FOREWORD

Some Personal Reminiscences of Donald Coxeter

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It is a great honor to have my name linked with that of Donald Coxeter.

As a mathematics and physics student in the 1960s and 1970s, I often ran across the intriguing name H. S. M. Coxeter. I knew that this man's books were world famous, had heard that they were elegant and concise, and, on flipping through them once or twice, had even seen that they were filled with beautiful, enticing diagrams. But somehow, I had other things on my mind and I paid them little heed. When, decades later, I finally came under the spell of Coxeter's words, images, and ideas, I fell in love with geometry.

What eventually launched me on a collision course with geometry was a spectacular course on complex analysis that I took at Stanford University way back in 1962. This course was given by a young professor named Gordon Latta, who hailed from Toronto, the city in which English-born Donald Coxeter eventually settled. Latta, without doubt the best mathematics teacher I ever had, was extremely visual in his teaching, and he conveyed the depth and power of calculus in the two-dimensional arena of complex numbers in an inimitable fashion. One image from that course stuck with me for three decades—that of a circle turning the complex plane inside out, flipping the finite disk inside the perimeter into the infinite region outside the perimeter, and vice versa.

One fateful morning in 1992—thirty years after Latta's course—I woke up with that image of circular inversion in my head, for God knows what reason, and in particular with the vague memory that any circle outside the disk was carried, by this strange but lovely operation, into a circle inside the disk (and vice versa). This weird geometric fact, which I knew Latta must have proven, struck me as so marvelous that I immediately decided to try to prove it myself. Actually, I wasn't entirely sure that I was remembering the statement correctly, and this made my idea of proving it a little dicier. Indeed, my first attempt, rather ironically, showed that a random circle did *not* become another circle! However, my sense of mathematical aesthetics insisted that this statement had the ring of truth, and compelled me to try again. The second time around, I caught my dumb mistake (the center doesn't go to the center!) and proved that circles indeed remain circles when flipped inside out by circles.

This small but joyful excursion into inversion was the tiny spark that ignited a forest fire in my brain, and over the next few months, as geometric imagery started cramming my head fuller and fuller, I knew I needed an external guide. Where else to turn but to the person whose name for me was synonymous with the word "geometry"—H. S. M. Coxeter? I bought a copy of the thin volume he had written with Samuel Greitzer, called *Geometry Revisited*, and went through it from beginning to end, absorbing the ideas with passion. Some of them, as it happened, I had already invented on my own, but by far the majority were brand new to me and served as springboards for countless geometrical forays that I made over the next several years. Thanks to Coxeter and Greitzer, I was flawlessly launched on one of the richest and happiest explorations in my life.

Somewhere around six months into my geometrical odyssey, I used a chain of analogies to make a discovery that excited me greatly, and I wrote up the story of this discovery in a short essay. I wanted to find out if my discovery was new or old, so I decided to seek the reaction of a number of geometers whose books I admired. First and foremost was Donald Coxeter, and so I took the plunge and sent him my essay along with a cover letter. Not wishing to impose, I tried to be very brief (a mere ten pages!), but felt I at least had to tell him how much his book had meant to me. In a most cordial and prompt reply, he suggested I take a look at a couple of books he had written on projective geometry, and so, without hesitation, I purchased them both.

The older of the two was a concise opus entitled *The Real Projective Plane*, and I have to say that reading this was another dazzling revelation to me. As Coxeter points out in his preface, the restriction to the real plane in two dimensions makes it possible for every theorem to be illustrated by a diagram. And not only is this *possible*, but in the book it is *done*. By itself, this simple fact makes the book a gem. Moreover, Coxeter strictly adheres to the philosophy of proving geometric theorems using geometric methods, not using algebra. This means that a reader of *The Real Projective Plane* comes to understand projective geometry through the ideas that are natural to it, building up an intuition totally unlike the intuition that comes through formulas. I am not impugning what is called the *analytic* style of doing geometry; I am just saying that coming to understand projective geometry using the *synthetic* style was among the most gratifying mathematical experiences I have ever had. I will never forget the many nights I spent in bed reading Coxeter's monograph with only a tiny reading light perched on it (in fact, inside it), in order not to wake up my wife, who had nothing against my infatuation with geometry but who seemingly couldn't sleep a wink if even a single photon impinged on her eyelids.

I cannot resist quoting a sentence in the preface to *The Real Projective Plane*. It says this: "Chapter 10 introduces a revised axiom of continuity for the projective line, so simple that only eight words are needed for its enunciation." I think Donald Coxeter must have felt not only pleased but also proud as he wrote this down, because he was so in love with simplicity, elegance, and economy of means. Here is the eight-word definition to which he was referring: "Every monotonic sequence of points has a limit." What a delight! As you probably can tell, my copy of *The Real Projective Plane* is one of my most lovingly read and most prized possessions.

Speaking of doing geometry with a minimum of photons, I have to relate one of the most absurd and yet enriching geometrical experiences I have ever had. Somewhere in my many readings on geometry, I came across a vignette about a famous nineteenth-century German geometer—probably Steiner, Plücker, von Staudt, or Feuerbach—who was so suspicious of the insidious dangers supposedly lurking in diagrams that he insisted on teaching his students geometry in a pitch-dark room, using words and words alone to convey all the ideas. When I first read about this, I was nonplussed, thinking it to be among the silliest notions I had ever heard of. But perhaps precisely because it was so silly, this scene kept bouncing around in my head for a long time, and eventually, years later, when I myself was teaching a course on triangle geometry that often met at my house at night, I couldn't resist pulling down all the shades, turning off all the lights, and trying out this technique myself. The room became absolutely pitch dark, so dark that the students couldn't even see my arms move when I traced geometric shapes in the air. All they ever knew about were my spoken words, not my physical gestures. And what theorem did I prove to them in that darkest darkness of night? None other than the gleaming jewel known as Morley's theorem, which states that the "taboo" trisectors of the three angles of a random triangle join each other at the corners of an equilateral triangle floating somewhere inside the random triangle. Did they see it in their mind's eyes? I am sure they did! And what proof did I relate to my assembled students? Well, naturally, it was the one I had found in the pages of Coxeter and Greitzer's little volume and had made my own, although of course I had to adapt it to fit my brave new light-free, diagram-free circumstances.

This whole episode may seem like an exercise in utter craziness, but in retrospect, I don't think so. Quite the contrary, it was an unforgettable exercise in visualization without vision. One has to remember that some of the greatest of all mathematicians have been blind, and yet that didn't stop them from making astounding discoveries. I was reminded of this as I perused Coxeter's famous book *Introduction to Geometry*, chock-full of literary quotes (the index includes Aeschylus, Aristophanes, Plato, Shakespeare, Goethe, Lewis Carroll, H. G. Wells, Dorothy L. Sayers, and even Tom Sawyer), and found the following sentence, which he took from E. T. Bell's book *The Development of Mathematics*: "Euler overlooked nothing in the mathematics of his age, totally blind though he was for the last seventeen years of his life."

There is a vast difference, I feel, between having no diagrams before one's eyes and having no diagrams inside one's head. They are not the same thing at all; indeed, internal imagery is indispensable. For that reason, one of the most regrettable and baffling tendencies in the mathematics of the twentieth century was a mad stampede toward obliteration of the visual and even the visualizable. Donald Coxeter, however, as everything he wrote vividly demonstrates, was among the people who most systematically opposed this madness.

I will never forget how, at age fifteen or so, I came across the book *General Topology* by John L. Kelley. This austere volume, the first treatise I had ever seen on "rubber-sheet geometry," that mysteriously alluring branch of mathematics I thought was populated by Möbius strips and distorted doughnuts, did not, in its hundreds of pages, contain a single diagram; instead, it was filled with incredibly dense and prickly notation using all sorts of arcane symbols (many of which, I realized years later, stood for rather simple, bland words, but were used in their place for the dubious sake of maximal symbolic compression). Being young and naïve and in love with mathematics, and not yet having had the experience of struggling with it, I merely thought to myself, "Oh, so this is the kind of thing I will have mastered in just a few years! Won't that be wonderful!" I wasn't dismayed in the least by the prospect of reading long and picture-free works of mathematics, and writing such things myself; it struck me as a natural part of the process of reaching the mythical status known as "mathematical maturity."

Within a few years, however, I discovered that I personally could not survive in such an arid atmosphere. Diagrams (or at least mental imagery that could be thought of as personal, inner diagrams) were the oxygen of mathematics to me, and without them I would simply die. And thus, when the air of abstraction for abstraction's sake became too thin for me to breathe, I wound up with no choice but to bail out of graduate school in mathematics. It was a terrible trauma. If, at that crucial moment in my life, someone had suggested that before abandoning mathematics, I take a look at geometry, I might have discovered the works of Donald Coxeter and followed a very different pathway in life.

In 2000, several years after my correspondence with Donald Coxeter, I went to the University of Toronto to give two colloquia in the Physics Department. After the first (a talk describing the key role played by analogies in physics), a very thin and well-dressed elderly gentleman walked up and softly said to me that he was Donald Coxeter. You could have knocked me over with a feather. At the time, he was ninety-three years old! We walked out to an informal reception together and ate cookies and chatted for a little while. Mentally speaking, he was completely at the top of his game, and we talked in a lively fashion about the importance of analogies in both math and physics. I was deeply touched by his presence at my lecture.

But the capper came at my second physics colloquium. Just as I started speaking, I spied Donald Coxeter once again in the audience. And after I had finished, we once again met and chatted for a little while. This time, after we had touched on the family of geometries about which I had written to him some eight years earlier, the conversation somehow veered to the topic of Coxeter's vegetarianism and his incredible daily exercise program, which at that time he was still religiously following.

How honored I felt that this great man, this icon of twentieth-century mathematics, had come to hear me not just once but twice, and had presented himself to me as if he were an admirer of mine rather than the reverse. The logic was simply upside down. Moreover, here was someone who for almost his entire life had stuck to a moral principle that I, too, had found central: the sacredness of life, whether that of humans or that of "lower" creatures. Altogether, the message that came straight to me was that this was a human being entirely without pretension, the kind of person that I had grown up hearing described as a "mensch"—the best kind of person that exists. I had the privilege of meeting this marvelous mensch face-to-face on only those two occasions, but they remain indelibly imprinted on my mind.

This concludes my personal reminiscences of Donald Coxeter, but I would like to add a few words about Siobhan Roberts's book. I have never met Siobhan, but we have corresponded a little bit. What I know of her comes almost entirely from reading her words about Donald Coxeter, and what emerges loud and clear is that she understands the man's spirit very deeply. She understands what drove him, and she knows just how to put into words the fire that always inhabits a great mathematician's soul. I hope that Siobhan's book will bring to many people not only a sense for the beauty of mathematics itself, but also a sense for how the very human love of hidden patterns and symmetries can result in a hundred years of exultant exploration.