2

TRANSCENDENTAL CIRCULATION: EXPONENTIATION AND LOGARITHMIZATION

2.1 EXTENSION BY INVERSION: FORGETTING AND REMEMBERING

One basic characteristic of the process of extension of the algebraic structure by means of metaphora through the scaffolding of sets is that the conceptually inverse bridges employed to achieve conjugation bear some particular meaning that is worth focussing on. More concretely, the encoding bridge is actually a *forgetful* bridge (*"lethe"*) in relation to to the initial algebraic structure, whereas the decoding bridge is one of *recollection* (*"anamnesis"*) that re-establishes the algebraic structure at the initial level. The reason is that the appropriate homeotic criterion of identity is established in terms of an equivalence relation at the level of set elements, following which, the algebraic structure is enforced in a suitable way. The other basic characteristic is that the extended algebraic structure, does not discard, but rather incorporates the initial one. This means that the restriction of the structure extended from our initial one serves to ensure agreement with it. This is clear already from the algebraic treatment of the Thalesian problem that required the extension of the integers into the rationals in order to solve the pertinent proportionality equation between magnitudes and their shadows. In the extended algebraic structure of the rationals the integers are qualified as a particular type of rationals, such that the restriction of the rationals to the integers is feasible.

From an algebraic viewpoint the metaphora realizing the extension of the integers to the rationals addresses the issue that division is not possible within the domain of the integers. It becomes possible only via the extension of the integers to the rationals, since the latter assume a group structure with respect to the operation of multiplication, which is absent in the case of the integers. In this sense, this extension in entailed by the necessity of determining the algebraic domain, where the inverse operation to multiplication becomes a total operation and can be performed without obstruction. The perspective of inverting an algebraic operation, which anyway proves indispensable in solving algebraic equations, is very fruitful and elucidating in understanding the emergence of arithmetics and algebraic structures. From this perspective, a structural algebraic metaphora provides the means to evade the obstacle of inverting an operation in an initially specified algebraic domain.

2.2 TWOFOLD INVERSION OF POWERS: ROOTS AND LOGARITHMS

A very important case presents itself in the consideration of the notion of taking powers. It is well known that the notion of power is defined by recursion on the operation of multiplication. The complexity in the notion of a power is that it involves two numerical entities assuming different operational roles. More concretely, we have the base of the power and the power itself, such that the operation of raising the base to a power is not a commutative operation, i.e. the result is not invariant where the roles of bases and powers are exchanged. In this sense, the non-commutativity appearing for the first time algebraically in the procedure of raising a base to a power, requires the consideration of two distinct inverses; one referring to the base, and the other referring to the power. If we call this non-commutative operation with respect to the base and the power the operation of *exponentiation*, then its inversion is twofold: Inverting with respect to the base is the procedure called *root extraction*, whereas inverting with respect to the power is the procedure called *logarithmization*.

Therefore, two distinct types of conjugation are needed in order to invert exponentiation. The first, referring to the powers with respect to a base obliges us to extend the field structure of the rationals to the field structure of the reals. As a result, logarithmization becomes a total operation in the domain of real numbers. The second, referring to the roots, necessitates the extension of the field structure of the rationals to the field structure of the complex numbers, if we include the roots of negative numbers. Both of these inversions are unified in the field domain of the complex numbers under the notion of the complex logarithm. It is important to highlight that both of these inversions are not purely algebraic, but entrain topological arguments for the effectuation of the respective metaphoras. The first requires an argument of continuity, whereas the second requires additionally a topological argument of evasion of the obstacle of multiple connectivity, to which we will come back later.

At this stage, it is worth considering first, real logarithmization in functional and algebraic terms. If we consider that b is any positive base different from the unit 1, then the exponentiation equation $b^y = x$, where $x > 0$, is solved in terms of y by logarithmization, i.e. $y = \log_b x$. Equivalently, the power y is expressed as the real logarithm of x in the base or root b. It is clearly not allowed to take the real logarithm of zero or a negative number. If we think of b^y as a function of y, then this function is a continuous (and differentiable) function of the variable y, whose inverse is the continuous (and differentiable) real logarithm function $x \mapsto \log_b x = y$. Note the intervention of this topological qualification required for the performance of the required inversion that requires the explicit consideration of the irrationals besides the rationals, in other words, the meaningful inclusion of limit processes, in order to achieve the extension to the domain of the reals. The real logarithm function is characterized as the unique monotonically increasing function from the positive reals to the reals, such that:

$$\log_b b = 1$$
$$\log_b(y_1 \cdot y_2) = \log_b y_1 + \log_b y_2$$

Note that the real logarithm function converts multiplication of positive reals to addition of reals and it is order preserving.

A natural question emerging in the functional context regarding the real logarithm function is how to express the procedure of raising to a power independently of the base employed. For this purpose, we define the exponential function $\exp : x \mapsto \exp(x)$, from the reals to the positive reals, i.e. the value is never zero and never negative, characterized by the property that

$$\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$$

meaning that it converts addition of reals to multiplication of positive reals. Then, the problem of raising to any power a with respect to a base $\circlearrowright$, where $\circlearrowright$ is thought of as a variable, is resolved by regarding the exponential and logarithm functions as inverse bridges between the group theoretic domains of the positive reals with respect to multiplication and the reals with respect to addition. More concretely, these inverse bridges, are inverse homomorphisms between these two groups, and thus, constitute the group of the reals under addition isomorphic to the group of the positive reals under multiplication. In other words, the real exponential function and the real logarithm function are not only inverse functions, but more important, they are inverse group homomorphisms.

In this fashion, we may define the real logarithm function $\log : \mathbb{R}^+ \to \mathbb{R}$ as the group homomorphism from the multiplicative group $(\mathbb{R}^+, \cdot)$ to the additive group $(\mathbb{R}, +)$ since $\log(y_1 \cdot y_2) = \log y_1 + \log y_2$ is satisfied for any positive reals y_1 and y_2. Inversely, the real exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$ is a group homomorphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(\mathbb{R}^+, \cdot)$ satisfying $\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$.

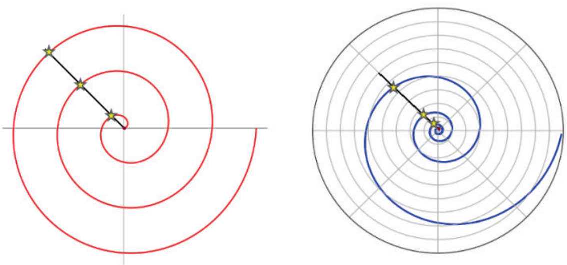
As such these two group homomorphisms are inverse to each other; they establish an isomorphism between these two different group structures. The most important consequence of this isomorphism is that the additive group structure of all real numbers, i.e. of the values of the logarithm function under addition, is indistinguishable from the multiplicative group structure of the positive reals, i.e. of the values of the exponential function under multiplication. Consequently, the difficult operation of raising to a power can be conjugated to the easy operation of multiplication by metaphora from the additive group of the reals to the multiplicative group of the positive reals, where $\exp$ and $\log$ play the role of the inverse bridges. Symbolically, we have:

$$\circlearrowright^{a} = \exp[a]\log(\circlearrowright) = \exp[a]\exp^{-1}(\circlearrowright)$$

Conversely, the capacity of the above metaphora to solve the problem of raising to a power by conjugating it to multiplication is equivalent to the group isomorphism induced by the inverse bridges identified with the real exponential and the real logarithm function.

2.3 TRANSCENDENTAL *GNOMONS*: BRIDGING THE HARMONIC WITH THE GEOMETRIC

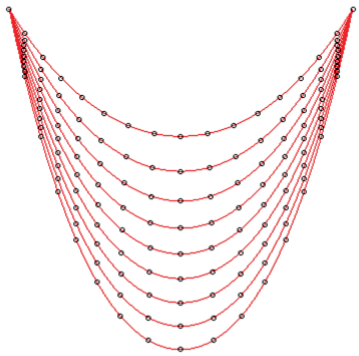
Both the exponential bridge and its inverse logarithmic bridge are characterized by self-similarity. Thus, they can be conceived in gnomonic terms. More concretely, since both of them are transcendental functions they act as inverse bridges between the transcendental or harmonic domain and the geometric domain, which is to say that the exponential is a bridge from the geometric to the harmonic, and inversely, the logarithm is a bridge from the harmonic to the geometric domain. If we consider the well-known example of the logarithmic spiral, it clearly provides an example of gnomonic growth, which is encountered in the natural world, for instance in the case of the Nautilus shell. The logarithmic or equiangular spiral differs from the *Archimedean* spiral in the sense that the distances between successive windings are not constant, but they increase in geometric progression.



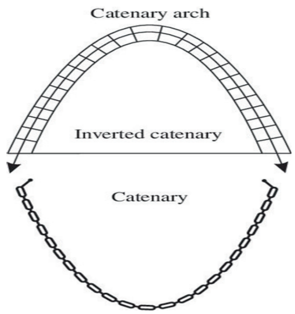
Archimedean Spiral

Logarithmic Spiral

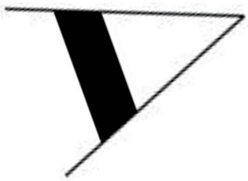
Coming to the exponential bridge, if we consider the arithmetic mean of the exponentials $\exp(x)$ and $exp(-x)$, i.e. $\frac{\exp(x)+\exp(-x)}{2}$, then we obtain a well-known curve, called the *catenary curve*, which can be also thought of in gnomonic terms. The origin of this curve is physical, and more precisely, it is the solution to the least action problem referring to a chain in a gravitational field. Put simply, the catenary curve composed by the arithmetic mean of two exponential bridges according to the above, is the natural shape of a hanging chain under the pull of gravity.

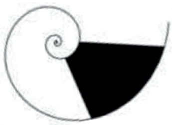


The inverted shape of the catenary is the well-known *catenary arch* in architectonics with myriad of applications. The catenary arch by its specification through the real exponential bridges stands by itself without any support, defying in a sense the pull of gravity as the inverse of the shape assumed by a hanging chain.



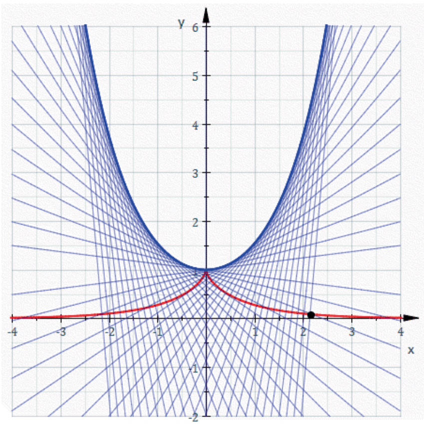
The most interesting aspect of these transcendental gnomons, which is absent from the initial rational conception of the geometric ones, is the appearance of curvature. Moreover, the pattern of gnomonic growth is not a linear trapezium as in the former case, but an angular trapezoidal sector, depicted for comparison below.



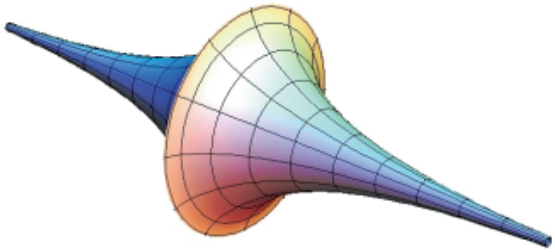


2.4 INTRINSIC GEOMETRIC CURVATURE

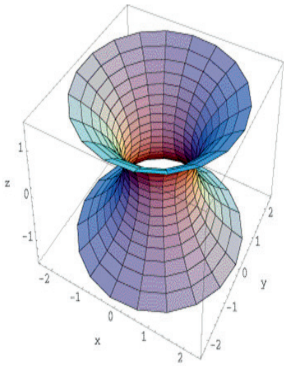
The blueprint of the different types of local curvature in two dimensions is already evident by considering the catenary curve. The geometric way of detecting the local curvature involves the consideration of the tangent and the normal at a point. The normal may be thought of as the radius of a circle at the specified point, whereas the tangent is the orthogonal to the normal, identified with the tangent of the circle at this point. We now imagine another curve that bears the inverse specification of tangents and normals, whereby the former tangents are the normals of the new curve and the former normals are the tangents of the new curve. Then, we obtain a geometric inversion with respect to the local curvature referring to these two curves. If we apply this to the case of the catenary, then we obtain another curve called the *tractrix* as depicted below:

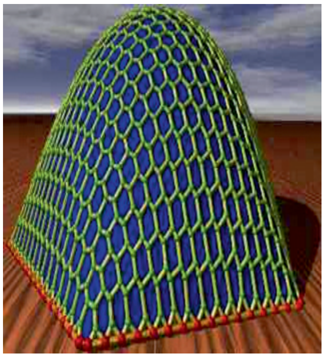


The surface of revolution emerging by rotating the tractrix about its asymptote is a *pseudosphere*, which is a surface with constant negative intrinsic curvature, characterized as a hyperbolic surface. The analogia with the sphere comes from the fact that a sphere has constant positive curvature $1/R^2$, where R is the radius of the sphere, whereas the pseudosphere has constant negative curvature $-1/R^2$. They can be treated on an equal footing by considering the radius of the pseudosphere as an imaginary radius, i.e. iR, such that its curvature becomes the negative magnitude $-1/R^2$. This is a non-trivial step that requires an imaginary metaphora between the harmonic and the geometric domain culminating in the role of the imaginary unit, a subject to which we will come later.

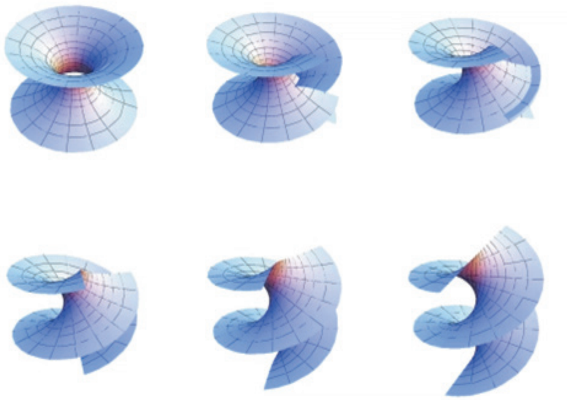


The revolution of the catenary around an axis can be performed in two ways, both concavely and convexly. The surface of revolution obtained in the first case is a *catenoid*, while in the second it is a *catenary dome*. The catenoid is a minimal surface; it occupies the least area when bounded from above and below, e.g. by two circular rings. Because of this fact, it has mean curvature zero everywhere. As such it should be thought of as the curved abstraction of the plane, which is also a minimal surface considered as a surface of revolution. The catenary dome should be thought of as the optimal correction to the shape of an ideally symmetric spherical dome when subject to acceleration due to gravity.

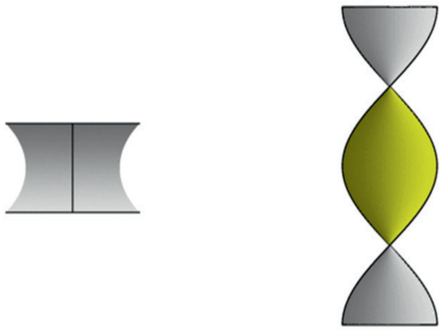




Topologically the catenoid is non-simply connected due to the hole it bears in the middle; if we make a cut, then it can be deformed periodically to a simply-connected helicoid, which is also a minimal surface, although not a surface of revolution. In particular, it occupies the least area when bounded sideways by two helices. In this way, the catenoid becomes locally isometric to the helicoid. A two-dimensional entity could not locally distinguish the catenoid from the helicoid. The fact that this locally isometric deformation exists is a strong motive to explore the implications of the exponential and logarithmic bridges when extended to the imaginary and complex number domains. The crucial observation is that after half a period a mirror image of the same helicoidal surface arises, which may be qualified topologically as a twisting. For example, we may think of a belt as a toy model whose two sides are coloured differently. The closed belt is an approximation to the region around the equator of the catenoid. If we open the belt and move the left end up and the right end down we have an approximate model of a helicoid. On the other side, if we move the left end down and the right end up we obtain the mirror or twisted image of the former helicoid.



Note that the rotation axis of both the helicoid and its mirror is orthogonal to the equator of the catenoid, since there is a $\pi/2$ rotation counterclockwise or clockwise in relation to the equator. This is a strong indication about the role of the imaginary unit from a transcendental viewpoint.



The extension of the real exponential function to the imaginary domain takes place via the complex exponential function $\exp:\mathbb{R}\to S^1$, where S^1 denotes the unit circle, whose elements are described by Euler's formula as $e^{i\theta}=\cos\theta+i\sin\theta$. Note the appearance of the imaginary unit, which is interpreted as a rotation by $\pi/2$ making the imaginary axis orthogonal to the real axis in the domain of the complex numbers. Given together with the imaginary unit, there is always its mirror image, described as its complex conjugate. Since the unit circle is coordinatized by means of the imaginary unit, we think of this circle as an imaginary ring. Its emergence will be elaborated as we go on. At this stage, it is useful not to adopt the more conventional *geometric*, but rather to favour the *harmonic* interpretation. Simply expressed, the imaginary ring is actually a harmonic ring, i.e. it descends not from the domain of geometry but from the transcendental domain of harmonics. Notwithstanding this fact, the image of the ring in the geometric domain of forms may be visualised as a circle, more precisely, as a circular shadow of a harmonic entity. The latter is expressed transcendentally through the complex exponential function as its imaginary power.

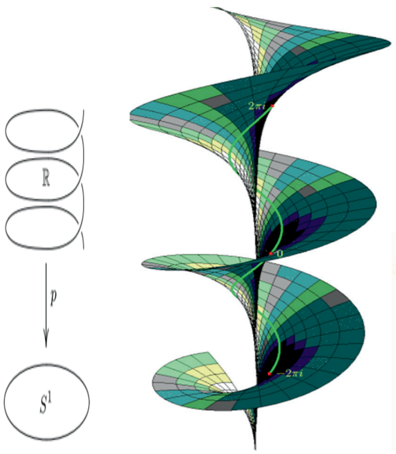
For the consistency of this metaphorical interpretation it is necessary to explicate the qualification of a harmonic entity as well as its expression as an imaginary power. The intuition comes from the dual consideration of the helicoid along with its mirror image as constituting a harmonic entity. Firstly, the helicoid unfolds continuously by parallel translation of its tangent planes, and after half a period of rotation a mirror image of the same helicoidal surface arises. We may think of the helicoid together with its mirror image as helical waves propagating in opposite directions such that the mirror image is the reflection of the first. This is possible if these helical waves are bounded from above and below for temporal length of one period so as to give rise to a helical standing wave. Here, this condition is equivalent to the requirement that within this bounded interval the helical wave is in unison with its reflection, its mirror image. Being in unison means that they are consonant in the fundamental harmonic frequency corresponding to the frequency ratio $1:1$, which in turn, would correspond to an angular temporal interval of one whole period 2π.

2.6 HARMONIC SERIES OF A TEMPORAL HELICAL CHORD AND FREQUENCY SPECTRUM

It is instructive to highlight the difference between a vibrating straight chord whose length is spatial, and a vibrating helical chord whose length is temporal as in the preceding. The unison ratio in the former case corresponds to a zero length spatial interval, whereas it corresponds to a 2π temporal interval in the latter case. Notwithstanding this fact, we are able to establish the whole *harmonic series* in the helicoidal case, such that there is an inverse relationship between frequency and temporal extent or duration. The visual imaginary ring in this context is the unit circle descending from the harmonic domain of relations between a variably bounded helicoid and its mirror image into the visible geometric domain as its observable shadow. It serves to spatialize temporal extents by means of the imaginary unit and its conjugate to allow for twofold directionality. The spatialization records temporal extents in a twofold imaginary axis, qualified by both a positive and a negative direction as usual, as simultaneously extended imaginary spatial lengths at the present of the emergent shadow. Equivalently, these spatialized extents can be viewed as angular sectors of the imaginary circle via the complex exponential function. In this manner, being in unison in the harmonic context of a helical standing wave has a shadow in the visible geometric domain quantified by the imaginary spatial length $2\pi i$, which is identical to the period of the complex exponential function. Alternatively, through the complex exponential function, being in unison corresponds to the whole 2π angular sector of the circumference of the imaginary ring.

It is worth pondering on some specific characteristics of the harmonic domain that make it different from the visible geometric one. If we think ontologically in terms of substances, then in the harmonic domain the twisted or mirror image, or simply the reflection is of the same substance as the original, since it can interact and interfere with it to produce a standing helical wave bounded from above and below. The latter is not traveling in space at all. In contrast, it resolves time in terms of the harmonics series and the concomitant harmonic ratios of frequencies, i.e. by means of *consonances* and *dissonances*. As such a standing helical wave in the context of its resonating environment is not an ontological entity in physical space, although it has a shadow quantified through the imaginary ring. Its most crucial aspect is that it resolves time periodically in terms of the harmonics, in such a way that time and frequency are reciprocally correlated. Thus, in the same way that time is spatialized via the imaginary axis, to give an imaginary length, frequency is spatialized orthogonally to the former as speed or momentum. What really matters is the orthogonal placement of frequency and spatialized temporal duration due to the intervention of the imaginary axis. As such, the opposite convention of indexing frequencies as imaginary quantities and spatialized temporal intervals as real is also valid and acceptable. Keeping in mind the above correspondences it becomes evident that the exponential function is qualified as eigenfunction of the differentiation operator, as well as a kernel of an integral transform, for example, the *Fourier* transformation between functions of these reciprocal variables. Note that an arbitrary angle in its expression as a power in the complex exponential is a product of the reciprocal variables.

Keeping the former convention, we identify the stairs of any bounded portion of the helical wave, unfolding orthogonally to the imaginary ring that constitutes its shadow, or present epiphaneia determined by the bounds of the resonator, with the harmonic series, being able to induce any harmonic ratio. In this manner, the harmonics are qualified as *powers* for the actualization of consonances and dissonances. The negative harmonics, setting up the whole frequency spectrum, correspond to the harmonic series of the reflection. Therefore, the whole frequency spectrum of a bounded helical standing wave does not bear any *ontological* role, but its role may be thought of as *teleological*. More precisely, the whole harmonic series depicted by the helical stairs is the *entelecheia* of the standing wave that accompanies the transcendental domain of time, as expressed in terms of the complex exponential function.



Considered structurally, which means here group-theoretically, the complex exponential is a group homomorphism from the additive group $(\mathbb{R},+)$ to the multiplicative group $(S^1,\cdot)$ satisfying $\exp(i(\theta_1+\theta_2))=\exp(i\theta_1)\cdot\exp(i\theta_2)$. The homeotic criterion of identity is encapsulated in the kernel of this group homomorphism, which is $2\pi\mathbb{Z}$. Note that the homeotic criterion of identity is established in terms of the angular temporal interval of one whole period 2π times the harmonic series, which belong to the group of the integers. In this sense a single moment of time, identified with the present of the imaginary ring shadow, making it a unity, is resolvable homeotically by the whole spectrum of harmonics, so that consonances can occupy this moment.

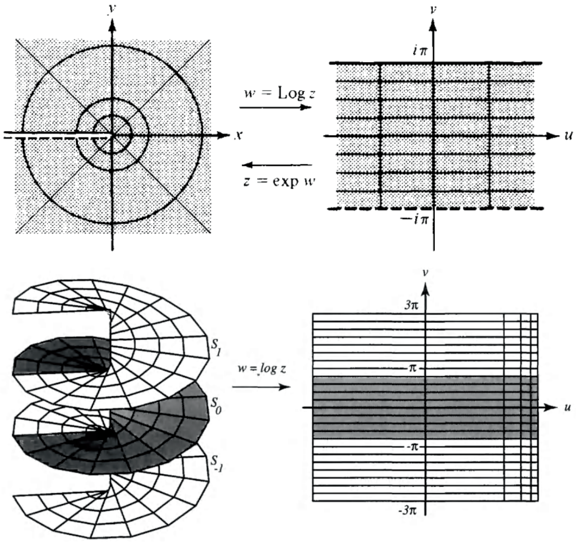
The existence of this homeotic kernel $2\pi\mathbb{Z}$ of the complex exponential group homomorphism $e^i:(\mathbb{R},+)\to(S^1,\cdot)$ has a price. The price is that the complex exponential is not invertible globally, but only locally. This broaches the significance of the domain of sheaves into which we will encounter later on. At present, the fact that it is not possible to have a well-defined global notion of a complex logarithm as the inversion of the complex exponential entails the novel phenomenon of branching. In other words, the projection from the helix to the circle, although it bears well defined local sections inverting it locally, does not possess a global inverse. We may assert that branching is the geometric way to engage with the issue of homeotic consonance in the harmonic or transcendental domain. Topologically, the latter gives rise to what is called multi-connectivity. Branching is the geometric way to evade multi-connectivity by a process of cutting, bounding, and unfolding, until everything becomes simply connected.

Considering the complex logarithm, we realize that an inverse homomorphism from the multiplicative group $(S^1,\cdot)$ to the additive group $(\mathbb{R},+)$ can be defined only locally, i.e. by restricting the values of the angle within a period, i.e. from $-\pi$ to $+\pi$, $-\pi<\theta\leq\pi$, or from 0 to 2π, $0\leq\theta<2\pi$, which depicts a branch by cutting. The meaning of the branch is that the complex logarithm is single-valued within this branch. The whole issue arises from the multi-valuedness of the angle, because the complex exponential has the same value for angle θ, and $\theta+2k\pi$, where k is an integer. This is precisely what is encapsulated in the homeotic kernel of the complex exponential group homomorphism that is understood, as established above, by the nature of the helicoid. Epigrammatically, we may say that if the harmonic domain is associated with multiplexing and knotting, the geometric domain is associated with branching and weaving. Topologically, the main theme is connectivity, and the metaphora pertains to the unfolding of harmonic multi-connectivity into geometric simple-connectivity.

2.8 CANONICS: *METAPHORA* FROM HARMONICS TO GEOMETRY VIA TOPOLOGY

Since the harmonic and the geometric domain incorporate different principles of organization, we may consider the transcendental exponential and logarithmic functions not only from a gnomonic standpoint, but more accurately, from the perspective of canonics. The notion of a canon incorporates the requirements for an analogia or metaphora between two structurally, or organizationally different, domains that can communicate to each other covariantly by conjugation, which means by the enforcement of appropriate encoding and decoding bridges. The notion of canonics emanates from Pythagoras' vibrating monochord that embodied the idea of descending from the harmonic to the geometric domain and ascending back. Conceptually it refers to the transfiguration of an acoustic chord to an optical fiber involving the instantiation of a scale that is able to transform acoustic frequency ratios to visual length intervals. The conception of relative frequencies as powers that can be perceived by the ear implies that the bridge from the harmonic to the geometric is of a logarithmic nature. In other words, the logarithm function transfigures frequency ratios to length intervals, since it converts division to substraction. Inversely, the transfiguration from the geometric to the harmonic domain is of an exponential nature. This fact has been implicated in the impossibility of setting up a rational scale of musical intervals. In turn, this bears the consequence that the discovery of the irrationals does not come from the geometric domain, most typically presented via the Pythagorean theorem, which targets the incommensurability of the diagonal with the sides of an orthogonal triangle, but derives rather from harmonics. The notion of the equally-tempered, based on the equipartition of musical intervals, ending up on the chromatic geometric scale, is a geometric solution to evade this problem at the price of sacrificing the pure harmonics.

The conceptualization of the imaginary ring set out here, allows the exemplification of canonics from the viewpoint of complex geometric function theory, especially as pertaining to the complex exponential and complex logarithm functions. More precisely, the complex exponential is an encoding bridge from the geometric to the harmonic domain, whereas the complex logarithm is a decoding bridge from the harmonic to the geometric domain, which is actually inverse to the former only locally, giving rise to the phenomenon of branching. The complex logarithm bridge of this metaphora may be thought of as the means of unfolding harmonic multi-connectivity into geometric simple-connectivity. From a structural algebraic viewpoint, the complex exponential defined in terms of a group homomorphism from the additive group $(\mathbb{R},+)$ to the multiplicative group $(S^1,\cdot)$ is extended now to a group homomorphism from the additive group of complex numbers $(\mathbb{C},+)$ to the multiplicative group of non-zero complex numbers $(\tilde{\mathbb{C}},\cdot)$. Considering the complex logarithm, an inverse homomorphism from the multiplicative group $(\tilde{\mathbb{C}},\cdot)$ to the additive group $(\mathbb{C},+)$ can be defined only locally, which obliges us to restrict the values of the angle within a period. The period in question is either, from $-\pi$ to $+\pi$, $-\pi < \theta \leq \pi$; or from 0 to 2π, $0 \leq \theta < 2\pi$, which depicts a branch where the complex logarithm is continuous and single-valued.

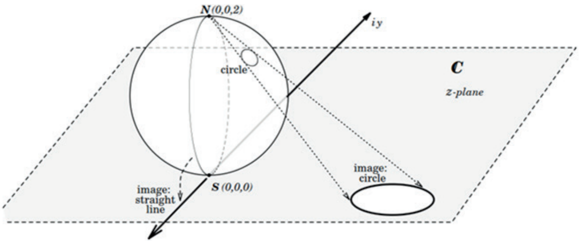


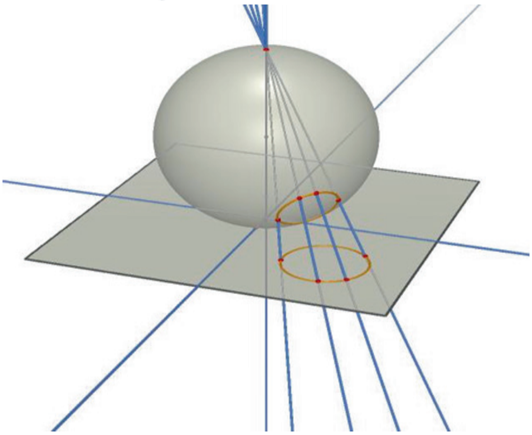
The harmonic multiple-connectivity is encapsulated precisely in the homeotic kernel $2\pi\mathbb{Z}$ of the complex exponential group homomorphism that is intrinsic to the nature of the helicoid. Consequently, the homeotic criterion of identity is expressed in terms of the angular temporal interval of one whole period 2π times the harmonic series, being identified structurally with the multiplicative group of the integers. We stress that the integers in this setting are manifested as powers. Topologically these powers are utilized for counting the number of windings. Thus, topologically, the above homeotic criterion of identity amounts to a homological criterion of identity. More concretely, the multiplicative group of the integers plays the role of the first homology group of the topological circle, e.g. the first topological structural invariant of multiple-connectivity expressed by means of a commutative group. Not only this, but additionally, in the case of the topological circle, the first homology group is isomorphic with the first *homotopy* group, or *fundamental* group, since both are identified with the multiplicative commutative group $\mathbb{Z}$. In this manner, the homeotic criterion of identity, serves not only as a homological criterion of identity, but also as a homotopic criterion of identity in relation to the topological circle. Of course, the imaginary ring endows the topological unit circle with the complex structure by which the complex exponential and logarithm functions are defined.

2.9 SPHERICS: THE PLANISPHERE PROJECTION

Given the identification of the topological unit circle endowed with the complex structure with the imaginary ring, culminating in the polar grid scaffolding of the complex plane, via the Euler representation, we realize that this grid actually descends from the sphere endowed with the complex structure, called currently, the *Riemann sphere*.

This is elucidating, since the original framework of Pythagorean harmonics and canonics was called *spherics*. A simple geometric perspective offering a glimpse to spherics consistent with our interpretation of metaphora is incorporated in the *stereographic projection* of a sphere onto a plane. This projection is described first in Ptolemy's *Planisphaerium*, called originally, the *planisphere projection*. This projection is defined on the whole surface of the sphere with the exception of a single point (usually taken as the North pole) that is identified with the locus of projection, or the point through which light rays enter the sphere, propagate through it, until eventually they emerge out of the sphere by crossing it at a point, which is mapped one-to-one on a point on the plane. The stereographic projection is bijective and smooth with the exception of the single point of projection. It preserves neither distances nor areas. Its major characteristic though is that it preserves oriented angles between any two paths on the sphere, hence it is not only isogonal but also *conformal*. It is precisely this characteristic emerging out of the metaphora from the sphere to the plane through the stereographic projection that provides the crucial insight on what geometrically qualifies the complex structure emanating from the imaginary ring, as we are going to clarify below. At this point, if we think of this projection as the encoding bridge from the sphere to the plane, it is just as important to consider the decoding bridge from the plane to the sphere.





The main observation is that the further out on the plane a point is, the closer its inverse image point on the sphere is to the North pole. But no point on the plane has as its inverse image the North pole itself. Rather, as a sequence of points move out towards infinity on the plane, their inverse images tend towards the North pole on the sphere. Therefore in the setting of this metaphora, the notion of infinity on the plane entrains the North pole of the sphere as its inverse image, so that there is a continuous one-to-one correspondence between the plane together with infinity and the sphere. In this sense, the sphere is homeomorphic topologically with the compactification of the plane emerging by the addition of a virtual point, called the ideal point at infinity.

2.10 STEREOGRAPHY: THE CONFORMAL QUALITY OF COMPLEX STRUCTURE

Next, the basic concept that we intend to pursue is that the conformal quality of the stereographic projection is equivalent to endowing both the plane and the sphere with a complex structure. For this purpose, it is worth pondering in more detail on the conformal character of this projection. The crucial aspect is that the projection preserves on the plane the angles at which paths on the sphere cross each other, and more precisely the angles at a crossing point between the tangent vectors of these paths at the crossing point. On the other side, the stereographic projection does not preserve area, i.e. the area of a region on the sphere is not generally equal to the area of its stereographic projection onto the plane. A precise geometric way to understand this phenomenon can be expressed by means of the notion of intrinsic Gaussian curvature.

More concretely, since the sphere and the plane have different intrinsic curvatures, there cannot exist a projection from the sphere to the plane that preserves both oriented angles and areas, since in that case, the curvature would be preserved. Therefore, a projection from the sphere to the plane can be either conformal or area-preserving, but not both simultaneously. According to the preceding the stereographic projection is only conformal. In consequence, circles on the sphere that do not pass through the North pole, i.e. the locus of projection, are projected to circles on the plane, whereas circles on the sphere that *do* pass through the North pole are projected to straight lines on the plane. Equivalently, these lines may be thought of as circles through the virtual point at infinity, or as circles of infinite radius. Inversely, all lines on the plane being transformed to circles on the sphere by the inverse of the stereographic projection meet at the North pole. In particular, parallel lines, which do not intersect on the plane, are transformed to circles, tangent at the North pole, whereas intersecting lines are transformed to circles intersecting transversally at two points on the sphere, one of which is the North pole. The *loxodromes* on the sphere, by which we mean the paths of constant compass bearing on the sphere, or the paths having invariant angle with the corresponding parallels of latitude and meridians of longitude, project onto paths intersecting radial lines on the plane in an equiangular way, i.e. they project onto logarithmic spirals on the polar grid.

The naturality of the complex differentiable structure on the sphere emerges as follows through the stereographic projection: We notice that although the stereographic projection from a single point on the sphere to the plane fails to map this single point of projection from the sphere to the plane, nevertheless if we consider two simultaneous projections from different points of the sphere to the plane, e.g. the first one from the North pole and the second from the South pole of the sphere, then the entire sphere can be mapped conformally on two copies of the plane. The first one may be thought of as tangent to the South pole, while the second as tangent to the North pole of the sphere. Clearly this double stereographic projection contains redundant information about the sphere.

The idea is that each copy of the plane is a local patch of the sphere, which actually covers the whole sphere with the exception of the projection point, such that each patch is identified with the inverse image of the corresponding projection. In this manner, it is evident that two distinct but overlapping patches afford a complete covering of the sphere in their descriptive terms. The implicated metaphora of structure from the plane to the sphere through these two locally covering patches is that they exchange information compatibly about the sphere, or else they are compatible on their overlaps. Each copy of the plane is endowed with a complex structure induced by the imaginary ring in two distinct ways.

For simplicity, we may identify both copies of the plane with the equatorial plane of the sphere and induce the complex structure in the first case by the complex parametrization $\alpha = W + iV$, while the second is given by the complex parametrization $\beta = W - iV$. Then, a transition map from one patch to the other, i.e. from the α-parameterized copy of the complex plane to the β-parameterized copy of the complex plane, both identified with the equatorial plane of the sphere, asserts how these two copies are glued together by restriction to their overlapping regions. The gluing takes place by the identification of each non-zero complex number α of the first copy with the non-zero complex number $\frac{1}{\beta}$ of the second copy, and conversely. In this sense, what plays the role of origin in the α-parameterized copy, assumes the role of infinity in the second β-parameterized copy, equally and conversely. The important thing is that the transition maps so-defined from one complex patch to the other are holomorphic maps. This means that a transition map as specified above is complex differentiable on its domain of definition.

Thus, we obtain a holomorphic atlas on the sphere endowing it with a complex differentiable structure, i.e. the sphere becomes a one-dimensional complex manifold, called the Riemann sphere.

It remains to show the equivalence of the *complex* structure on the sphere with the *conformal* characterization of the sphere emanating from the stereographic projection. For this purpose, let us denote the complex structure by J, such that analogously with the action of the imaginary unit on the complex plane, it rotates vectors in each tangent plane at a point of the sphere by $\pi/2$. Then, if we consider the stereographic projection from the North pole of the sphere, denoted by P, the conformal characterization in complex differentiable terms with respect to a tangent vector X at a point of the sphere, amounts to the prescription that $dP(JX) = idP(X)$, i.e. dP is a complex linear map.

Equivalently, this informs us that the conformal projection map is a non-degenerate *holomorphic* map; a complex differentiable map such that the differential never becomes zero. Henceforth, the complex structure J on the sphere is equivalent to the conformal characterization of the sphere by the stereographic projection, in the precise sense that the differential map dP is complex linear. This implies a further equivalence: The operation of first rotating a tangent vector at a point of the sphere by J followed by the operation of pushing forward this vector from the sphere to the plane by dP is indistinguishable from the operation of first pushing forward the tangent vector from the sphere to the plane by dP followed by the operation of rotating the latter by the imaginary unit i. In other words, these two distinct operations commute, such that on the complex plane of projection $dP(JX) = idP(X)$, called the *Cauchy-Riemann* equation.

Thus, the complex structure on the sphere making it a complex manifold is equivalent to the conformal characterization of the sphere via the stereographic projection, which in turn, is algebraically equivalent to the commutativity of the operations of imaginary rotation and pushing forward by the differential of the projection with respect to the plane of this projection. Note that if P is the stereographic projection from the North pole of the sphere, then dP is the complex linear map from the tangent plane at a point of the sphere to the plane of projection to be thought of as pushing forward tangent vectors from the former to the latter.

It is important now to stress that the metaphora induced by the stereographic projection of the sphere should not be considered as independent from the original framework of harmonics and canonics that made this same metaphora necessary in the first place. This means that the actual depth of the stereographic projection, together with the threefold correspondence established previously, can be properly appreciated from a temporal standpoint, rather than a spatial one. The objective is to think of the sphere in temporal or chronological terms so as to achieve an insight on the temporal status of the projection plane. What really matters is the fact that the projection plane is capable of representing the sphere completely by inverse stereographic projection, if and only if we employ two distinct complex-parameterized copies of it, standing for projection planes with respect to the North and South pole of the sphere correspondingly, and so long as they are amalgamated compatibly -continuously and holomorphically- together, on their overlapping regions. With these conditions, the projection plane, identified with the equatorial plane of the sphere, binds together antipodally the northern with the southern hemisphere, such that the imaginary ring in the patch corresponding to the projection from the North pole is glued together antipodally with the imaginary ring in the patch corresponding to the projection from the South pole, and both are identified with the equator of the sphere.

Thus, in view of the doubly articulated stereographic projection, the projection disk bounded by the equator of the sphere with the complex structure, i.e. in its function as an imaginary ring, is conceptualized temporally as an epiphaneia of the present, which binds together what is included inside the ring with what is outside the ring by means of inversion with respect to this ring. This is geometrically termed *circle inversion*, meaning geometric inversion with respect to the equator in our context, which is accomplished by means of complex inversion followed by reflection on the W-axis, where the latter is simply complex conjugation. Taking into account that complex inversion is actually physically implemented by a π-rotation about the W-axis, we conclude that what appears on the epiphaneia of the present is a binding of a point inside the projection disk with a point outside it obtained from the first via a π-rotation about the W-axis followed by reflection in the W-axis.

There are now two interrelated issues that we have to elucidate in order to make viable the sought after temporal interpretation. The first issue targets the nature of the above binding; what is the physical process corresponding to the shadow of this binding that is accomplished on the epiphaneia of the projection disk bounded by the imaginary ring of the equator of the sphere? The second issue targets the nature of what we call the present in the context of the temporal interpretation of the stereographic projection. It turns out that these two issues are not independent of each other. We just have to focus carefully on the domain of harmonics in order to make sense of the implicated geometry on the epiphaneia of the disk.

What is actually involved is the helicoid together with its mirror image as constituting a harmonic entity. We recall that the helicoid unfolds continuously by parallel translation, orthogonally and away from the epiphaneia, and after half a period of rotation a mirror image of the same helicoidal surface arises. In this way, we think of the helicoid together with its mirror image as helical waves propagating in opposite directions such that the mirror image is the reflection of the first. This is possible if these helical waves are bounded from above and below for a temporal length of one period so as to give rise to a helical standing wave. Equivalently, within this bounded interval the helical wave is in unison with its reflection, its mirror image. Being in unison means that they are consonant in the fundamental harmonic frequency corresponding to the frequency ratio $1:1$, pertaining to an angular temporal interval of one whole period 2π. Therefore, the pertinent kind of binding appearing on the epiphaneia of the disk expresses the harmonic resonances of the helical standing wave in question, whence the bounding of this standing wave is provided by the sphere in terms of the temporal length of its periods of rotation.

The above leads to a novel meaning in relation to the notion of the present. A single moment of time is resolvable homeotically by the whole harmonic series capable of being instantiated within the posed bounds at this moment. Moreover, the harmonics persist diachronically, that is for each conceivable present, being able to induce any harmonic ratio in that present. In this way the harmonics are qualified as powers for the actualization of consonances in each present. Consequently, the diachronic, harmonically persistent present manifesting on the epiphaneia, constituted by the invariants of the helical standing wave homeorhesis, that is the points of homeostasis depicted by the harmonics, is identified with the bounding imaginary ring, which in turn is the equator of the sphere endowed with the complex structure by virtue of the conformality of the stereographic metaphora. Hence, in the context of the present, the harmonics appear through the complex roots of unity of the imaginary ring equator.

2.12 *ARCHIMEDEAN* SPIRAL: *METAPHORA* FROM THE CIRCULAR TO THE LINEAR

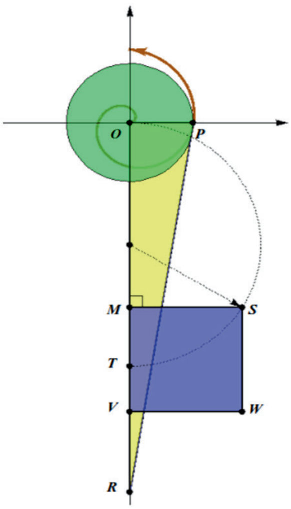
There are two fundamental questions that have to be addressed in the preceding framework. The first concerns the clarification of the precise manner that the sphere bounds the helical standing wave. This is fundamental because it elucidates the temporal interpretation of the sphere in terms of its periods of rotation. The second concerns the problematic of how the present can be thought of as the imaginary axis of the complex plane in its rectangular manifestation in relation to a possible conformal projection of the sphere derived from the stereographic projection that manifests in the polar grid of the projection plane.

Both of these questions can be dealt with in a satisfactory manner if we pay attention to the significance of the major problem of ancient Greek mathematical enquiry, i.e. the problem of squaring the circle. We view this problem as a problem of natural communication, and for this reason the method proposed by *Archimedes* to address it bears great significance. We call this method Archimedes' metaphora because the Archimedean method does not supply a constructible solution to this problem by straightedge and compass. Instead, Archimedes having realized that there does not exist a constructible solution, he invents a metaphora from the circular to the linear domain. It is this metaphora that deserves a proper emphasis and appreciation.

The problem of squaring the circle, refers to the instantiation of a square that has the same area as that of the disk bounded by a circle. In the first stage, Archimedes considers an isomorphic problem. Namely, the problem of geometrically unfolding the perimeter of a circle to a linear length. This more fundamental problem conceptually can be cast isomorphic to the original as follows: If the geometric unfolding of the perimeter of a circle to a linear length is possible, then the area of a circle can be made equivalent to the area of an orthogonal triangle whose sides are given by the radius of the circle and the perimeter of the circle.

The main problem arises from the irrationality of π, which is actually a transcendental number. For every conceivable circle of some radius, π is an invariant characterizing the perimeter through the radius. The incommensurability of the circular domain with the linear domain is precisely captured by the irrationality of π. In the "Measurement of the Circle" Archimedes devised an ingenious approximation to the perimeter of the circle involving the method of exhaustion by means of inscribed and superscribed polygons. This is in effect to march toward the perimeter both from inside and outside using polygonal approximations involving up to 96 sides. In relation to the pertinent problem of squaring the circle, Archimedes devised the means of metaphora from the circular domain to the linear domain in terms of the Archimedean *spiral*. In other words, the Archimedean spiral is introduced relationally with respect to these incommensurable domain; in our terms as a bridge of metaphora from the circular to the linear and inversely.

The spiral is conceived in physical terms by Archimedes. He considers a point particle, located initially at the centre of the circle, which starts to move uniformly from the centre to the periphery of the circle along the radius. Simultaneously, Archimedes considers that the radius rotates uniformly counterclockwise around the centre of the circle. Thus, the particle movers according to the composition of these two uniform motions, the first linear, and the second circular. The composition of these two uniform motions is a non-uniform motion, which describes the trajectory of the considered particle. It is this trajectory that bears the geometric form of the Archimedean spiral. The spiral is devised as a means of metaphora from the circular to the linear domain, more precisely, as a means of geometric linear unfolding of the perimeter of the circle into a measurable linear length. This is accomplished by realizing the tangent to the spiral after one turn, i.e. at the point where it intersects the circle after one turn. Archimedes showed that the tangent line to the spiral at this point crosses the vertical axis at a point whose distance from the origin is exactly $2\pi r$, where r is the radius of the circle. As a result, the tangent to the spiral at the point of its intersection with the circle corresponding to a 2π rotation, accomplishes the required unfolding of the perimeter into a linear length, which is provided by the distance of the point of intersection of this tangent with the vertical axis from the origin. The important thing to notice is that the recording of this linear length corresponds to the time needed by the particle to complete one turn of its spiral trajectory; to all intent the perimeter of the circle is unfolded as a temporal length. This form of temporal unfolding is periodic and can be analogously recorded for all higher turns of the spiral. Note also that the radius of the spiral at each point of the trajectory of the particle is determined by the angle with respect to the horizontal axis.



According to the above, the perimeter of the circle is unfolded into the linear length $2\pi r$, which is recorded at the vertical axis as the vertical side of an orthogonal triangle whose horizontal side is the radius r of the circle. Then, the area of this triangle is half of the area of the parallelogram having the same sides, which is clearly $2\pi r^2$. Thus, the area of the circle is the same as the area of the above triangle, i.e. πr^2.

2.13 HARMONIC RESOLUTION OF TIME: IMAGINARY IMPRINT OF THE *POLYSTROPHIC* SPIRAL

It is elucidating to attempt an interpretation of Archimedes's method in the terms of the imaginary ring. For simplicity, we may consider the circle as the unit circle in the complex plane. Then, the two-dimensional Archimedean spiral unfolds the unit circle into the imaginary axis by the above procedure. The question is how we qualify the imaginary axis in this setting. There are two stages to address this issue. The first refers to the conception of the imaginary axis as a spatialized temporal dimension through the intervention of the imaginary unit. Accordingly, the spatialized temporal length corresponding to the time needed for the completion of one turn of the spiral becomes imaginary, and thus, negative if squared. The second stage takes into account the periodicity that is implicit in the successive turns of the spiral. More precisely, Archimedes' method allows the unfolding of the perimeter of the circle multiple times recorded by the turns of the spiral, which means the Archimedean spiral is *polystrophic* and not only *monostrophic*. This fact forces the conception of time in this setting as a helix in three dimensions unfolding orthogonally to the complex pane, and which is projected epimorphically on the imaginary ring, i.e. on the unit circle on the complex plane, endowed with the polar grid via Euler's coordinatization. This is nothing else than the exponential group homomorphism $\exp : \mathbb{R} \to S^1$, whose kernel is $2\pi\mathbb{Z}$. Thus, topologically the winding number counts the integer number of turns around the origin, which is excluded from the complex plane.

If we consider Archimedes' spiral as the means of metaphora from the circular to the linear domain, according to the preceding, the counterclockwise oriented spiral is the encoding bridge, whereas the inversely oriented clockwise spiral is the decoding bridge. Thus, we may invoke harmonic considerations in our setting, as pertaining to the helical conception of time, by means of a helix and its mirror image, if bounded appropriately. The leading idea again is that a single moment of time, identified with the present of the imaginary ring shadow, can be resolved homeotically by the whole spectrum of harmonics, giving a precise meaning to the polystrophic quality of the Archimedean spiral in the two-dimensional projection.

The effect of this manoeuvre through the domain of harmonics is that we can now qualify the integers as a quantum spectrum of frequencies on the imaginary axis of the complex plane via branches of the complex logarithm. We emphasize that this is viable due to the fact that both the complex exponential and its local inversion in terms of a branch of the complex logarithm are conformal, meaning they preserve oriented angles. In light of this fact, the homeotic criterion of identity can be imprinted on the imaginary axis in terms of the angular temporal interval of one whole period 2π times the harmonic series. Consequent on the periodic resolution of time in terms of the harmonics, time and frequency become reciprocally correlated, and represented orthogonally to each other. Time in the form of the *helix* in three dimensions unfolds orthogonally to the complex plane. The helix considered together with its mirror image give rise to a helical standing wave bounded by temporal intervals of integer periods. The latter projects down to the complex plane with the origin removed on an annular strip of the polar grid. Applying a corresponding branch of the complex logarithm transforms this strip conformally to a rectangular region on the complex plane. The imaginary axis is marked in this way by the harmonic frequencies corresponding to the integer number of cycles per unit of time, where the latter is taken to correspond to the temporal length of one whole period 2π.

Henceforth, it is instructive to note that the bounding in the complex analytic setting takes place via logarithmic branch cutting, which stems from the fact that no global complex logarithm function exists by which to invert the complex exponential function. Notwithstanding this fact the restriction to branches preserves the conformal character mapping annular strips to rectangular regions on the complex plane and inversely. As a side remark, we come to an understanding of the finite topological coverings of the circle by itself corresponding to all different integer powers as emanating from the universal unfolding of the circle by the helix, considered together with the above described process of bounding that reveals the harmonics. Therefore, the topological winding number physically descends from the harmonics, which are qualified as powers for the actualization of consonances, i.e. harmonic ratios. Finally, it is the action of the logarithm through its single-valued branches that transforms these ratios into spatialized spectral intervals measured on the imaginary axis, as outlined above.

2.14 HARMONICS OF THE *ARCHIMEDEAN* SCREW AND THE EQUIAREAL PROJECTION

An adequate mechanical model that captures all of the above aspects is the Archimedean screw in three-dimensional space. If we consider a finite portion of the screw, then the projection of the screw onto the plane thought of as perpendicular to the central axis of the screw, depicts an annular strip of the polar grid on the complex plane with the origin removed. Again, if we apply the complex logarithm this strip is transformed conformally to a rectangular region on the complex plane. We may now imagine the bounding of such a finite portion of the screw by an open cylinder.

This is particularly elucidating in relation to Archimedes's method of determination of the surface of a sphere on the basis of his method of unfolding the perimeter of the circle into a linear length. We recall that the latter leads to the conclusion that the area of a circle is equal to the area of an orthogonal triangle whose sides are equal to the radius of the circle and the perimeter of the circle respectively. Since the area of the sphere is $4\pi r^2$, this area is the same as the area of four circles of the same radius r, or equivalently, the same as the area of four orthogonal triangles fitting compatibly together, whose big side is $2\pi r$ and small side is r, where r is identified with the radius of the sphere.

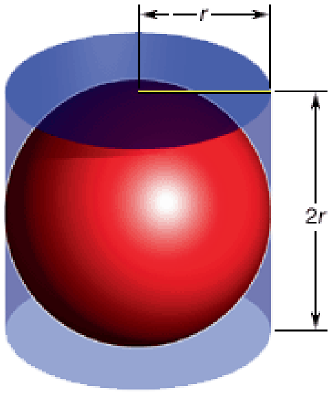
Necessarily these circles should be considered as great circles passing through the North and South pole of the sphere, so as their radius is the same as the radius of the sphere. The unfolding of any such circle into a linear length equals the equatorial length of the sphere, or the length of the perimeter of the equator, given by $2\pi r$. Thus, all four orthogonal triangles should have a big side equal to the equatorial length of the sphere $2\pi r$ and small side equal to the radius of the sphere.

It follows directly that these four triangles fit together in a plane region divided in two halves by the horizontal equatorial line of length $2\pi r$, such that the small side of each triangle r equals the vertical side of a half of this plane region. Conclusively, this plane region has a horizontal side equal to the equatorial length of the sphere $2\pi r$ and a vertical side equal to $2r$. Each horizontally conceived half divided by the equatorial line has sides $2\pi r$ and r respectively. In each half there fit two orthogonal triangles sharing the same diagonal of sides $2\pi r$ and r respectively. Thus, the area of the sphere equals the area of these four orthogonal triangles, each one of which equals πr^2.

In order to obtain a proper insight on the above Archimedean method of determining the area of a sphere it is significant to realize what it implies. Precisely, it implies that there exists an *equiareal* *projection* of a sphere onto a cylinder, which we call the Archimedean projection. Note that this projection of the sphere is not conformal, as is the stereographic projection, but it does preserve areas. It is realized as a horizontal radial projection emerging by placing the sphere within an open cylinder touching it along the equator. In this sense, the Archimedean projection should be considered as complementary to the stereographic projection of the sphere onto a plane.

Topologically, we may easily see that if we cut the sphere along a meridian passing through both the North and South pole of the sphere, we unwrap the sphere onto an open cylinder of height $2r$. The area of this cylinder equals the area of a plane region on which it can roll for the temporal length corresponding to one rotation, $2\pi r$, identified with the equatorial length of the sphere.

Hence, the sphere excluding its North and South pole can be projected in an area-preserving manner onto a open cylinder, and vice versa. Rolling the cylinder as above, we obtain an equiareal projection of the sphere on a planar region whose horizontal side is the equatorial length of the sphere $2\pi r$ and whose vertical side is $2r$, together with a rectangular and straight weaving grid of meridians and parallels.



The inverse equiareal projection from the open cylinder to the sphere is very important in relation to the first of the previously posed questions concerning the clarification of the precise manner that the sphere bounds a helical standing wave. It was conceivable simply as a finite portion of an Archimedean screw, characterized by its harmonics. As we mentioned before, this is fundamental because it elucidates the temporal interpretation of the sphere in terms of its periods of rotation, recorded eventually in terms of spectral lengths on the imaginary axis of the complex plane due to the Archimedean linear unfolding of the equatorial circle of the sphere. Since the latter bounds the epiphaneia of the present by the conformal stereographic projection of the sphere onto the plane, the inverse equiareal projection reveals the resolution of the present into the pertinent harmonics according to the bounding role of the sphere. For these reasons, all bearing on the posited temporal interpretation, the conformal stereographic projection of the sphere onto the plane cannot be considered independently from the equiareal Archimedean projection of the sphere onto the cylinder unrolling onto another plane.

2.15 TEMPORAL DIASTASIS: SYNTHESIS OF CONFORMAL WITH EQUIAREAL *METAPHORA*

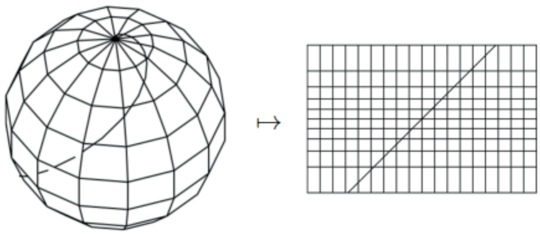
We argued that the present manifesting on the epiphaneia is expressed by the equator of the sphere, endowed with the complex structure by virtue of the conformality of the stereographic metaphora, and thus identified with the bounding imaginary ring of the horizontal disk of this projection. If we think in terms of Leon Battista Alberti's veil metaphora of *Renaissance perspectivism*, then the veil is the epiphaneia of the metaphora, whereas the eye corresponds to the projection point, i.e. either the North pole or the South pole in the case of the stereographic projection. If we identify the veil with the equatorial disk of the sphere, the interesting thing is that the sphere, which plays the role of the scene in this metaphora, lies both in front and behind the veil.

We recall that the pertinent kind of binding appearing on the epiphaneia of the disk descends from the harmonic domain; it expresses the harmonic resonances of a helical standing wave, whence the bounding of this standing wave is provided by the sphere in terms of the temporal length of its periods of rotation. It is precisely this fact that is encoded in the Archimedean equiareal projection of the sphere onto the cylinder, according to the preceding.

We emphasize that the consequence of this synthesis combining the stereographic with the Archimedean metaphora from the sphere to the plane and inversely, implies that the present is resolvable homeotically by the whole harmonic series capable of being instantiated within the pertinent temporal spherical bounds. Additionally, since the harmonics persist diachronically, that is, for each conceivable present, they are qualified as powers for harmonic resonance in each present.

Hence, in the context of the present, the harmonics appear through the complex roots of unity on the imaginary ring equator. The diachronic persistence of the harmonics, elevating them to invariants from a homology-theoretic topological viewpoint, has the effect that they appear as points of *stasis* geometrically. In this sense, the temporal helical axis of unfolding perpendicular to the epiphaneia bears the meaning of a temporal *diastasis*. As we have shown, any appropriate finite bounded portion of this diastasis can be transfigured conformally, by means of a branch of the complex logarithm on the imaginary axis of the complex plane equipped with the rectangular grid, as a spatialized spectral interval. This settles completely the second question concerning the problematic of how the present can be thought of in terms of the imaginary axis of the complex plane in its rectangular manifestation.

A significant observation regarding the complementary roles of the stereographic and Archimedean projections of the sphere is that the achieved synthesis actually pertains to any conformal and equiareal projection of the sphere. We should nevertheless recognize that a projection of the sphere on the plane that is both conformal and equiareal is not possible due to the *curvature* of the sphere. Reciprocally, the geometric characteristic of the curvature can be explicated from the synthesis of these two complementary types of metaphora, which are both indispensable in order to derive the notion of the present introduced here, e.g. from the temporal interpretation of these projections. For instance, we may consider another conformal projection of the sphere on the plane different from the stereographic one, but bearing the same oriented angle-preserving character, called the *Mercator projection*.



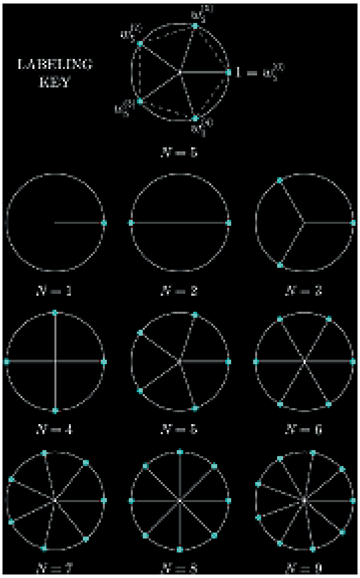
In this projection parallels of latitude on the sphere correspond to horizontal lines and meridians of longitude to vertical lines on a rectangular grid. The importance of this projection is that loxodromes on the sphere correspond to straight lines, making this projection very useful in navigation. Note that in the context of the stereographic projection loxodromes appear as logarithmic spirals whose center is either the North or the South pole. Loxodromes are not defined at the poles of the sphere, but they spiral from one pole to the other. They may be thought of as winding around each pole an infinite number of times as they approach it, but the distance they cover is finite. The Mercator projection is a conformal projection that maps the unit sphere within rectangular strips of width 2π excluding the North and South pole of the sphere, such that loxodromes appear as straight lines.

Using both the Mercator and stereographic projections we can easily realize the conformal character of the complex exponential and the complex logarithm. For this purpose, first we consider the inverse Mercator projection from the plane to the sphere, and then the stereographic projection from the sphere to the plane. Both of them are conformal, thus their composition is conformal as well. This composition can be expressed in terms of the complex exponential, which maps a rectangular strip of width 2π to an annulus on the complex plane with the origin removed, since the pole is excluded from the Mercator projection.

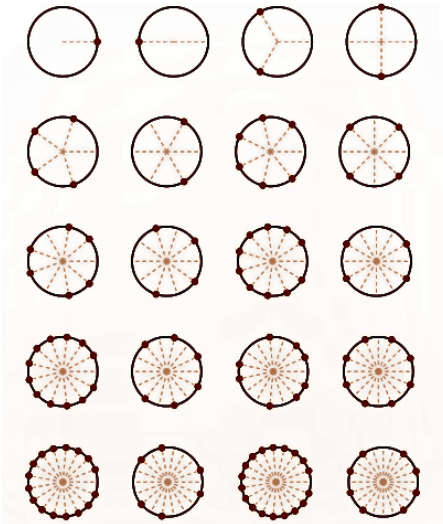
This argument shows that the complex exponential is a conformal map through the considered conformal projections of the sphere. The same argument can be used in reverse to show that the complex logarithm, through its principal branch for instance, is also conformal from the plane of the stereographic projection to the plane of the Mercator projection. With this approach, our basic conclusions pertaining to the synthesis of any conformal and equiareal projection of the sphere, thought of as complementary types of metaphora from the sphere to the plane, are confirmed for the proposed temporal interpretation of these projections.

2.16 CYCLOTOMY: COMPLEX ROOTS OF UNITY ON THE IMAGINARY RING

We recall that since the harmonics persist diachronically they bear the status of powers for harmonic resonance in each present. Therefore, in the context of the present, the harmonics appear through the complex roots of unity on the imaginary ring equator. There are always n different complex n-th roots of unity, that is, complex numbers whose n-th power is equal to unity, equally spaced around the perimeter of the unit circle in the complex plane. Since, they are equally spaced they constitute a well-tempered scale on the epiphaneia of the present. Roots of unity are manifested geometrically as the vertices of a regular polygon that connects them together.



Of particular importance are the primitive roots of unity. More precisely, on the unit circle with n equally spaced rays, there is now a mark on ray k, denoting a primitive root of unity, if and only if k and n are relatively prime, having no common divisors other than 1.



An equally-tempered scale marking the unit circle leads to *cyclotomy* and is manifested geometrically on the epiphaneia in terms of regular polygons inscribed in the unit circle. We consider now the temporal helical axis of unfolding perpendicular to the epiphaneia bearing the meaning of a temporal diastasis, as above. It is important to examine the means of subdivision of this diastasis.

For this purpose, it suffices to adopt a topological standpoint and consider the integer winding numbers of this helical diastasis evenly covering the circle on the epiphaneia. We recall that finite bounded portions of this diastasis qualify these windings in terms of harmonics, corresponding to finite covering spaces of the circle, and expressed as powers in the complex analytic setting. First, we point out that the square power in relation to the unit circle corresponds to doubling the angle, and so on for all higher integer powers. Let us consider the finite double covering of the circle by the circle. This corresponds spectrally to doubling the frequency, and thus by inversion dividing the unit circle into half. Similarly, if we consider the finite triple covering, it corresponds to tripling the frequency, thereby dividing the circle into three parts.

We treat all higher integer powers analogously, and by inversion, that is in terms of the roots of unity, we are able to subdivide the circle. The cyclotomy corresponds spectrally to the generation of regular polygons. The deeper the resolution of the cyclotomy is, the higher the number of vertices appearing equi-distantly on the unit circle, giving an ever-higher number of sides of the inscribed regular polygon. In this manner, the harmonic subdivision of the unfolding helical diastasis, is manifested geometrically as regular polygons inscribed in the circle. The further this subdivision is pursued by ascending to higher harmonics, resolving the circle in a more refined way, the higher the number of polygonal sides inside the circle.

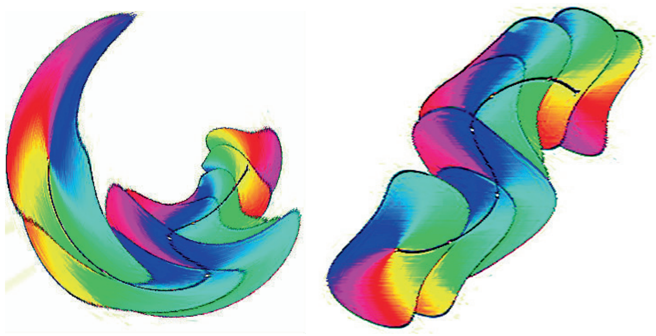
2.17 SHEAVES: *METAPHORA* FROM DESIGN TO ARCHITECTURE AND COLUMN CANONICS

The previous analysis provides an ideal starting point in order to think of the relation between design and architecture. The basic idea is that design descends from the domain of harmonics, whereas architecture descends from the domain of geometry. In this light, they can be characterized as reciprocally related to each other. Equivalently, there exists a metaphora from design to architecture, and vice versa, which should be though of as a metaphora from the domain of harmonics to the domain of geometry, as mediated through the topological level. From this perspective, the notion of a purely geometric design as well as its antipode, i.e. the notion of a purely harmonic architecture appear as degenerate conceptions that ignore the metaphora from the one to the other. Our distinction will be elaborated, in particular, by considering the fundamental notion of a *column* in terms of the proposed metaphora.

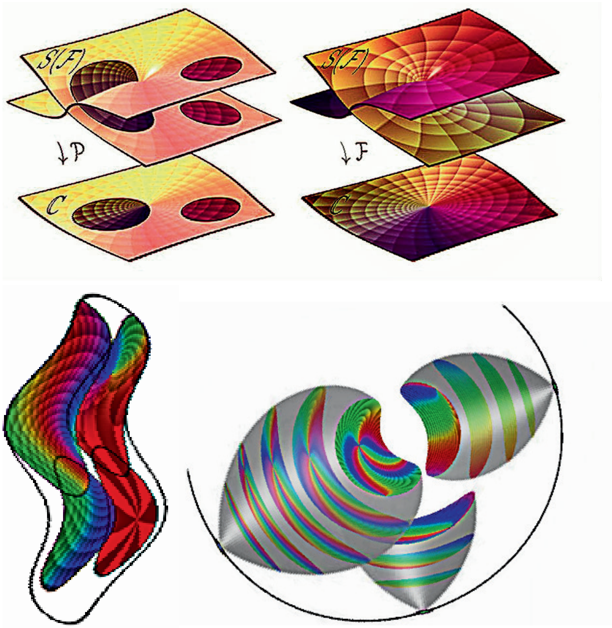
For this purpose, it is preferable to start working at the topological domain, considered as the mediating level between harmonics and geometry. The basic topological distinction is the part-whole, or more concretely, the local-global distinction. In the case of design, the global is envisioned in its totality, and the local parts are organized in a way to fulfill the global. The basic constraint of design is tantamount to the restriction of the local parts by the global. Thus, the local parts are passive, and the global is active restricting the local parts appropriately. Reciprocally, in architecture the local parts are selected in the form of atomic elements, which have to be jointly organized together geometrically so as to open up a new space.

In this sense, the basic constraint of architecture is how a local part can be extended by joining it together with another compatible local part, and so on toward the global. In consequence, in the latter case, the local parts are active, and the global is passive. Of course, in order that the opening up of a new space becomes possible, the local parts cannot be assembled randomly together. Rather their assemblage in space should conform to some pre-conceived global vision topologically, which allows the opening up of such a new space. In short, what is required is a metaphora from the domain of design to the domain of architecture.

If we characterize topologically the architectural assemblage that achieves the opening up of a new space as a sheaf, then the natural communication scheme between these domains, requires that at the level of design this assemblage bears the character of a *presheaf*. The difference between these notions is that in the case of the presheaf only the *global-to-local* compatibility is required, whereas in the case of the *sheaf*, the reverse *local-to-global* compatibility must also hold. Note that the first does not necessarily imply the second.



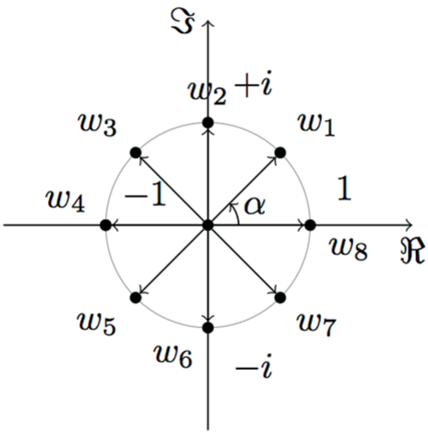
The idea that design, as conceptualized above, descends from harmonics is based on the fact that design is characterized by finality, or better, by *entelecheia*, since it carries its final purpose implicitly within itself according to the Aristotelian conception. Reciprocally, architecture is characterized by elementarity, and *efficient causality* in the organization of the local parts towards the global, subject to the constraints of geometry. Thus, architecture, although operating within the domain of geometry, implicating simple connectivity, is capable of opening up a new space only if the atomic elements are able to achieve a certain consonance with the whole, which in turn, presupposes the envisioning of such a whole: in other words the metaphora from design to architecture.



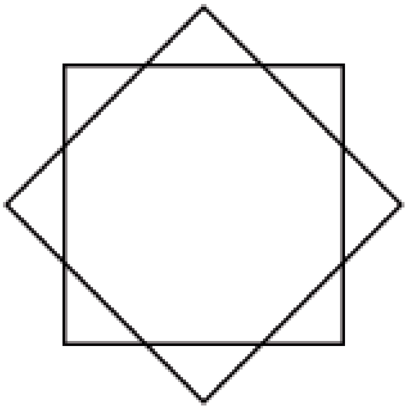
In this sense, the action of design may be though of as the action of a resonator that accomplishes the harmonic resonance of the actively envisioned global with the local through powers or spectral frequencies. Also in this sense, the assemblage of the active atomic elements geometrically should comply with the spectral compatibility of these elements in order that the opening up of a new geometric space in agreement with the harmonically envisioned global becomes feasible.

Note that beyond the topological level, in architecture, the active local parts can be joined together by admissible geometric transformations, in particular by translation and rotation, which in three-dimensional space can combine together in the form of a screw, a simply-connected finite portion of a helicoid. In the case of design, the envisioned global may be thought of harmonically as bearing the characteristic of the imaginary ring. We emphasize that this is not a *visual*, but a *harmonic* characterization, which means that the active global in design is not an already formed geometric entity, since this would contradict the essence of the metaphora. The active global in its harmonic conception should be able to restrict the suitability of the local parts only on the condition of harmonic resonance with them at each present, i.e. diachronically. This condition can be met only if the local parts can be qualified as complex roots of unity, which they synchronize as a polygonal totality with the global via powers up to a certain depth of spectral frequential resolution.

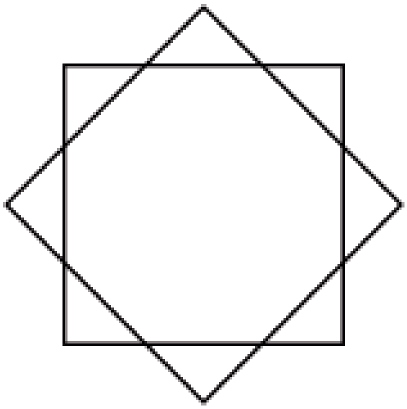
As a particular example of the above metaphora from design to architecture, together with their topological characterization in terms of the notions of a presheaf and a sheaf respectively, we consider the case of a column. We are going to examine the active global in design through the 8-th complex roots of unity, displayed below.

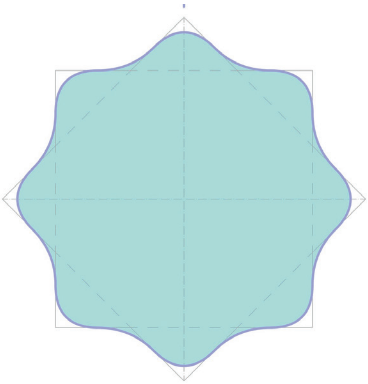


The basic idea is that the active local parts in this architectural implementation of the harmonics instantiated by design synchronize as a polygon of eight sides with the global via corresponding powers up to the depth of this spectral resolution. These active local parts can be amalgamated by the admissible geometric transformations of translation and rotation, which combine to a helicoidal screw in three dimensions. If we assume that these atomic elements admit the geometric form of graphs of primitive roots of unity, expressed as polygons, then in the considered case of the 8-th complex roots of unity, we instantiate a geometric template consisting of the superposition of a square together with another tilted square, together comprising an octagon.

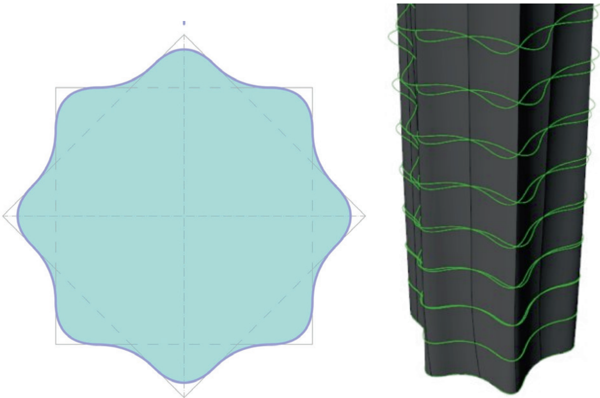


In this manner, the active local parts in architecture give rise to a geometric template. We consider the screw motion of this template along the temporal diastasis orthogonally to the superscribing imaginary ring marked by the 8-th complex roots of unity. Since we view the template as a superposition of two squares, it is instructive to consider the geometric template as an 8-star, whose sharp nodes may be rounded. The idea is that this star constitutes the basis of a column. Note that the 8-star template as a constellation is amenable to a screw motion with respect to two orientations; it can ascend according to the counterclockwise orientation and descend back according to the clockwise orientation, thereby providing the *ichnography* of a helical standing wave comprised of the harmonics under consideration.

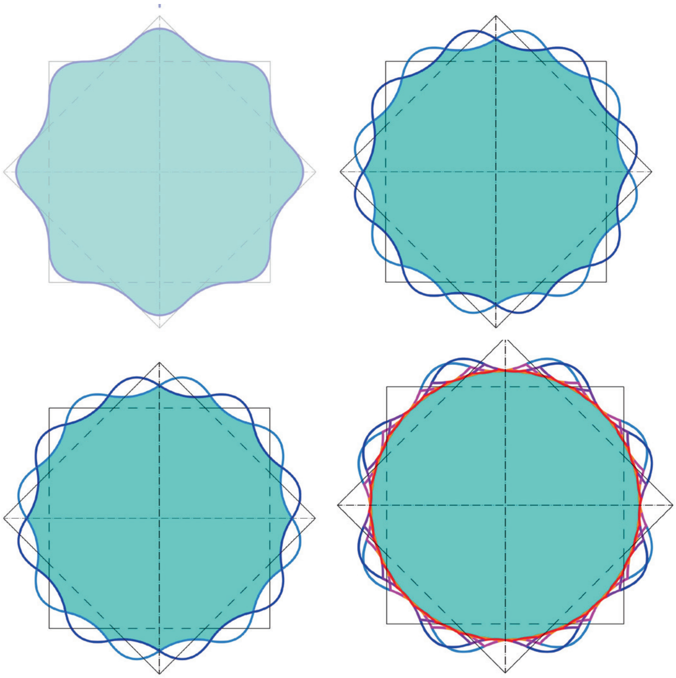




Therefore, the double screw motion of the 8-star template in three dimensions gives rise to a weaving pattern of two oppositely oriented helicoids, which topologically they overlap compatibly on the harmonics. Thus, they comprise a sheaf that admits the precise geometric manifestation of a part of a column.



Note that the above realized part of a column bears 8 striations. Clearly, it emerges from a quite low spectral resolution of the global imaginary ring involving only eight roots. We bear in mind that a pretty faithful approximation, according to the original construction of Archimedes, would involve 96 striations. Thus, either we increase the resolution implementing more and more striations, all at once, following geometrically the same pattern of double screw motion, or we treat the above realized part of a column as a branch of the whole column.



The latter is possible, since the whole column comprises a sheaf, and thus, we may legitimately consider its amalgamation through joining together distinct branches compatibly. We emphasize that joining branches together furnishes a resembling a tree. The basic notion is that when different logarithmic branches, constituted according to the preceding, join together, the number of roots doubles at each joint section. This branching tree-like structure of a column is iterated until the global section approximates a circle, completing in this sense the metaphora from design to architecture.