Conventions and Notation

In a definition or when a word is defined in the text, the concept defined is italicized. Italics in the running text is also used for emphasis. The definition of a word, phrase, or symbol is to be understood as an "if and only if" statement. Lower-case letters such as x denote vectors, upper-case letters such as A denote matrices, upper-case script letters such as $\mathcal{S}$ denote sets, and lower-case Greek letters such as α denote scalars; however, there are a few exceptions to this convention. The notation $\mathcal{S}_1 \subset \mathcal{S}_2$ means that $\mathcal{S}_1$ is a proper subset of $\mathcal{S}_2$, whereas $\mathcal{S}_1 \subseteq \mathcal{S}_2$ means that either $\mathcal{S}_1$ is a proper subset of $\mathcal{S}_2$ or $\mathcal{S}_1$ is equal to $\mathcal{S}_2$. Throughout the book we use two basic types of mathematical statements, namely, *existential* and *universal* statements. An existential statement has the form: there exists $x \in \mathcal{X}$ such that a certain condition C is satisfied; whereas a universal statement has the form: condition C holds for all $x \in \mathcal{X}$. For universal statements we often omit the words "for all" and write: condition C holds, $x \in \mathcal{X}$. The notation used in this book is fairly standard. The reader is urged to glance at the notation below before starting to read the book.

$\mathbb{Z}$	set of integers
$\overline{\mathbb{Z}}_+$, $\mathbb{Z}_+$, $\overline{\mathbb{Z}}_-$, $\mathbb{Z}_-$	set of nonnegative, positive, nonpositive, negative integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^{n\times m}$	set of $n \times m$ real matrices
$\mathbb{R}^n$	$\mathbb{R}^{n\times 1}$ (real column vectors)
$\overline{\mathbb{R}}_+$, $\mathbb{R}_+$, $\overline{\mathbb{R}}_-$, $\mathbb{R}_-$	set of nonnegative, positive, nonpositive, negative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^{n\times m}$	set of $n \times m$ complex matrices
$\mathbb{C}^n$	$\mathbb{C}^{n\times 1}$ (complex column vectors)
$\overline{\mathbb{C}}_+$, $\mathbb{C}_+$, $\overline{\mathbb{C}}_-$, $\mathbb{C}_-$	set of complex numbers with nonnegative, positive, nonpositive, negative real parts
$\mathbb{F}$, $\mathbb{F}^n$, $\mathbb{F}^{n\times m}$	$\mathbb{R}$ or $\mathbb{C}$, $\mathbb{R}^n$ or $\mathbb{C}^n$, $\mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$
CLHP, OLHP	closed, open left half plane

CRHP, ORHP	closed, open right half plane
$\jmath$	$\sqrt{-1}$
$\jmath\mathbb{R}$	imaginary numbers
$\mathrm{Re}\, z$, $\mathrm{Im}\, z$	real part; imaginary part of a complex number z
$\triangleq$	equals by definition
$\emptyset$	empty set
$\{\ \}$, $\{\ \}_{\mathrm{m}}$	set, multiset
$\cup$, $\cap$	union, intersection
$\in$, $\notin$	is an element of, is not an element of
$\subseteq$, $\subset$	is a subset of, is a proper subset of
$\to$	approaches
0, $0_{n\times m}$, 0_n	zero matrix, $n\times m$ zero matrix, $0_{n\times n}$
I_n, I	$n\times n$ identity matrix
$\mathcal{R}(A)$, $\mathcal{N}(A)$	range space of A, null space of A
x_i, $x_{(i)}$	ith component of vector $x\in\mathbb{R}^n$
$A_{(i,j)}$	(i,j) entry of A
$\mathrm{col}_i(A)$, $\mathrm{row}_i(A)$	ith column of A, ith row of A
$\mathrm{diag}[A_{(1,1)},\ldots,A_{(n,n)}]$	diagonal matrix $\begin{bmatrix} A_{(1,1)} & & 0 \\ & \ddots & \\ 0 & & A_{(n,n)} \end{bmatrix}$
$\text{block–diag}[A_1,\ldots,A_k]$	block-diagonal matrix $\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$, $A_i\in\mathbb{R}^{n_i\times m_i}$, $i=1,\ldots,k$
A^{T}	transpose of A
$\bar{A}$	complex conjugate of A
A^*	$\bar{A}^{\mathrm{T}}$
A^{-1}	inverse of A
$A^{\dagger}$	Moore-Penrose generalized inverse of A
$A^{\#}$	group generalized inverse of A
$A^{-\mathrm{T}}$, A^{-*}	$(A^{\mathrm{T}})^{-1}$, $(A^*)^{-1}$
$\mathrm{tr}\, A$	trace of A
$\det A$	determinant of A
$\mathrm{rank}\, A$	rank of A
$\mathbb{S}^n$	set of $n\times n$ symmetric matrices
$\mathbb{N}^n$	set of $n\times n$ nonnegative-definite matrices
$\mathbb{P}^n$	set of $n\times n$ positive-definite matrices
$A\geq\geq 0$ $(A>>0)$	$A_{(i,j)}\geq 0$ $(A_{(i,j)}>0)$ for all i and j
$A\geq\geq B$ $(A>>B)$	$A_{(i,j)}\geq B_{(i,j)}$ $(A_{(i,j)}>B_{(i,j)})$, where A and B are matrices with identical dimensions
$A\geq 0$ $(A>0)$	nonnegative (respectively, positive) definite matrix; that is, symmetric matrix with

	nonnegative (respectively, positive) eigenvalues
$A \geq B$	$A - B \in \mathbb{N}^n$
$A > B$	$A - B \in \mathbb{P}^n$
$\mathbb{R}^n_+$, $\overline{\mathbb{R}}^n_+$	$\{x \in \mathbb{R}^n : x >> 0\}$, $\{x \in \mathbb{R}^n : x \geq\geq 0\}$
$\otimes$, $\oplus$	Kronecker product, Kronecker sum
$x^{[k]}$	$x \otimes \cdots \otimes x$ (k times)
$\overset{k}{\oplus} A$	$A \oplus A \oplus \cdots \oplus A$ (k times)
$\mathcal{N}^{(k,n)}$	$\{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geq 0,\ x \in \mathbb{R}^n\}$
vec	column-stacking operator
spec(A)	spectrum of A including multiplicity
$\rho(A)$	spectral radius of A
$\alpha(A)$	spectral abscissa of A
$\lvert\alpha\rvert$	absolute value of α
$\sigma_i(A)$	ith singular value of A
$\sigma_{\min}(A)$, $\sigma_{\max}(A)$	minimum, maximum singular value of A
$\lVert\cdot\rVert$, $\vert\vert\vert\cdot\vert\vert\vert$	vector or matrix norm, vector or matrix operator norm
$\lVert x\rVert_2$	Euclidean norm of x $(= \sqrt{x^*x})$
$\lVert x\rVert_p$	Hölder vector norms, $[\sum_{i=1}^n \lvert x_i\rvert^p]^{1/p}$, $1 \leq p < \infty$
$\lVert x\rVert_\infty$	$\max_i \lvert x_{(i)}\rvert$
$\lVert A\rVert_p$	Hölder matrix norms, $\left[\sum_{i=1}^m \sum_{j=1}^n \lvert A_{(i,j)}\rvert^p\right]^{1/p}$, $1 \leq p < \infty$
$\lVert A\rVert_\infty$	$\max_{i,j} \lvert A_{(i,j)}\rvert$
$\lVert A\rVert_{\sigma p}$	$[\sum_{i=1}^r \sigma_i^p(A)]^{1/p}$, $1 \leq p < \infty$, $r = \operatorname{rank} A$
$\lVert A\rVert_{\sigma\infty}$	$\sigma_{\max}(A)$
$\lVert A\rVert_{\mathrm{s}}$	spectral norm of A $(= \sigma_{\max}(A))$
$\lVert A\rVert_{\mathrm{F}}$	Frobenius matrix norm of A $(= (\mathrm{tr}\ AA^*)^{1/2})$
$\lVert A\rVert_{q,p}$	induced matrix norm
$\lambda_i(A)$	ith eigenvalue of $A \in \mathbb{R}^{n\times n}$
$\lambda_{\min}(A), \lambda_{\max}(A)$	minimum, maximum eigenvalues of the Hermitian matrix A
He A, Sh A	$\frac{1}{2}(A + A^*)$, $\frac{1}{2}(A - A^*)$
$E_{(i,j)}$	elementary matrix with unity in the (i,j) entry and zeros elsewhere
$\log_e$	logarithm with base $e = 2.71828\cdots$
e_i	vector with unity in the ith component and zeros elsewhere
$\mathbf{e}$	$[1,\ 1,\ \ldots,\ 1]^{\mathrm{T}}$
$\mathcal{L}_p$	Lebesgue space, $1 \leq p \leq \infty$
$\mathcal{L}_2$	space of square-integrable Lebesgue measurable functions on $[0,\infty)$
$\mathcal{L}_\infty$	space of bounded Lebesgue measurable functions

	on $[0,\infty)$
$\lVert f \rVert_{p,q}$	$\{\int_0^\infty \lVert f(t)\rVert_q^p \mathrm{d}t\}^{1/p}$, $1 \le p < \infty$
$\lVert f \rVert_{\infty,q}$	$\operatorname{ess\,sup}_{t\ge 0} \lVert f(t)\rVert_q$
$\langle f, g\rangle$	$\int_0^\infty f^{\mathrm{T}}(t) g(t)\mathrm{d}t$
ℓ_p	sequence space, $1 \le p \le \infty$
ℓ_2	space of square-summable sequences on $\overline{\mathbb{Z}}_+$
ℓ_∞	space of bounded sequences on $\overline{\mathbb{Z}}_+$
$\mathcal{H}_p$	analytic function space
$\mathcal{H}_2$	Hardy space of real-rational transfer function matrices square-integrable on the imaginary axis (unit disk) with analytic continuation in the right half plane (outside the unit disk)
$\mathcal{H}_\infty$	Hardy space of real-rational transfer function matrices bounded on the imaginary axis (unit disk) with analytic continuation in the right half plane (outside the unit disk)
$\Re\mathcal{H}_2$	real-rational subspace of $\mathcal{H}_2$
$\Re\mathcal{H}_\infty$	real-rational subspace of $\mathcal{H}_\infty$
$[a,b]$	closed interval
(a,b)	open interval
$\mathcal{X}\times\mathcal{Y}$	Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$
$f:\mathcal{X}\to\mathcal{Y}$	function f with domain $\mathcal{X}$ and codomain $\mathcal{Y}$
$\frac{\partial f}{\partial x_i}(x_0)$	partial derivative of f with respect to x_i at x_0
$f'(x_0)$	Fréchet derivative of f at x_0
$f^{(k)}(x_0)$	kth Fréchet derivative of f at x_0
$D^+f(x_0)$	upper right Dini derivative of f at x_0
$D_+f(x_0)$	lower right Dini derivative of f at x_0
$f^{-1}(\mathcal{D})$	inverse image of the set $\mathcal{D}$
$f_2\circ f_1$	composition of two functions; $(f_2\circ f_1)(\cdot) = f_2(f_1(\cdot))$
$\mathcal{L}[z(t)]$	Laplace transform of $z(\cdot)$
$G(s) \sim \left[\begin{array}{c\|c} A & B \\ \hline C & D\end{array}\right]$	state space realization of transfer function $G(s) = C(sI-A)^{-1}B + D$
$G(s) \overset{\min}{\sim} \left[\begin{array}{c\|c} A & B \\ \hline C & D\end{array}\right]$	minimal state space realization of $G(s)$
$\mathcal{B}_\varepsilon(\alpha)$	$\{x\in\mathbb{R}^n : \lVert x-\alpha\rVert < \varepsilon\}$
$\mathcal{B}_\varepsilon[\alpha]$	$\{x\in\mathbb{R}^n : \lVert x-\alpha\rVert \le \varepsilon\}$
$\mathcal{X}\backslash\mathcal{Y}$	$\{x\in\mathcal{X} :\ x\notin\mathcal{Y}\}$ for sets $\mathcal{X}$ and $\mathcal{Y}$
$\partial\mathcal{S}$	boundary of the set $\mathcal{S}$
$\overset{\circ}{\mathcal{S}}$	interior of the set $\mathcal{S}$
$\overline{\mathcal{S}}$	closure of the set $\mathcal{S}$
$\mathcal{S}^{\mathrm{c}}$ or $\mathcal{S}^{\sim}$	complement of the set $\mathcal{S}$

inf	infimum; greatest lower bound
sup	supremum; least upper bound
$\liminf_{n\to\infty} f(x_n)$	limit inferior of $f(x_n)$; $\liminf_{n\to\infty} f(x_n) = \sup_n \inf_{k\geq n} f(x_k)$
$\limsup_{n\to\infty} f(x_n)$	limit superior of $f(x_n)$; $\limsup_{n\to\infty} f(x_n) = \inf_n \sup_{k\geq n} f(x_k)$
min, max	minimum, maximum
C^0	continuous functions
C^r	functions with r-continuous derivatives
C^∞	infinitely differentiable functions
$\mathcal{C}[a, b]$	space of continuous functions
$\mathbb{E}$	expectation
a.e.	almost everywhere
$\triangle$	end of example
$\square$	quod erat demonstrandum or end of proof