Preface

A dynamical system is mainly used to describe the state of the motion of dynamic systems. More precisely, it is used to study the law of the solutions to systems changing with time. When $t = 0, 1, 2, ...,$ the system is called to be discrete and when $t \in \mathbb{R}$, the system is called to be continuous. If the dynamical system is classified according to whether the nonlinearity of the system depends on time, it can be divided into two categories. When the nonlinearity is independent of time, the system is called to be autonomous and when the nonlinearity is dependent on time, the system is called to be non-autonomous. Generally speaking, non-autonomous systems are more difficult to study than autonomous systems because their nonlinearities are more complex.

For autonomous systems, the long-time behavior of solutions is generally studied by investigating global attractors. A global attractor possesses invariance under the corresponding semigroup, compactness, and attractiveness. To determine the existence of a global attractor based on semigroup theory, three factors need to be verified: the continuity of the semigroup, dissipativity (the existence of an absorbing set), and compactness in some sense. It has a complex structure and is composed of bounded orbits, including equilibrium points, periodic orbits, quasi-periodic orbits, fractal dimensions, etc. It may not be a smooth manifold and has non-integer dimensions. At present, there are many monographs and papers on global attractors (exponential attractors) (see [8, 60, 69, 113, 115, 124], etc.). For non-autonomous systems, the global attractors are no longer applicable, thus many scholars turn to study more complex attractors such as pullback attractors, uniform attractors, etc. (see [24, 27, 34]).

For the past decades, many scholars have mostly focused on some common Sobolev spaces to discuss attractors. For concepts and definitions about Sobolev spaces, please refer to [52]. However, in the last decade, scholars no longer confined their research spaces to the conventional Sobolev spaces and began to study the time-dependent attractors in the space related to time. They considered their

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problems in the time-dependent space $\mathcal{H}_t(\Omega)$, which is endowed with the norms $\| u\|^2_{\mathcal{H}_t} = \| u\|^2_2 + \varepsilon(t) \| \nabla u\|^2_2$, for any $t \in \mathbb{R}$. If the time-dependent term $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function and satisfies $\lim_{t\to+\infty} \varepsilon(t) = 0$, and there exists a constant $L > 0$ such that $\sup_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \le L$, then these assumptions make it possible to study the existence of the time-dependent global attractors in $\mathcal{H}_t(\Omega)$.

Since the norm of the time-dependent space contains time-dependent term $\varepsilon(t)$, it is easy to see that the study of attractors in this space is more complex than the study of usual attractors in Sobolev spaces without time-dependent terms. It is necessary to carry out more complex calculations on the absorbing sets and asymptotic compactness to obtain the desired results. Although it is difficult to calculate and analyze, the existence of a time-dependent term broadens the previous research framework, which enables us to carry out research in models that are closer to physical reality and promotes the research process on the long-time behavior or well-posedness of solutions to dynamical systems, which is of great significance.

In addition, inspired by some works on time-dependent global attractors, we shall adjust the assumptions on the time-dependent function $\varepsilon(t)$, such as re-assuming its lower bound and monotonicity, and shall obtain some new works on the existence and regularity of pullback attractors and upper semicontinuity of pullback attractors and global attractors in some function spaces similar to $\mathcal{H}_t(\Omega)$, which are new attempts (see [111, 112, 144]).

When the density of the fluid inside is not uniform, the substance inside the object will be diffused from the high concentration of the place to the low concentration of the phenomenon, which is a common phenomenon in life, resulting in the diffusion of gases, liquid penetration, the diffusion of impurities in semiconductor materials and other related issues, involving such as aviation, meteorology, marine, biology, materials, and many other fields. Compared with classical diffusion equations, non-classical diffusion equations cover a wider range of physical phenomena, and the relevant mathematical theories and applications of these models have attracted much attention from scientists, and have great important scientific and applied value.

This book aims to present some latest results on attractors for non-classical diffusion equations and Kirchhoff wave equations in time-dependent spaces, which are divided into five parts. Firstly, the first part includes chapter 1, in which we shall systematically survey the works till 2023 on the time-dependent global attractors, and strong attractors and pullback attractors in the time-dependent spaces. Then, the second part includes chapter 2, where time-dependent global attractors for a class of reaction–diffusion equations are considered. To be more specific, chapter 2 will state the long-time behavior of weak solutions to the non-classical diffusion equations with fading memory when the nonlinear term satisfies critical exponential growth in the framework of weighted time-dependent spaces. In addition, the third part includes chapters 3–7, in which the existence of strong attractors and pullback attractors for non-classical reaction–diffusion equations will be investigated in several different time-dependent spaces. Chapter 3 will cover the long-time behavior of solutions to the non-classical diffusion equation with fading memory when the nonlinear term fulfills the polynomial growth of arbitrary order. Chapter 4 will study the long-time behavior of solutions to the non-classical diffusion equations with nonlocal diffusion when the nonlinear term satisfies critical exponential growth. Chapter 5 will consider the existence and upper semicontinuity of pullback attractors for non-autonomous nonlocal diffusion equations with the nonlinear term containing a perturbation term. Chapter 6 will discuss the asymptotic behavior of solutions to non-autonomous diffusion equations with delay containing some hereditary characteristics and a nonlocal diffusion term, and chapter 7 will prove the existence and regularity of pullback attractors for non-classical non-autonomous diffusion equations with delay under a new framework. Then the fourth part includes chapter 8, in which we shall survey the existing works on Kirchhoff wave equations with strong damping in detail. Finally, chapter 9 is the last part, in which we shall prove the existence, regularity, and fractal dimension of global attractors for a Kirchhoff wave equation with a strong damping and a memory.

We sincerely wish that the readers will know the main ideas and essence of the basic theories and methods of attractors of several different reaction–diffusion equations and Kirchhoff wave equations in time-dependent spaces. We also wish that the readers can be stimulated by some ideas from this book and continue to undertake further study and research after having read this book.

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