam* of local Boolean patches, where each patch covers the extent of a maximal commutative algebra of simultaneously measurable observables. Non-commutative observables like position and momentum belong to two different Boolean patches, which cannot be amalgamated together simultaneously under a bigger Boolean patch. Notwithstanding this fact, a position observable can be transformed to a momentum observable by means of a unitary transformation and conversely; these are the well-known *Joseph Fourier's transform* and its inverse. Hence, the position and momentum Boolean patches constitute two conjugate logical domains, which *cannot* be subsumed under a universal Boolean domain, and thus are *complementary* in the standard terminology.

These conjugate Boolean domains correspond to conjugate Boolean projection-valued measure algebras. Note that each Boolean algebra of projectors gives rise, using Cohen's gnomon in this context, meaning logical conjugation through the level of variable sets as above, to a generic set of forcing conditions. Then, a proposition is true in the generic extension, obtained as we have seen, if and only if it is forced by some generic forcing condition. This is suited to understanding the *measurement process* of an observable in quantum mechanics, where a proposition refers to the result of a measurement on this observable and the generic forcing condition corresponds to the projection operator of a measurement device which clicks upon registration of this result.

The difference in comparison to the previous case, appearing for the first time in quantum mechanics, is that distinct local generic sets of forcing conditions corresponding to conjugate observables exist, which cannot be subsumed under a universal global generic set. Hence, in a well-defined sense, which can be made precise using the theory of sheaves, the logical treatment of quantum mechanics requires a *localization* of Cohen's gnomon of forcing, with respect to local Boolean domains, thereby giving rise to generalized *local models* of set theory called *topoi*. In turn, this logical localization with respect to conjugate Boolean valued sets gives rise to the phenomena of *contextuality* in quantum theory. We interpret Heisenberg's uncertainty principle as setting the bound (in terms of Planck's constant) of the simultaneous determination of two conjugate observables with respect to the same Boolean domain of measurement. This is expressed in terms of the standard deviations in the expectation values of conjugate observables in the form $\delta x \cdot \delta p \geq \hbar / 2$, where $\hbar := h / 2\pi$ in the case of position and momentum observables. Each of these observables is considered as a Boolean homomorphism from the *Emile Borel measure algebra* of the real line (where the results of measurements are recorded) to the corresponding Boolean patch containing the respective projections in the spectral resolution of these observables.

4.6 UNCERTAINTY IN A SELF-DELIMITING UNIVERSAL TURING MACHINE

Let us now examine if Chaitin's gnomon can be presented in a form giving rise to an uncertainty relation between two conjugate Boolean domains. The first Boolean domain we consider is the domain of random real numbers in the continuum $[0,1]$. Bearing in mind that we identify 2^N with the interval of real numbers $[0,1]$ by associating each real number with its binary representation. Moreover, if we regard μ as the Lebesgue measure on $[0,1]$, we have that $\mu(E(\gamma)) = 2^{-|\gamma|}$, where γ is a finite binary string, to be thought of as a program of a self-delimiting universal Turing machine ψ. For an output χ of this machine, we have immediately that the probability of χ is given by:

$$P(\chi) := \mu(\chi) = \sum_{\gamma:\psi(\gamma)=\chi} 2^{-|\gamma|}$$

Chaitin's $\Omega = \sum_{\chi} P(\chi)$ is a random infinite sequence of bits, and thus a *random real* in $[0,1]$ of Lebesgue measure one. It is interpreted as the *halting probability* of ψ, defined as the probability that ψ halts when its binary input is chosen randomly bit by bit, such as by flipping a coin. In practice, we may only compute finitely many digits of Ω.

The second Boolean domain we consider is the domain of program-size complexity. If $\psi : 2^{[N]} \to 2^{[N]}$ is a partial recursive function with prefix-free domain, that is, computable by a self-delimiting universal Turing machine, the Chaitin or program-size complexity of χ, or even the algorithmic *information content* of χ is defined by:

$$I(\chi) = min\{|\gamma| : \psi(\gamma) = \chi\}$$

The complexity measure $I(\chi)$ is the length of the shortest program γ of the self-delimiting universal Turing machine that outputs χ. Moreover, an infinite sequence x is random if and only if there exists a constant k, such that:

$$(\forall n)[I([x]_n) \geq (n-k)]$$

The infinite random sequences that satisfy this condition form a set of measure one, and thus for Chaitin's Ω we obtain:

$$(\forall n)[I([\Omega]_n) \geq (n-k)]$$

The above inequality is interpreted clearly as an uncertainty relation pertaining to the conjugate Boolean domains of random real numbers in $[0,1]$ and program-size complexity length measures. Since it is an *uncertainty* relation between two *conjugate* Boolean domains, these domains cannot be embedded in a universal Boolean domain simultaneously subsuming both of them. Thus, the constant k is interpreted as setting the *bound* of the simultaneous determination of two conjugate observables, viz. the random real Ω in $[0,1]$ and the program-size complexity length measure I.

4.7 LOGICAL CONJUGATION CYCLES

We have shown previously that both Heisenberg's and Chaitin's logical conjugation methods give rise to uncertainty relations between two conjugate or complementary Boolean domains which cannot be subsumed under a common universal Boolean domain simultaneously with absolute precision. Moreover, if we consider each Boolean domain separately we may interpret it as a Boolean algebra of generic forcing conditions, descend to the level of Boolean valued sets, then apply a Cohen-type criterion of homeotic identity with respect to these forcing conditions, and finally ascend back to the initial level of constant sets, obtaining in this manner a generalized model internally distinguishable from the one we started with. The latter reflects the intervention of a suitable measurement procedure for obtaining information with respect to an observable logically classified by this Boolean domain. The logical classification takes place via the procedure of spectral resolution in terms of a Boolean algebra of projectors in the context of operator functional analysis, or more generally according to, the procedure of measurability in terms of a Borel measure algebra, which can even be projection-valued. The important point to be emphasized is that the Cohen-type strategy of logical conjugation cannot be implemented simultaneously with respect to two complementary Boolean domains.

A natural question arising in this context is if it possible to implement the strategy of logical conjugation in such a way that circumvents the above obstacle. We may think of each logical Boolean domain as giving rise to a separate gnomon of conjugation. If we consider two complementary Boolean domains, we cannot apply the method of logical conjugation with respect to both of them simultaneously, but the possibility remains of *composing* these two gnomons in an appropriate way. Since we consider these two gnomons as complementary in a precise sense, justified by the existence of an uncertainty relation as above, then the most economical hypothesis is to assume that each gnomon may conjugate the complementary one. In other words, the hypostatic levels between which each gnomon operates should function as the encoding/decoding bridges of the complementary gnomon.

We may explicate this idea in more detail as follows: We recall that the method of logical conjugation expressing a metaphora requires, first, a certain stratification into different levels and, second, the delineation of encoding/decoding bridges between these levels in order to be able to descend and re-ascent. Each Boolean domain of discourse provides a natural stratification as well as a natural descending/ascending bridging between the strata, which can be conceptualized in accordance with Cohen's gnomon. But, what if there is *no* intrinsic way of *distinguishing* between *strata* and *bridges*? Reciprocally put, the distinction between strata and bridges is meaningful only under the specification of a Boolean domain. If two complementary gnomons pertaining to two complementary Boolean domains are utilized simultaneously the only way that logical conjugation can function is by reversing the role of strata and bridges with respect to these two gnomons, such that a closure is achieved. Algebraically, the only way that these two complementary gnomons may be amalgamated together simultaneously is by temporarily *suspending* the rigid distinctions between strata and bridges, and just *iterating* the process of logical conjugation with respect to the composition of these two gnomons until we reach a *closure*. The closure corresponds to a non-trivial *logical cycle* of compositions. It turns out that the formation of this cycle is equivalent to composite logical conjugation where the levels of one gnomon correspond to the bridges of its complementary gnomon. We present this simple algebraic argument as follows:

A logical conjugation is generally expressed in the symbolic form

$$X = S \circ A \circ S^{-1}$$

which defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S. Now we consider the first two symbols of the conjugation $S \circ A \circ S^{-1}$, that is, $S \circ A$, as a string, and extend this string by adding new symbols at the end, such that every three consecutive symbols pertain to a logical conjugation, or equivalently, establish a metaphora. We iterate this operational procedure until we generate a cycle, which means until the last two symbols are $S \circ A$ again that we started with. In more detail we obtain successively:

$$S \circ A \dashrightarrow S \circ A \circ S^{-1} \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \dashrightarrow$$
$$\dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A$$

Since the iteration has produced the string $S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A$, where the last two symbols are $S \circ A$ again, our initial terms, we have generated a closure, viz. a non trivial *conjugation cycle* that in linear sequential unfolding reads as follows:

$$S \circ A \circ S^{-1} \circ A^{-1} := [S, A] := \circlearrowleft (S, A) := S \circlearrowleft A$$

By a slight abuse of notation we may identify the complementary gnomons by the symbols S, A correspondingly, whence their composition or gluing is denoted by the conjugation cycle $S \circlearrowleft A$. Note that the order of composition cannot be reverted, viz. $S \circlearrowleft A \neq A \circlearrowleft S$, hence the operation of composition of complementary gnomons is non-commutative. Thus, it is significant to impose an orientation on the conjugation cycle, which reflects the specified cyclic order of composition.

In the case that S, A are elements of a non-commutative group, the composition $[S, A]$ is referred as the commutator of S, A. In this case the symbols S^{-1} and A^{-1} stand for the group-theoretic inverses of S, A respectively. This observation leads to the conjecture that the complementarity of conjugate Boolean domains pertains to their Boole group theoretic structures, or else it is of a group-theoretic origin. A Boole group is a group-structure on the topological spectrum of a Boolean algebraic domain. Thinking of two complementary Boolean group domains as local patches of a non-abelian global structure the notion of a conjugation cycle provides a natural method of logically gluing them together simultaneously.

4.8 SOLVABILITY VIA NILPOTENCY: CIRCUMVENTING NON-COMMUTATIVITY

Before we examine the aspects of this logical gluing by conjugation cycles of complementary gnomons it is instructive to start from a reciprocal viewpoint and leverage the existing knowledge about the structure of groups. This will provide the method to locate the existence of complementary gnomons from a group-theoretic perspective. The central notion of significance for our problem has to do with the *Galoisian* notion of *solvability* of a group. In particular, the understanding of *Évariste Galois'* theory of groups by the strategy of logical conjugation uses the gnomon of solvability. This will be discussed in more detail as we go on, but for the time being it is enough to convey the basic idea.

The triumph of Galois theory is based on the theorem that a polynomial equation is solvable by radicals if and only if the corresponding Galois group of the equation is solvable. Now a general group is solvable if it can be derived by the method of group *extensions* of *Abelian* (commutative) groups. Reciprocally, a solvable group is a group whose *derived series* terminates in the trivial subgroup. Intuitively, the derived series is a stratification into group levels together with a descending staircase among these strata formed by identifying each subgroup in the descending series with the *commutator* subgroup of the previous one. In turn, the commutator subgroup of a group is the group generated by all the commutators of this group. The importance of the commutator subgroup of a group rests on the fact that it provides the most economical way, i.e. it is the smallest normal subgroup, such that the quotient of the initial group by the commutator subgroup is an abelian group. Thus, a group is solvable if by descending into lower and lower subgroup strata by division with the commutator subgroup we end up with the trivial subgroup that completely annihilates the complexity of the group we started with.

It is well-known that all Abelian groups are solvable, as well as that all *nilpotent* groups are solvable. The first is trivial, but the second is very important, for example, in quantum mechanics. It is worth explaining the latter in more detail. A nilpotent group is one that may be thought of as an almost-Abelian group, in the sense that the commutator subgroup is almost trivial. For instance, we know that in quantum mechanics we have complementary Boolean algebraic domains, like those pertaining to position and momentum. The bounded form of these conjugate observables, called the *Hermann Weyl form*, are constrained to obey the canonical commutation relations expressed by means of the infinitesimal Planck's constant, and hence almost commute. These give rise to a nilpotent group, called the *Heisenberg group*. The Heisenberg group is of fundamental importance in quantum mechanics and essentially constitutes the solvability of the theory in group-theoretic terms. In other words, the non-commutativity induced by any two conjugate or complementary Boolean domains in quantum mechanics is *circumvented* in an almost-commutative manner by the nilpotency of the Heisenberg group, and its attendant solvability. This circumvention is technically possible in all cases where we have at our disposal the structure of a *vector space* equipped with a *symplectic form*. In other words, the structure of a nilpotent group, induced symplectically, transforms the intrinsic insolvability of two conjugate domains into a solvable case. From the perspective of logical conjugation this amounts to considering conjugation cycles as *infinitesimally* small, and thus, behaving like covariant *differential* operators in a precise differential geometric sense.

The above analysis requires the investigation of the source of intrinsic insolvability in groups. It is enough to consider the case of finitely generated linear groups, i.e. matrix groups, which are used as a concrete representation of abstract groups. In this case, according to a well-known theorem of *Jacques Tits*, called the Tits alternative, a finitely generated linear group is either virtually solvable, meaning that it contains a solvable subgroup involving a finite descending staircase, or it contains a non-Abelian (non-commutative) *free* subgroup in *two* generators. Thus, we are able to locate the free group in two non-commuting generators, denoted by Θ_2, as the actual *source* of intrinsic insolvability. From the view of logical conjugation, Θ_2 should be associated with non-trivial and non-reducible logical conjugation cycles between two complementary Boolean domains. The only way that non-solvability can be leveraged or circumvented is through nilpotency, as in the case of the Heisenberg group in quantum mechanics. We bear in mind that uncertainty relations will always pertain between the observables of two complementary Boolean domains. If the associated constant of interrelation can be made either infinitesimally small or reciprocally very big, then the resultant logical conjugation cycles vanish in higher order iterations and the complexity is reducible. It is not an accident that both of our fundamental physical theories, to wit the theory of relativity and quantum mechanics involve this type of constants between conjugate Boolean domains. Thus, from the perspective of logical conjugation, the free group in two generators is the *source* of logical conjugation cycles and the group-theoretic property of nilpotency is the “golden mean” between non-commutativity and commutativity.

4.9 CANONICS FROM THE LOGICAL TO THE TEMPORAL DOMAIN

Therefore, it is of high priority to focus our attention on the fundamental significance of the non-Abelian free group Θ_2. The surprising and counterintuitive result is that the non-Abelian free group in two generators contains *copies* of all other non-Abelian free groups in any finite number of generators as *finite* index subgroups! Thus, the complexity of non-reducible logical conjugation cycles and their iterations generated by two complementary (in some appropriate sense) gnomons subsumes the *whole* complexity we may get from *any* number of obstacles! A way to qualify this proposition, whose reference is algebraic and derives from logical considerations, is to consider the representation of Θ_2 in three dimensions.

From this representation, we obtain a valuable and novel connection between logic and time, thereby providing the canonics from the logical domain to the temporal domain. More specifically, the notion of a logical conjugation cycle is mirrored on the notion of a tripodal link that expresses the quality of a prime temporal bond between two temporal diastases. In the same way that a logical conjugation cycle amalgamates two complementary Boolean domains simultaneously, a temporal bond amalgamates two unlinked temporal diastases in the present. Moreover, and most importantly, the nilpotency condition of solvability transferred to the temporal domain through bonds, provides the origin of geometric differential calculus under the algebraic commutation rule of two infinitesimal flows at a *fulcrum* point, being *bounded* at this point. We will examine this path later on in detail, together with its ramifications, starting from *Hermann Grassmann's* theory of extension and its relation with *Gottfried Wilhelm Leibnitz's* infinitesimal analysis, and culminating with *sheaf cohomology*.

The canonics from the logical to the temporal spectral domain, through the group Θ_2, is particularly important in relation to algorithmic information theory, and more generally, the concept of programs and computability. There are two reasons on which we base our claim. The first is the fact that elements of Θ_2 can be assigned complexity lengths. Since every element of Θ_2 can be uniquely expressed as a freely reduced word in the generators and their inverses, we may simply define the length of an element as the number of terms in this freely reduced expression. This notion of length has the property that the length of an element equals that of its inverse element in this group. The second is related to the fact that, although the group Θ_2 has *exponential* growth rate, a deep theorem of *Mikhail Gromov* shows that a nilpotency circumvention, in agreement with the preceding, reduces the growth rate to a *polynomial* one, and thus proves economical for computational purposes.

4.10 QUBIT COMPUTABILITY: SELF-DELIMITING PROGRAMS AS CYCLES ON THE SPHERE

The initial motivation of this investigation is based on the profound idea conceived by Chaitin, according to which, a key technical point that must be stipulated in order for Ω to make sense is that an input program must be self-delimiting. Its total length in bits must be given *within* the program itself. Chaitin points out essentially that this seemingly minor point, which paralyzed progress in the field for nearly a decade, is what entailed the redefinition of algorithmic randomness. Real programming languages are self-delimiting, because they provide constructs for beginning and ending a program. Such constructs allow a program to contain well-defined subprograms, which may also have other subprograms nested in them. Because a self-delimiting program is built up by concatenating and nesting self-delimiting subprograms, a program is syntactically complete only when the last open subprogram is closed. In essence the beginning and ending constructs for programs and subprograms function respectively like left and right *parentheses* in mathematical expressions.

If programs were not self-delimiting, they could not be constructed from subprograms, and summing the halting probabilities for all programs would yield an infinite number. If one considers only self-delimiting programs, not only is Ω limited to the range between 0 to 1 but also it can be explicitly calculated in the limit from below.

Our main interest in this section focusses on the metaphora considering the beginning and ending constructs of self-delimiting programs and subprograms in analogy to the left and right parentheses in mathematical expressions. It is true that our linear representation of strings or words implicates the self-delimiting property by means of left and right parentheses. A natural generalization would be to *complete* each such pair of parentheses in the 1-dim line to a *circle* in the 2-dim plane, or equivalently the 1-dim *complex line*. This extremely simple generalization generates two conjugate domains immediately, where each one of them corresponds to the choice of orientation on the circle. If we do not impose any orientation on a circle, it is as though we are working in the *modular* arithmetic $\mathbb{Z}_2$, that is, we recover the bit representation of linear strings. Even better, we may complete each pair of parentheses in the 1-dim line to a circle in the one-point *compactification* of the 1-dim complex line, i.e. on the 1-dim *complex projective* space, or equivalently the *Bernhard Riemann sphere* S^2. Can we imagine representing self-delimiting programs by means of *circular strings* on the sphere S^2?

The choice of the sphere S^2 is not accidental. Without loss of generality we may consider the unit sphere S^2, that is imply the normalization according to which all points lying on the sphere are of unit distance from the origin. The unit 2-sphere S^2 constitutes the space of pure states, or equivalently *rays*, of a 2-level quantum mechanical system, called currently a *qubit*. The unit 2-sphere may be thought of as embedded in the usual 3-dimensional space $\mathbb{R}^3$. The *David Hilbert space* of normalized unit state vectors of a qubit is the 3-sphere S^3, and thus the unit 2-sphere is considered as the base space of the topological *Heinz Hopf fibration*:

$$S^1 \rightarrowtail S^3 \twoheadrightarrow S^2$$

We note that each pair of *antipodal* points of S^2 corresponds to mutually orthogonal state vectors. The north and south poles are chosen to correspond to the standard orthonormal basis vectors $|0\rangle$ and $|1\rangle$ correspondingly. In the case of a *spin*-$\frac{1}{2}$ system, these simply correspond to the spin-up and spin-down states of this system.

We consider the unit sphere S^2 as the set of points of 3-dimensional space $\mathbb{R}^3$ that lie at distance 1 from the origin. Then, the non-commutative group $SO(3)$ denotes the group of *rotation operators* on $\mathbb{R}^3$ with center at the origin, viz. linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented as 3×3 matrices with determinant one. These are called *orthogonal* matrices, characterized by the fact that their columns form an orthonormal basis of $\mathbb{R}^3$. Rotations around an axis going through the origin are the isometries of 3-dimensional Euclidean space $\mathbb{R}^3$ leaving the origin fixed. Note that a 3×3 orthogonal transformation preserves the inner product for any pair of vectors in $\mathbb{R}^3$, and moreover it is an *isometry* of $\mathbb{R}^3$ that takes the unit sphere S^2 to itself.

In this context, we ask the following question: Does there exist a *representation* of the non-Abelian free group in two generators Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3? We recall that the existence of such a representation would imply the *action* of non-trivial logical conjugation cycles on S^2 and S^3 respectively. Moreover, these logical conjugation cycles would be representable by means of the Tripodal link topology. Such a representation definitely exists if we are able to locate a subgroup of the non-commutative group $SO(3)$, which is *isomorphic* to Θ_2.

We will show further on that this is indeed the case. The proof is based on the observation that there actually exist rotation operators A and B about two *independent* axes through the origin in $\mathbb{R}^3$ generating a non-commutative subgroup of $SO(3)$, which is isomorphic to the free group Θ_2. In other words, there exists an isomorphic copy of Θ_2 in $SO(3)$ generated by two independent rotations A and B. The term independent refers to the requirement that all rotations performed by sequences of A and B and their inverses are distinct strings in Θ_2.

Actually, we realize that *most* pairs of rotations in $SO(3)$ are independent in the above sense, so that even picking A and B randomly would do. For instance, one could consider two counterclockwise rotations A and B about the z-axis and the x-axis respectively of the same angle $arccos(3/5)$. The proof is based on showing that no reduced string in the symbols A and B and their inverses collapses to the identity transformation (3×3 identity matrix). Intuitively, if we choose two counterclockwise rotations A and B about the z-axis and the x-axis of the same angle, then this specific angle needs to be an irrational number of degrees. More precisely, given an initial orientation, if the specified angle *is* an irrational number of degrees, then none of the distinct strings of rotations in Θ_2 performed by sequences of A and B and their inverses can give back the initial orientation. Thus, no reduced word in Θ_2 collapses to the identity transformation.

The existence of an isomorphic copy of Θ_2 in $SO(3)$ has the following consequence: Each rotation belonging to the non-commutative free subgroup Θ_2 of $SO(3)$ fixes two points in the unit sphere S^2, namely the intersection of S^2 with the axis of rotation passing through the origin. If we take the union of all these points, they form a countable set of points. This reveals not only that an action of Θ_2 on the unit sphere S^2 (as a subgroup of $SO(3)$ generated by A and B) must exist, but that this action is actually free on S^2 *modulo* the countable set of fixed points K.

Thus, we can partition $S^2 \setminus K$ into a disjoint union of orbits for the action of Θ_2. If we choose a base point for an orbit we may identify this orbit with Θ_2 due to the freeness of the action. Moreover, if a countable collection K of points as above is removed from S^2 they can be restored by rotations around an axis through the origin which has zero overlap with K. In this way, the action of the group Θ_2 via strings of rotation operators allows us to *resolve* the whole unit sphere S^2. The crucial point again is that the algebraic irreducibility of the *commutator* $[A,B]$ of the rotations A and B generating an isomorphic copy of Θ_2 in $SO(3)$ expresses a non-trivial logical conjugation cycle. In turn, such a logical conjugation cycles express the fundamental property of topological Tripodal non-splittability, or non-separability, of these three rotations belonging to the subgroup of $SO(3)$ that is isomorphic with Θ_2.

Most important, this interpretation provides a topological justification of the fact that one cannot specify a finitely additive rotation-invariant probability measure on all subsets of the unit sphere S^2 simultaneously. In the same vein of ideas, if we consider S^2 embedded in 3-dim space $\mathbb{R}^3$, we deduce that it is not possible to specify a finitely additive measure on $\mathbb{R}^3$ that is both translation and rotation invariant, which can measure every subset of $\mathbb{R}^3$, and which gives the unit ball a non-zero measure. This explains why the Lebesgue measure, which is countably additive and both translation and rotation invariant, and gives the unit ball a non-zero measure, cannot measure every subset of $\mathbb{R}^3$. Thus, it has to be carefully restricted to only measuring subsets that can be Lebesgue measurable.

According to the preceding analysis, since the group of rotation operators $SO(3)$ contains an isomorphic copy of the free non-commutative group Θ_2 is unsolvable.

An immediate consequence of the above is that the group of 2×2 *complex unitary* matrices with unit determinant $SU(2)$ is also unsolvable, that is it also contains an isomorphic copy of Θ_2. The reason is that topologically, the simply-connected special unitary group $SU(2)$ is a *covering* space of the non-simply connected group of rotations $SO(3)$, and in particular it is a *double cover*. More concretely, there exists a two-to-one surjective homomorphism of groups:

$$\Delta : SU(2) \twoheadrightarrow SO(3)$$

whose kernel is given by $Ker\Delta = \mathbb{Z}_2 = \{+1, -1\}$.

Hence, it follows that there must be an isomorphic copy of Θ_2 in $SU(2)$. More precisely, if A and B are rotations generating an isomorphic copy of Θ_2 in $SO(3)$, and $\Delta : SU(2) \twoheadrightarrow SO(3)$ is the covering projection, then $\overline{A}$ and $\overline{B}$ generate a free subgroup of the form Θ_2 in $SU(2)$, for any $\overline{A}$ and $\overline{B}$ with $\Delta\overline{A} = A$ and $\Delta\overline{B} = A$. Since $SU(2)$ is a double cover of $SO(3)$ there can only be exactly two elements of the form $\overline{A}$, namely U and $-U$ such that $\Delta U = \Delta(-U) = A$ (the same holds for $\overline{B}$ respectively).

We conclude that there exists a representation of the group Θ_2 on the unit sphere S^2, which *lifts* to a unitary representation on S^3.

The representation of the group Θ_2 on the unit sphere S^2 is given by the free subgroup of rotations of $SO(3)$ generated by A and B according to the above. Concomitantly, this representation lifts to a *unitary* representation on S^3 by the free subgroup of unitary operators of $SU(2)$ generated by $\bar{A}$ and $\bar{B}$.

Thus, the Hilbert space of normalized unit state vectors of a qubit or of a spin-$\frac{1}{2}$ system carries a unitary representation of the group Θ_2. This means that the algebraic irreducibility of the commutator $[\bar{A},\bar{B}]$ of the unitary operators $\bar{A}$ and $\bar{B}$ generating an isomorphic copy of Θ_2 in $SU(2)$ expresses non-trivial conjugation cycles. Moreover, since the action of the group Θ_2 by strings of rotations in two generators allows S^2 to resolve, such that the same lifted action resolves S^3 as well, by strings of corresponding unitary operators, we can make a conclusion. It is the Tripodal link topological connectivity - by means of conjugation cycles – that is transferred through these actions to the space of rays S^2 and the space of unit state vectors S^3 of a qubit. This is the crux of the *non-classical* behavior of a qubit and the problem arising here is whether the existence of non-trivial conjugation cycles can be turned to a *novel* computational possibility.

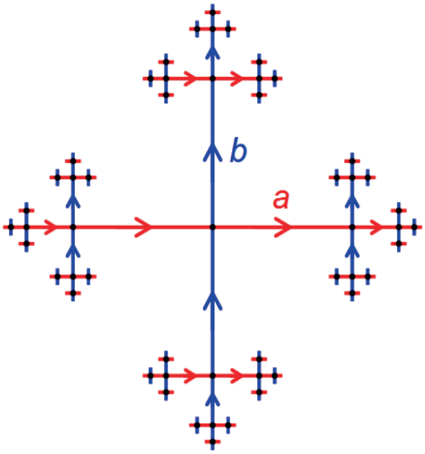
We will outline the first steps towards implementing such a computational paradigm. For this purpose, our guiding principle will be the implementation of Chaitin's uncertainty relation. We recall that the form of Chaitin's uncertainty relation reads:

$$(\forall n)[I([\Omega]_n) \geq (n-k)]$$

where the constant k is interpreted as setting the bound of the simultaneous determination of two conjugate observables, viz. the random real Ω and the program-size complexity length measure I.

We have shown that isomorphic copies of the group Θ_2 exist within the group of rotation operators $SO(3)$ and the group of unitary operators $SU(2)$ leading to the realization of logical conjugation cycles, or equivalently Tripodal loops, on S^2 and S^3 respectively. Since the group $SU(2)$ is a subgroup of the group $GL(2,\mathbb{C})$, the matrix group of 2×2 matrices with complex coefficients, and of the group $SL(2,\mathbb{C})$, viz. the group of 2×2 matrices with complex coefficients and unit determinant, they also contain a copy of the group Θ_2. So we are going to identify two complex matrices acting as the generators of this copy of Θ_2 using Chaitin's uncertainty relation in the present setting. For this purpose, we assume the existence of a positive integer N playing the role of string length measure, such that for all $m \geq N$, the powers G^m and H^m, where G, H are complex matrices, generate a copy of Θ_2. This is possible using the method of *dominant eigenvalues* and *dominant eigenvectors* of matrices.

We observe that for this purpose we have to *diagonalize* these matrices, a technique which is also based on logical conjugation. In particular, we look for two matrices G and H, such that: G has the dominant eigenvalue μ corresponding to a dominant eigenvector u. This means that the eigenspace of $G-\mu I$ is 1-dimensional and all other eigenvalues of G have modulus less that $|\mu|$. Similarly, let H take the dominant eigenvalue ν corresponding to a dominant eigenvector v. Finally, we denote the dominant eigenvalues and corresponding dominant eigenvectors of G^{-1} and H^{-1} by ρ, w, and υ, z, respectively. Next, we consider the dominant eigenvectors as points on the 1-dim complex projective space, viz. equivalently on S^2. Then, the dominant eigenvalues/eigenvectors implement the requirement that there exist disjoint open sets containing the points u, v, w, z, denoted by U, V, W, Z, respectively, such that: There is some $m \geq N$ with the property that, G^m sends each of these open sets to U, and correspondingly for the others, viz. H^m to V, G^{-m} to W and H^{-m} to Z. Now, we think of a *finite state computer*, with four states labelled by U, V, W, Z and an *alphabet* $G^m := a$, $H^m := b$, $G^{-m} := a^{-1}$, $H^{-m} := b^{-1}$ and transitions rules as described above. It is clear that the matrices a and b now *generate* a copy of the free group Θ_2, and thus we obtain logical conjugation cycles for the formation of strings using our alphabet with the prescribed transition rules.


b
a