Appendix C

Notations for point and crystallographic groups

The notation used for symmetry groups varies depending on the science domain and on the class of groups used in applications.

Point symmetry groups of three-dimensional space are the most widely used in different concrete applications. In fact, they are not the abstract groups but their representations in three-dimensional Euclidean space.

There are seven infinite families of groups and seven exceptional groups. The different notations for these groups are given in Table C.1.

We characterize shortly these groups here using Schoenflies notation.

The seven infinite series of point groups are:

C_n - group of order n generated by rotation over $2\pi/n$ around a given axis; $n = 1, 2, 3, \ldots$ C_1 is a trivial, "no symmetry" group. In the limit $n \to \infty$ we get the $C_\infty = SO(2)$ group.

S_{2n} - group of order $2n$ generated by rotation-reflection over π/n around a given axis; $n = 1, 2, \ldots$ For n-odd, the group S_{4k+2} is often noted as $C_{2k+1,i}$. i.e. as an extension of the C_{2k+1} group by inversion. In particular $S_2 = C_i$. In the limit $n \to \infty$ we get the $C_{\infty h}$ group.

C_{nh} - Group of order $2n$ obtained by extension of C_n by including reflection in plane orthogonal to the symmetry axis. $n = 1, 2, \ldots$ For $n = 1$, the notation $C_s \equiv C_{1h}$ is used. In the limit $n \to \infty$ we get the $C_{\infty h}$ group.

C_{nv} - Group of order $2n$ obtained by extension of C_n by adding reflection in plane including the symmetry axis. $n = 2, 3, \ldots$ In the limit $n \to \infty$ we get the $C_{\infty v} = O(2)$ group.

D_n - Group of order $2n$ obtained by extension of C_n by including symmetry axes of order two orthogonal to the C_n axis. $n = 2, 3, \ldots$ In the limit $n \to \infty$ we get the D_∞ group.

D_{nd} - Group of order $4n$ obtained by extension of D_n by including reflection in the symmetry plane containing the C_n axis, but not containing orthogonal C_2 axes. $n = 2, 3 \ldots$ In the limit $n \to \infty$ we get the $D_{\infty h}$ group.

TAB. C.1 – Different notations for point groups.

Schoenflies	C_n	S_{2n}	C_{nh}	C_{nv}	D_n	D_{nd}	D_{nh}
ITC (even n)	n	$\overline{(2n)}$	n/m	nmm	$n22$	$\overline{(2n)}2m$	n/mmm
ITC (odd n)	n	$\overline{n}$	$\overline{(2n)}$	nm	$n2$	$\overline{n}m$	$\overline{(2n)}2m$
Conway	nn	$n\times$	$n*$	$*nn$	$n22$	$2*n$	$*n22$
Schoenflies	T	T_d	T_h	O	O_h	I	I_h
ITC	23	$\overline{4}3m$	$m\overline{3}$	432	$m\overline{3}m$	235	$m\overline{3}\overline{5}$
Conway	332	$*332$	$3*2$	432	$*432$	532	$*532$

D_{nh} - Group of order $4n$ obtained by extension of D_n by including reflection in the symmetry plane orthogonal to the C_n axis and containing all orthogonal C_2 axes. $n = 2, 3\ldots$ In the limit $n \to \infty$ we get the $D_{\infty h}$ group.

The seven exceptional groups:

T - A group of order 12 contains all rotational symmetry operations of a regular tetrahedron.

T_d - Group of order 24. The symmetry group of a regular tetrahedron.

T_h - Group of order 24 obtained by extension of group T by adding an inversion symmetry operation.

O - A group of order 24 contains all rotational symmetry operations of a regular octahedron (or cube).

O_h - Group of order 48. The symmetry group of a regular octahedron (or cube).

I - A group of order 60 contains all rotational symmetry operations of a regular icosahedron (or dodecahedron).

I_h - Group of order 120. The symmetry group of a regular icosahedron (or dodecahedron).

C.1 Two-dimensional point groups

There are two families of finite two-dimensional point groups.

A C_n, group of order n, is generated by rotation over $2\pi/n$. $n = 1, 2, 3, \ldots$

Another family of groups is the extension of C_n by reflection in line passing through the rotation axis. There is no universal notation for groups in this family. D_n or C_{nv} notation is used because of obvious correspondence with notation for three-dimensional point groups.

Two continuous two-dimensional point groups $SO(2)$ and $O(2)$ can be described equally as C_∞ and D_∞ ($C_{\infty v}$) groups respectively.

C.2 Crystallographic plane and space groups

For the notation for two- and three-dimensional crystallographic groups we simply refer to the International Tables of Crystallography [14] or to any basic book on crystallography.

C.3 Notation for four-dimensional parallelohedra

We give here correspondence between different notations used for four-dimensional lattices. Delone was the first to give in 1929 a list of 51 combinatorial types of four-dimensional lattices. In [41] he gave figures of three-dimensional projections for all 51 found types, numerated consecutively by numbers from 1 to 51. For each of these 51 types he gave also numbers of facets of each type and in cases when several polytopes have the same numbers, he added information which allows us to make a distinction between different polytopes. In 1973 Shtogrin [87] found one combinatorial type missed by Delone. We give in Tables C.2 and C.3 characterization of all 52 types. Column "Delone" gives numbering used by Delone in [41] together with his description of the set of facets in the form used by Delone, namely: $(n_1)_{k_1} + (n_2)_{k_2} + \ldots$ where $(n_i)_{k_i}$ gives the number n_i of facets with k_i 2-faces. The combinatorial type discovered by Shtogrin is denoted as **St**.

In the tables we refer also to two types of notations used by Engel [11, 49, 53]. A short notation indicates the number of facets and uses consecutive numbers $1, 2, \ldots$ to label polytopes within the subset of polytopes with the same number of facets. The more detailed notation uses symbol $\mathbf{N_f}.\mathbf{N_v}$-$n_6$ where $\mathbf{N_f}$ is the number of facets, $\mathbf{N_v}$ is the number of vertices, and n_6 is the number of hexagonal 2-faces. When such labeling is insufficient, a full description uses 2-subordinate and 3-subordinate symbols $K_{\alpha_{n_\alpha}} \ldots$ giving numbers K_α of 2-faces with n_α edges in case of the 2-subordinate symbol and numbers K_α of 3-faces with n_α 2-faces in the case of 3-subordinate symbol.

For zonohedral polytopes we give also the notation used by Conway [32] and slightly different but essentially the same notation used by Deza and Grishukhin [44] (see column DG). For non-zonohedral polytopes we do not use Conway notation which is based on a rather different principle and give notation used by Deza and Grishukhin for a zonotope contribution $Z(U)$ which allows us to write a non-zonohedral polytope as a Minkowski sum of $P_{24} = \mathbf{24}.\mathbf{24}$-0 and a zonotope $Z(U)$.

TAB. C.2 – Combinatorial types of four-dimensional zonohedral lattices. Correspondence between notations.

m	Engel	Engel (full)	Delone	Conway	DG
10	30-2	30.120-60	1	K_5	K_5
		$4_{90}6_{60}$; $8_{20}14_{10}$	$10_{14}+20_8$		
9	30-1	30.102-36	19	$K_{3,3}$	$K^*_{3,3}$
		$4_{108}6_{36}$; $6_{12}12_{18}$	$18_{12}+12_6$		
	28-4	28.96-40	4	K_5-1	$K_5-\mathbf{1}$
		$4_{90}6_{40}$; $6_6 8_{12}12_6 14_4$	$4_{14}+6_{12}+12_8+6_6$		
8	24-16	24.72-26	6	K_5-2	$K_5-\mathbf{2}$
		$4_{76}6_{26}$; $6_8 8_{10}12_4 14_2$	$2_{14}+4_{12}+10_8+8_6$		
	26-8	26.78-24	5	K_5-1-1	$K_5-2\times\mathbf{1}$
		$4_{92}6_{24}$; $6_8 8_8 12_{10}$	$10_{12}+8_8+8_6$		
7	16-1	16.48-16	8	K_4+1	$K_4+\mathbf{1}$
		$4_{48}6_{16}$; $6_6 8_8 14_2$	$2_{14}+8_8+6_6$		
	20-3	20.54-16	10	K_5-3	$K_5-\mathbf{3}$
		$4_{64}6_{16}$; $6_8 8_8 12_4$	$4_{12}+8_8+8_6$		
	22-2	22.54-12	11	C_{2221}	C_{2221}
		$4_{72}6_{12}$; $6_{16}12_6$	$6_{12}+16_6$		
	24-12	24.60-12	7	K_5-2-1	$K_5-\mathbf{1}-\mathbf{2}$
		$4_{86}6_{12}$; $6_{10}8_8 12_6$	$6_{12}+8_8+10_6$		
6	12-1	12.36-12	16	C_3+C_3	C_3+C_3
		$4_{36}6_{12}$; 8_{12}	12_8		
	14-2	14.36-8	13	K_4	K_4
		$4_{44}6_8$; $6_8 8_4 12_2$	$2_{12}+4_8+8_6$		
	20-2	20.42-6	12	C_{321}	C_{321}
		$4_{66}6_6$; $6_{12}8_6 12_2$	$2_{12}+6_8+12_6$		
	22-1	22.46-0	9	C_{222}	C_{222}
		4_{84}; $6_{16}12_6$	$6_{12}+16_6$		
5	10-1	10.24-4	17	C_3+1+1	$C_3+2\times\mathbf{1}$
		$4_{30}6_4$; $6_6 8_4$	4_8+6_6		
	14-1	14.28-0	15	C_4+1	$C_4+\mathbf{1}$
		4_{48}; $6_{12}12_2$	$2_{12}+12_6$		
	20-1	20.30-0	14	C_5	C_5
		4_{60}; 6_{20}	20_6		
4	8-1	8.16.0	18	$1+1+1+1$	$4\times\mathbf{1}$
		4_{24}; 6_8	8_6		

TAB. C.3 – Combinatorial types of four-dimensional lattices obtained as a sum $P_{24}+Z(U)$ of 24-cell $P_{24} = \mathbf{24.24}$-0 and a zonotope $Z(U)$. Correspondence between notations for polytopes and for zonotope contribution to the sum.

m	Engel	Engel (full)	Delone	$Z(U)$, [DG]
10	30-3	30.120-42	2	$K_5 - \mathbf{1}$
		$4_{72}5_{36}6_{42}$; $6_6 8_2 10_{12} 12_6 14_4$	$4_{14}+6_{12}+12_{10}+2_8+6_6$	
	30-4	30.120-36	3	$K^*_{3,3}$
		$3_6 4_{54} 5_{54} 6_{36}$; $6_6 8_6 12_{18}$	$18_{12}+6_8+8_6$	
9	28-6	28.104-24	21	$K_5 - 2\times\mathbf{1}$
		$3_6 4_{52} 5_{54} 6_{24}$; $6_4 8_6 19_8 12_{10}$	$10_{12}+8_{10}+6_8+4_6$	
	28-5	28.104-30	20	$K_5 - \mathbf{2}$
		$4_{70} 5_{36} 6_{30}$; $6_4 8_4 10_{14} 12_4 14_2$	$2_{14}+4_{12}+14_{10}+4_8+4_5$	
8	26-9	26.88-12	24	$K_5 - \mathbf{1} - \mathbf{2}$
		$3_{12} 4_{38} 5_{60} 6_{12}$; $6_2 8_{10} 10_8 12_6$	$6_{12}+12_{10}+6_8+4_6$	
	26-10	26.88-18	22	$K_5 - \mathbf{3}$
		$3_6 4_{56} 5_{42} 6_{18}$; $6_2 8_8 10_{12} 12_4$	$4_{12}+12_{10}+8_8+2_6$	
	26-11	26.88-24	25	$K_4 + \mathbf{1}$
		$4_{74} 5_{24} 6_{24}$; $6_2 8_8 10_{14} 14_2$	$2_{14}+14_{10}+8_8+2_6$	
	28-3	28.94-12	24	$K_5 - \mathbf{1} - \mathbf{2}$
		$3_6 4_{60} 5_{54} 6_{12}$; $6_4 8_6 10_{12} 12_6$	$6_{12}+12_{10}+6_8+4_6$	
	28-2	28.94-18	23	C_{2221}
		$4_{78} 5_{36} 6_{18}$; $6_4 8_6 10_{12} 12_6$	$6_{12}+12_{10}+6_8+4_6$	
7	24-17	24.72-0	31	C_{222}
		$3_{24} 4_{12} 5_{72}$; $8_{18} 12_6$	$6_{12}+18_8$	
	24-18	24.72-12b	30	$C_{221} + \mathbf{1}$
		$3_{12} 4_{48} 5_{36} 6_{12}$; $8_{14} 10_8 12_2$	$2_{12}+8_{10}+14_8$	
	24-19	24.72-12a	32	$C_3 + C_3$
		$3_{12} 4_{48} 5_{36} 6_{12}$; $8_{12} 10_{12}$	$12_{10}+12_8$	
	24-20	24.72-24	33	K_4
		$4_{84} 6_{24}$; $8_{16} 10_6 14_2$	$2_{14}+6_{10}+16_6$	
	26-6	26.78-6	28	$C_{221} + \mathbf{1}$
		$3_{12} 4_{52} 5_{48} 6_6$; $6_2 8_{10} 10_{12} 12_2$	$2_{12}+12_{10}+10_8+2_6$	
	26-7	26.78-12	27	C_{321}
		$3_6 4_{70} 5_{30} 6_{12}$; $6_2 8_{10} 10_{12} 12_2$	$2_{12}+12_{10}+10_8+2_6$	
	28-1	28.88-0	29	C_{222}
		$3_6 4_{72} 5_{54}$; $6_4 8_6 10_{12} 12_6$	$6_{12}+12_{10}+6_8+4_6$	

m	Engel	Engel (full)	Delone	$Z(U)$, [DG]
6	24-15	24.62-0	37	$C_4 + \mathbf{1}$
		$3_{24}4_{32}5_{48}$; $8_{18}10_4 12_2$	$2_{12} + 4_{10} + 18_8$	
	24-13	24.62-6	38	$C_3 + 2 \times \mathbf{1}$
		$3_{18}4_{50}5_{30}6_6$; $8_{16}10_8$	$8_{10} + 16_8$	
	24-14	24.62-12	39	C_{221}
		$3_{12}4_{68}5_{12}6_{12}$; $8_{18}10_4 12_2$	$2_{12} + 4_{10} + 18_8$	
	26-3	26.68-0	35	C_5
		$3_{18}4_{54}5_{42}$; $6_2 8_{12} 10_{12}$	$12_{10} + 12_8 + 2_6$	
	26-4	26.68-6	34	$C_3 + 2 \times \mathbf{1}$
		$3_{12}4_{72}5_{24}6_6$; $6_2 8_{12} 10_{12}$	$12_{10} + 12_8 + 2_6$	
	26-5	26.72-0	36	$C_4 + \mathbf{1}$
		$3_{12}4_{70}5_{36}$; $6_2 8_{10} 10_{12} 12_2$	$2_{12} + 12_{10} + 10_8 + 2_6$	
5	24-8	24.52-0	41	$4 \times \mathbf{1}$
		$3_{30}4_{40}5_{30}$; $8_{20}10_4$	$4_{10} + 20_8$	
	24-9	24.52-6	42	$C_3 + \mathbf{1}$
		$3_{24}4_{58}5_{12}6_6$; $8_{20}10_4$	$4_{10} + 20_8$	
	24-10	24.56-0c	43	$4 \times \mathbf{1}$
		$3_{24}4_{56}5_{24}$; $8_{16}10_8$	$8_{10} + 16_8$	
	24-11	24.56-0d	44	C_4
		$3_{24}4_{56}5_{24}$; $8_{18}10_4 12_2$	$2_{12} + 4_{10} + 18_8$	
	26-2	26.62-0	40	$4 \times \mathbf{1}$
		$3_{18}4_{78}5_{18}$; $6_2 8_{12} 10_{12}$	$12_{10} + 12_8 + 2_6$	
4	24-6	24.42-0	47	$3 \times \mathbf{1}$
		$3_{42}4_{36}5_{18}$; 8_{24}	24_8	
	24-5	24.42-6	**St**	C_3
		$3_{36}4_{54}6_6$; 8_{24}		
	24-7	24.46-0	48	$3 \times \mathbf{1}$
		$3_{36}4_{52}5_{12}$; $8_{20}10_4$	$4_{10} + 20_8$	
	26-1	26.56-0	45	$3 \times \mathbf{1}$
		$3_{24}4_{90}$; $6_2 8_{12} 10_{12}$	$12_{10} + 12_8 + 2_6$	
3	24-3	24.36-0	49	$2 \times \mathbf{1}$
		$3_{54}4_{36}5_6$; 8_{24}	24_8	
	24-4	24.40-0	48	$2 \times \mathbf{1}$
		$3_{48}4_{52}$; $8_{20}10_4$	$4_{10} + 20_8$	
2	24-2	24.30-0	50	$\mathbf{1}$
		$3_{72}4_{24}$; 8_{24}	24_8	
1	24-1	24.24-0	51	24-cell
		3_{96}; 8_{24}	24_8	itself