Preface

Imagine a subject that is so pervasive that almost all of us see some of it during our high school education, one whose story goes back well into ancient times. This subject crosses most major cultures and places; indeed, it is not easy to identify societies that contributed significantly to science without using it in some way. It might even be argued that this discipline was the link that brought geometric models of the cosmos together with numerical computation in a synthesis that allowed theory to be converted into prediction: the birth of the exact sciences. All this makes it hard to believe that trigonometry has never been given a proper book-length historical treatment in English.

Trigonometry is particularly interesting because it is rare among mathematical subjects: it was originally motivated almost entirely by outside concerns, in this case, astronomy and geography. Many theorems that we now call trigonometric had been known before in other mathematical contexts; others came to our awareness only through the need to solve particular scientific problems. Thus trigonometry often retooled existing mathematics for new purposes, but partly through its interface with the sciences it blossomed into an independent discipline, providing structures and ways of thinking that eventually returned to inspire mathematics itself.

The most recent effort to write the history of trigonometry in a Western language was over a century ago, with Anton von Braunmühl's two-volume classic *Vorlesungen über Geschichte der Mathematik*.[1] The *Vorlesungen* was an outstanding example of the best work of its time: rigorous and thorough, yet concise and clear. However, most of the professional discipline of the history of mathematics has happened since it was written, and after one hundred years it is well past time to revisit the topic. In no area is this more true than in the non-Western cultures most relevant to us, India and Islam. The great nineteenth-century historians of mathematics, I am sure, would be astounded and impressed to see what manuscripts and discoveries have come to light in these fields.

I am proud to say that this book owes much to the efforts of the historians of von Braunmühl's generation. It is fashionable to criticize them these days, and not without cause. Current historiographic trends emphasize

[1] [von Braunmühl 1900/1903]. If one considers Russian to be a Western language, then there is also the notable [Matvievskaya 1990].

cultural contextualization of mathematical history,[2] and looking back, it is clear that our predecessors too often brought their nineteenth-century contexts into their interpretations. It is part of the job of the historian to attempt to disengage from the modern context, and to try to live the past the way the past lived it. Surely this task is quixotic, but there is much to be gained in the perpetual struggle to shed our own skins. Nevertheless, our success in this is elusive, and really can only be evaluated after much time has passed. I wonder whether twenty-second-century historians will judge us as harshly as we judge the historians of one hundred years ago.

What I find admirable about the great historians who did so much to establish the foundation of our field is their unremitting attention to rigor and reliability, and their commitment to both the history and the mathematics. We may sometimes question their historicity, but we can trust their honest attempts to face up to the material. They did their best to place their findings in an appropriate context, but they did not shy away from the mathematics itself. One cannot genuinely practice the history of a scientific subject without also living and breathing the science itself. Von Braunmühl knew the mathematics of which he wrote, and I have done my best to live up to that heritage.

This is not to say that I will stick slavishly to the style of the past masters. The value of placing results in their original context is well taken; one of my efforts in this direction is to include a number of extracts from (translations of) the original texts in the narrative. This should give the reader the chance to experience directly what the texts say, and to judge how they should be read. Explanations are provided to help decode passages that would otherwise be obscure. These should also help to illustrate that the modern mathematical notation used throughout this book is a transformation of the original texts; hopefully the inclusion of excerpts will allow the reader to decide how great this transformation is.[3] I hope that some of the excerpts will prove useful in the classroom: most of them stand on their own, and

[2] The paper that is sometimes said to have sparked the current trend of searching for and destroying modern mathematical conceptions in Greek mathematics is Sabetei Unguru's "On the need to rewrite the history of Greek mathematics" [Unguru 1975/76]; see also [Saito 1998] for an update. The issue of appropriate contextualization is felt particularly strongly in the study of the most ancient mathematical cultures; see [Imhausen 2003b] on Egypt, and [Robson 1999] and [Robson 2001] on Mesopotamia.

[3] The degree to which the arguments in historical mathematical texts should be translated to modern notation is a matter of current debate. For instance, many ancient and medieval texts will refer to "the rectangle between *AB* and *CD*." The arguments often require merely the product of the lengths of the line segments, but writing it as such changes the mental process by removing the geometric content, possibly changing our perception of what the author thought in some significant way. In this book—a scholarly survey, but not a research paper—I have tried to respect the original sources as much as possible while making the arguments accessible to the modern reader.

some might help to enliven trigonometric or astronomical concepts by providing historical context.

So, the reader will find here some of the flavor of an “episodes” book; it is not necessary to read it from cover to cover. Nevertheless there is enough of a flow of ideas that the completist reader should also be satisfied.

The question arises who I intend to read this book. I would be delighted if the interested lay public finds it to be useful, and I hope that they are given the chance. However, my first loyalty is scholarly: I aim to provide a solid, reference-supported account of the best understanding of the history of trigonometry that the academic community can provide. Hence the hundreds of footnotes: for those readers who wish to delve into the issues in more depth, the notes contain appropriate links to the literature, much of the back story, and details on the mathematics. However, I have been careful to keep scholarly debates out of the main text. My goal is to give the historical characters the stage. Nevertheless, this is a mathematical rather than a social history. There are plenty of other places to go to find out about the lives and contexts of the scientists. Instead, I have chosen to emphasize the mathematical arguments. Many of them are beautiful; most are practical. They are the heart of trigonometry, and I will not apologize for asking the reader to experience their pleasures.

The practical nature of trigonometry is a distinctive feature compared to most other mathematical disciplines and is worth highlighting, especially in this first of two volumes. Through the sixteenth century, trigonometry existed mostly (often exclusively) to serve the needs of mathematical astronomers. Without astronomy, there wouldn’t have been much point. Thus the reader should expect to encounter at least the basics of spherical and planetary astronomy within these pages. I have done my best to avoid requiring any outside astronomical reading, pleasurable though that might be. The brief opening chapter on the ancient heavens should convey almost all the spherical astronomy that is needed to make sense of the book.

The dependence of trigonometry on outside sources of inspiration informs other distinctive choices I have made that might surprise some. Following the theme of contextualization, I focus on topics that the historical practitioners—not necessarily the modern scholars—found interesting and useful. Since these practitioners were astronomers, they were often less interested in harmonious theory than in practical results. So, in addition to the trigonometric identities and theorems, I have gone into some depth for instance on the methods that were developed to compute trigonometric tables. And in addition to some beautiful direct solutions to astronomical problems, I also discuss the numerical methods that were applied when geometry was not up to the task. Some modern readers find a pleasing aesthetic quality in the geometry and consider table-making a thing of the past, but this is after all a work of history. The past is my business.

I do find occasion to apologize in one respect. When considering what to include in a survey of this scope, the author must eventually draw a line in the sand. This leads inevitably to regret that some valuable topics missed the cut. In my case this includes trigonometric traces that may be found outside of my major cultural groupings—in particular, Jewish, Byzantine, and Chinese. In the case of China the story is episodic early and much happens well after the end date of this volume, so it will be treated in the sequel. There are also certain topics that I wish had received a proper scholarly treatment before the writing of this book (the role of the analemma in Islam comes to mind). However, this will always be the case, and eventually one has to put pen to paper.

How to Read This Book

I will be grateful to the enterprising reader who starts on page one and works faithfully to the end; there is reward in following a tale to its conclusion. But I do not expect everyone to have the same zeal that I do for my subject. Many people will use this book as a reference, concentrating on one or two of the major chapters at a time, or even dipping into individual applications that look intriguing. I have done my best to accommodate this, but there is a problem: a certain depth of astronomical understanding is required for a number of the episodes in this book. I cannot repeat the same astronomical introduction again and again. So, the text itself contains only brief explanations of the astronomy involved in each particular problem. If you find the text too terse, I have included a preliminary chapter which orients newcomers to the ancient heavens. That chapter should give all the context that is needed.

Some formatting details to keep in mind while reading: throughout the time periods covered in this volume trigonometric functions were defined not in the unit circle, but in circles of different radii. These are notated by capitalizing the first letter of the function ("Sin" rather than "sin"); the value of the radius R should be indicated somewhere nearby. Secondly, many of the calculations reported here were performed in the sexagesimal (base 60) system. Numbers are represented as follows: for example,

$$7,8;12,34,56 = 7 \cdot 60 + 8 + \tfrac{12}{60} + \tfrac{34}{60^2} + \tfrac{56}{60^3}.$$

Sometimes the whole parts of numbers are given in decimal format; for instance, the number above can be written as 428;12,34,56. Finally, errors in trigonometric values are given in units of the last place. For instance, if the value given in the historical text is 12;34,56 while the correct value is 12;34,59, the error is displayed as [-3].

I originally intended to cover the entire history of trigonometry in one book, but soon realized that task to be overwhelming. This volume is book-ended, more or less, by the developments of the Earth-centered and the Sun-centered universe. Trigonometry proper began with the origins of the Ptolemaic geocentric system, and a nice breaking point is found with the beginning of Copernicus's heliocentric model. From this point trigonometry began to move with the rest of mathematics toward symbolic algebra and (eventually) analysis, losing some of its geometric character and picking up a more familiar modern flavor. As the medieval Islamic scientists often wrote in their conclusions: there is much more to say, and I hope some day with the will of God to write the necessary sequel. It took von Braunmühl three years to publish his second volume; I hope that readers will be more patient with me. It might take a little longer.

Reference Sources

This project would not have been possible without access to a number of literature databases, both modern and old-fashioned. The opportunity to acknowledge these sources does not present itself in the main text, so I would like to give them a nod here. I turned most frequently to the following:

- *MathSciNet* (the online version of *Mathematical Reviews*);
- *Zentralblatt für Mathematik*;
- Kenneth O. May's *Bibliography and Research Manual of the History of Mathematics* [May 1973];
- The *Dictionary of Scientific Biography* reference sections;
- Dauben and Lewis's *The History of Mathematics from Antiquity to the Present: A Selective Annotated Bibliography*, revised edition on CD-ROM [Dauben 2000];
- The Research Libraries Group's History of Science, Technology and Medicine Database;
- Rosenfeld and Ihsanoğlu's *Mathematicians, Astronomers & Other Scholars of Islamic Civilisation and Their Works (7th–19th c.)* [Rosenfeld/Ihsanoğlu 2003];
- A private literature database on Islamic astronomy maintained by Benno van Dalen.

Although I have found ways of coping with most languages relevant to this study, Russian was a problem and I wish I could have handled this literature more effectively. A good Russian reference source is the extensive bibliography in [Matvievskaya 1990].

Acknowledgments

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The Mathematics of the Heavens and the Earth