Chapter 5

Lattices and their Voronoï and Delone cells

In this section we study lattices from the point of view of their tilings by polytopes.

5.1 Tilings by polytopes: some basic concepts

Definition: polytope A polytope P is a compact body with a nonempty interior whose boundary ∂P is the union of a finite number of facets, where each facet is the $(n-1)$-dimensional intersection of P with a hyperplane.

Two-dimensional polytopes are called polygons; three-dimensional polytopes are called polyhedra.

Definition: k-face (of a polytope) For $k = 0, \ldots, n-2$, a k-dimensional face (or k-face, for short) of a polytope is an intersection of at least $(n-k)$ facets that is not contained in the interior of a j-face for any $j > k$.

Thus a 0-face of a polytope is a point that lies in the intersection of at least n facets but not in the interior of any 1-face, 2-face, etc. As a customary, we use the terms vertex and edge, respectively for the 0-dimensional and 1-dimensional faces of tiles, and facets for faces of dimension $n-1$.

In the tilings we will study, the tiles will be convex polytopes in E^n. Remember that the polytope P is convex if P contains the line segments joining any two points in P or on its boundary.

Definition: tiling A tiling $\mathcal{T}$ of E^n is a partition of E^n into a countable number of closed cells with non-overlapping interiors:

$$\mathcal{T} = \{T_1, T_2, \ldots\}, \quad \bigcup T_i = E^n, \quad \text{int } T_i \cap \text{int } T_j = \emptyset \text{ if } \ i \neq j. \tag{5.1}$$

The words *tiling* and *tessellation* are used interchangeably; similarly, *tiles* are often called cells.

Definition: prototile set A prototile set $\mathcal{P}$ for a tiling $\mathcal{T}$ is a set of polytopes such that every tile of $\mathcal{T}$ is an isometric copy of an element of $\mathcal{P}$.

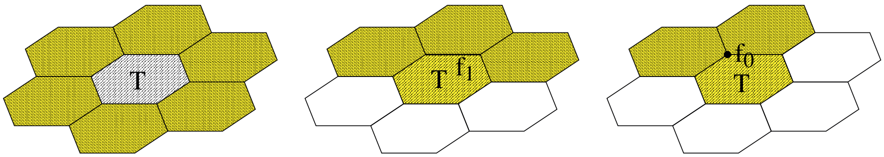

FIG. 5.1 – Two-dimensional tiling with a single prototile T. Left: Corona of a tile T. Middle: Corona of a 1-face f_1 (facet) of a tiling. Right: Corona of a 0-face f_0 (vertex) of a tiling.

When the prototile set contains a single tile T, the tiling is said to be *monohedral*. A prototile set does not, in general, characterize a tiling completely. Indeed a single prototile may admit different tilings. There are uncountably many Penrose tilings of the plane with the same prototile set of two rhombs.

Definition: convex, facet-to-facet, locally finite (tilings) A convex tiling is one whose tiles are convex. A tiling is said to be facet-to-facet if the intersection of the interior of any two facets is either empty or coincides with both facet interiors. A tiling is said to be locally finite if every ball in E^n of finite radius meets only finitely many tiles.

We state without proof the important fact [59]:

Proposition 5 *(Gruber and Ryshkov) A locally finite convex tiling in E^n is facet-to-facet if and only if it is k-face-to-k-face ($k = 0, 1, \dots, n-2$).*

Definition: corona of a k-face. Let $\mathbf{f_k}$ be a k-face of a tiling $\mathcal{T}$, where $0 \le k \le n$. The (first) corona of $\mathbf{f_k}$ is the union of $\mathbf{f_k}$ and the tiles that meet it, i.e., the tiles whose intersection with $\mathbf{f_k}$ is nonempty. When $k = 0$, the corona is called a vertex corona. When $k = n$ (i.e. when $\mathbf{f_k}$ is a tile T) the corona is called the corona of T.

Figure 5.1 shows different corona for an example of a two-dimensional tiling.

Definition: parallelotope A convex prototile P of a monohedral tiling in which the tiles are translates of P is called a parallelotope. Every convex parallelotope admits a facet-to-facet tiling; this is a corollary of the Venkov-McMullen's theorem [92, 67] characterizing convex parallelotopes in arbitrary dimension. To formulate this theorem, we need the concept of a *belt*:

Definition: belt A belt of a parallelotope P is a complete set of parallel $(n-2)$-faces of P.

Note that when $n = 3$, the $(n-2)$-faces of P are edges. Figure 5.2 shows the two belts of a hexagonal prism.

Theorem 5 *(Venkov, McMullen) A convex polytope P is a parallelotope if and only if it satisfies the following three conditions:*

1. *P is centrosymmetric;*
2. *all facets of P are centrosymmetric;*
3. *all belts of P have length four or six.*

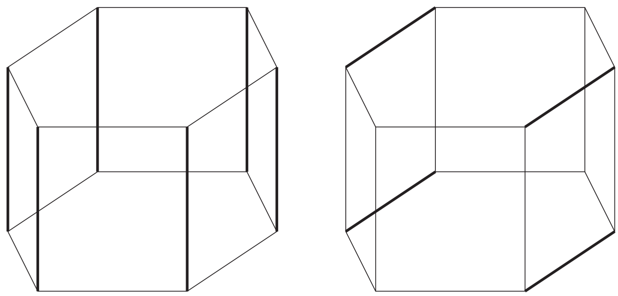

FIG. 5.2 – Different belts for a hexagonal prism. Left: Belt formed by six edges, i.e. by six $(d-2)$-faces. Right: One of three belts formed by four edges.

Corollary 3 *Of the five Platonic (regular) solids in E^3, only the cube is a parallelotope.*

It follows immediately that all other Platonic solids have triangular or pentagonal facets which are not centrosymmetric. (See figure 5.10.)

Central symmetry of faces implies also that within a belt the number of $(n-2)$-faces equals the number of facets.

5.1.1 Two- and three-dimensional parallelotopes

Two-dimensional parallelotopes are called parallelogons; in three dimensions they are parallelohedra. Since a monohedral tiling of the plane by convex polygons can have at most six edges, parallelogons are either parallelograms or centrosymmetric hexagons. To characterize their combinatorial type it is sufficient to use single labels indicating the number of edges (1-faces) or number of vertices (0-faces) which coincide. In order to use the same notation for two-, three-, and arbitrary d-dimensional parallelohedra we prefer to use symbols $\mathbf{N}_{(d-1)}.\mathbf{N}_0$ indicating both, the number of facets, i.e. $(d-1)$-faces, and the number of 0-faces.

Two combinatorial types of two-dimensional parallelogons are therefore **4.4** and **6.6**. They were described by Dirichlet in 1850 [45]. For $n = 3$ Fedorov found five combinatorial types of parallelohedra in 1885 [12].[1] We label combinatorial types of three-dimensional parallelohedra by $\mathbf{N}_2.\mathbf{N}_0$ showing number of 2-faces and of 0-faces of a parallelohedron. The five combinatorial types of three-parallelohedra are: the cube **6.8**, the hexagonal prism **8.12**, the rhombic dodecahedron **12.14**, the elongated dodecahedron **12.18**, and the truncated octahedron **14.24**. They are shown in Figures 5.4-5.8.

These five combinatorial types of parallelohedra can be related by the operation consisting in shrinking one of the belts. Such operation is very

[1] In 1929 Delone found 51 combinatorial type for $n = 4$; this was corrected to 52 by Shtogrin in 1972 [41, 87]. The number, 103769, of combinatorial types in five dimensions was determined by Engel [51].

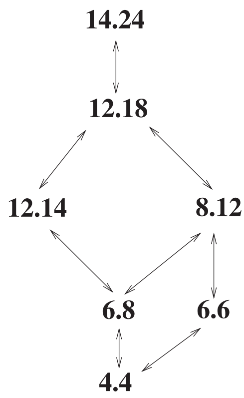

FIG. 5.3 – Zone contraction family of three- and two-dimensional parallelohedra. For three-dimensional polytopes the zone contraction can be equivalently described as belt shrinking.

important for a general classification of parallelohedra in arbitrary dimension. But instead of belts (set of parallel $(n-2)$-faces) one needs to consider zones (the set of all edges (1-faces) parallel to a given vector). Obviously, for three-dimensional parallelohedra zones are equivalent to belts. Nevertheless, to be consistent with more general treatment we prefer to name the operation of shrinking of belts for three-dimensional parallelohedra the zone contraction operation. The zone contraction family of three-dimensional parallelohedra is represented in Figure 5.3. It includes the zone contraction operation which reduces three-dimensional polytopes to two-dimensional ones and also the zone contraction between two-dimensional polytopes. Concrete geometrical visualization of a zone contraction for all possible pairs of three-dimensional Voronoï parallelohedra is shown in Figures 5.4-5.8. Contractions for three dimensional parallelohedra are complemented in Figure 5.3 by zone contraction operations transforming three-dimensional cells into two-dimensional: These are $\mathbf{8.12} \to \mathbf{6.6}$ and $\mathbf{6.8} \to \mathbf{4.4}$. Also there is one zone contraction between two-dimensional cells: $\mathbf{6.6} \to \mathbf{4.4}$. Note, that with each zone contraction operation we can associate inverse operation which is named zone extension.

5.2 Voronoï cells and Delone polytopes

We return to Delone sets Λ and to Voronoï cells and Delone polytopes introduced briefly in Chapter 3.

First we note that the Voronoï cells of the points of Λ tile E^n; that is, they fill E^n without gaps or overlapping interiors. We denote the tiling by $\mathcal{T}_\Lambda$. This follows from the fact that every point of E^n is closer to a unique point

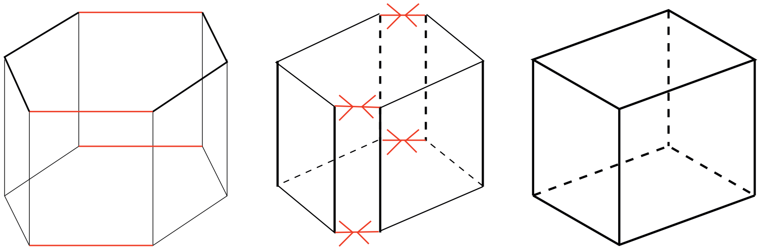

FIG. 5.4 – Contraction of the **8.12** cell (hexagonal prism) into the **6.8** cell (cube). Four edges shrink to zero, two quadrilateral facets disappear, two hexagonal facets transform into quadrilateral ones. There are three 4-belts to shrink.

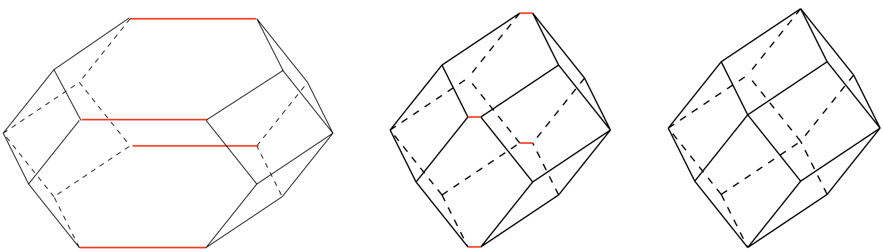

FIG. 5.5 – Contraction of the **12.18** cell (elongated dodecahedron) into the **12.14** cell (rhombic dodecahedron). Four edges shrink to zero and four hexagonal facets transform into quadrilateral ones. There is only one 4-belt to shrink.

of Λ, or is equidistant from two or more of them. The tiling $\mathcal{T}_\Lambda$ is locally finite and facet-to-facet.

Theorem 6 *The vertices of the Voronoï cells of* Λ *are the centers of its holes.*

Proof. A vertex v of a Voronoï cell $D(p)$ is the intersection of at least $n+1$ hyperplanes bisecting the vectors from p to other points $q_1, \ldots, q_k$ of Λ, where $k \geq n$. Consequently, the distances $r_1 = \ldots = r_k = r$ between p and q_i, $i = 1, \ldots, k$ are all the same and v is the center of a ball of radius r. By construction, there is no other point of Λ in this ball. $\square$

Figure 5.9 illustrates construction of the Voronoï cell for a Delone set. The construction consists of two steps:

i) construct the $2R_0$ star for a chosen point p,

ii) construct the orthogonal bisectors of the arms of the star.

Then the Voronoï cell is the intersection of the half-spaces containing p determined by these bisectors.

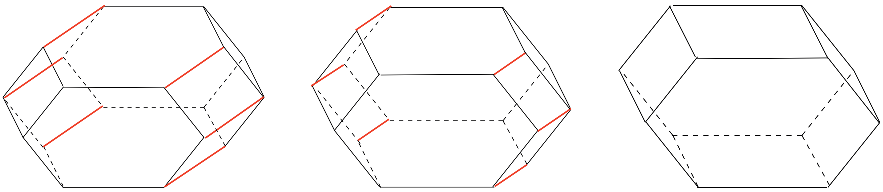

FIG. 5.6 – Contraction of the **12.18** cell (elongated dodecahedron) into the **8.12** cell (hexagonal prism). Six edges shrink to zero and four quadrilateral faces disappear. There are four 6-belts to shrink.

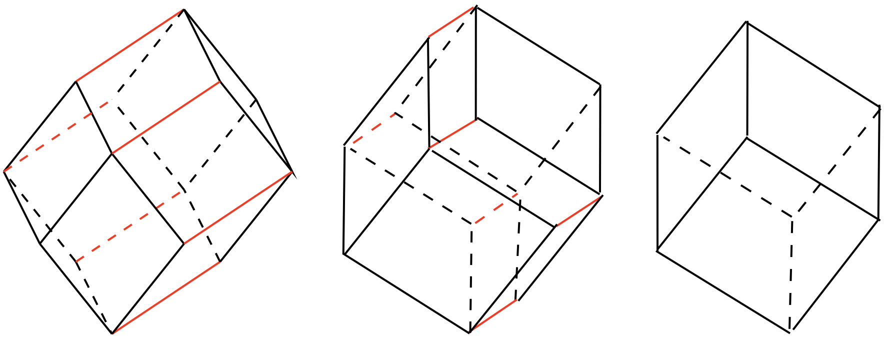

FIG. 5.7 – Contraction of the **12.14** cell (rhombic dodecahedron) into the **6.8** cell (cube). Six edges shrink to zero and six quadrilateral facets disappear. There are four 6-belts to shrink.

Definition: corona vector A vector $\vec{f} \in \Lambda$ is said to be a *corona vector* of the Voronoï cell $D(o)$ if it joins o to the center of a Voronoï cell in the corona of $D(o)$.

We denote the set of corona vectors of $D(o)$ by $\mathcal{C}_o$.

Definition: facet vector A facet vector $\vec{f} \in \Lambda$ is a corona vector joining o to a Voronoï cell with which it shares a facet.

Alternatively we can say that a vector $\vec{f} \in \Lambda$ is a facet vector of the Voronoï cell $D(o)$ if a facet $\mathbf{f}$ of $D(o)$ is contained in its orthogonal bisector. We denote the set of facet vectors of $D(o)$ by $\mathcal{F}_o$.

The equation of the bisecting hyperplane is $(\vec{f}, \vec{x}) = \frac{1}{2}N(\vec{f})$. Thus the Voronoï cell of the point o is the set

$$D(o) = \{x \in E^n | (\vec{x}, \vec{f}) \leq \frac{1}{2}N(\vec{f}), \quad \forall \vec{f} \in \mathcal{F}\}. \tag{5.2}$$

When $x \in \partial D(o)$, equality must hold in (5.2) for at least one $\vec{f} \in \mathcal{F}$. The definition of facet vector does not imply that the midpoint $\frac{1}{2}\vec{f} \in \mathbf{f}$; it may lie outside of $D(o)$. But $\frac{1}{2}\vec{f} \in \mathbf{f}$ when Λ is a lattice.

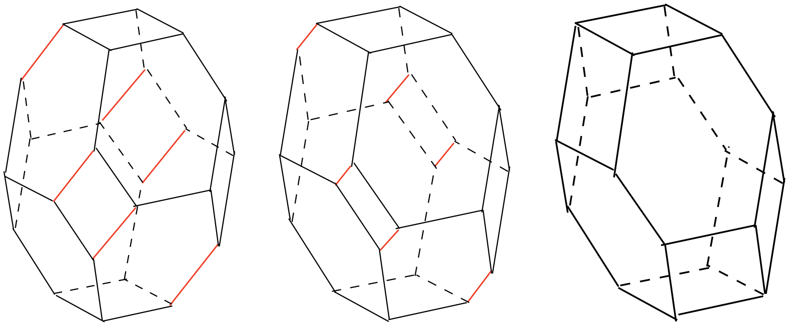

FIG. 5.8 – Contraction of the **14.24** cell (truncated octahedron) into the **12.18** cell (elongated dodecahedron). Six edges shrink to zero and two quadrilateral facets disappear. Four hexagonal facets transform into quadrilateral facets. There are six 6-belts to shrink.

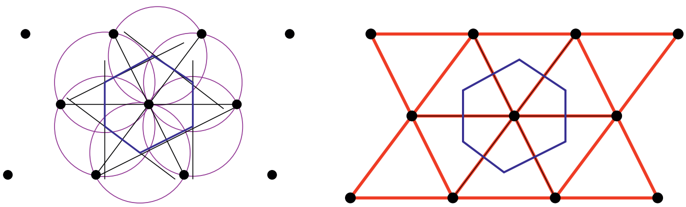

FIG. 5.9 – Construction of the Voronoï and Delone cells for a two-dimensional Delone set.

Proposition 6 *Let* $\mathbf{f}_k$ *be a* k*-face of* $D(o)$, $0 \leq k \leq n-1$, *and let* $o, p_1, \ldots, p_m$ *be the centers of the Voronoï cells in its corona. The* m *vectors* $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m$ *span an* $(n-k)$*-dimensional subspace* E^{n-k} *orthogonal to* $\mathbf{f}_k$, *so* $\mathbf{f}_k \cap E^{n-k}$ *is a single point* x_o, *and* $o, p_1, \ldots, p_m$ *lie on a sphere in* E^{n-k} *centered at* x_o.

Proof. Since $\mathbf{f}_k$ is the intersection of at least $n-k$ facets of $D(o)$, and since $D(o)$ is convex and compact, the corresponding facet vectors span an $(n-k)$-dimensional subspace of E^n. Thus $m \geq n-k$. By construction, these vectors and thus this subspace are orthogonal to $\mathbf{f}_k$. The intersection $E^{n-k} \cap \mathbf{f}_k$ is a single point x_o (otherwise the points of $\mathbf{f}_k$ could not all be equidistant from $o, p_1, \ldots, p_m$). Thus $o, p_1, \ldots, p_m$ lie on a sphere about x_o. □

Next we describe the Delone tiling Δ_Λ, obtained by connecting points of Λ. This tiling was in fact first introduced by Voronoï; later it was thoroughly studied by Delone. Today it is known as the Delone tessellation induced by Λ, except in Russian literature, where Delone tessellations are called L-tessellations, the name that Voronoï had given them.

Definition: Delone polytope The Delone polytope of a hole of Λ is the convex hull of the points of Λ that lie on its boundary.

From this definition it follows immediately that the set $\Delta_\Lambda = \Delta(x_i, r_i)$, $i \in \mathbb{Z}$ of Delone polytopes is a facet-to-facet tiling of E^n.

To make the connection with the Voronoï tiling induced by Λ, we remember that the center of any empty hole must be a vertex of the Voronoï tiling. For, the vertices y_j, $j = 1, \ldots, k$ of $\Delta(x_i, r_i)$, all points of Λ lie on the sphere about x_i and are the closest points of Λ to x_i. Thus x_i belongs to the all Voronoï cells $D(y_j)$. Since there are $n+1$ independent points among the y_j, $\cap_{j=1}^{k} D(y_j) = \{x_i\}$.

We will denote the Delone polytope associated with the vertex v of a Voronoï tiling by $\Delta(v)$, the set of vertices of $D(o)$ by $V(o)$, and the set of vertices of the Voronoï tiling $\mathcal{T}_\Lambda$ by $\mathcal{V}_\Lambda$.

Proposition 7 *For each $v \in V(o)$, the polytope $\Delta(v)$ is circumscribable, and so are its k-faces, $k = 0, \ldots, n-1$.*

Proof. The first statement follows immediately from the fact that the vertices of $D(o)$ lie on the boundary of an empty hole; the second is immediate since the intersection of a ball with a plane of lower dimension is again a ball. □

5.2.1 Primitive Delone sets

Definition: primitive (Delone set and Voronoï tessellation) A Delone set and the Voronoï tessellation it induces are said to be primitive if all of its Delone polytopes are simplices.

By the definition of the Delone polytope, we have

Proposition 8 *A Delone set is primitive if and only if every vertex of the Voronoï tessellation belongs to exactly $n+1$ Voronoï cells.*

More generally we have

Proposition 9 *In the Voronoï tessellation of a primitive Delone set, every k-face, $k = 0, \ldots, n-1$ belongs to exactly $n+1-k$ adjacent Voronoï cells.*

Proof. Voronoï proved this proposition for the case when Λ is a lattice but it is true more generally. If a k-face $\mathbf{f}_k$ is shared by exactly $n+1-k$ cells, then it lies in the intersection of exactly $n+1-k$ hyperplanes. Now let $\mathbf{f}_{(k+1)}$ be a $(k+1)$-face containing $\mathbf{f}_k$. It lies in the intersection of $m \leq n-k$ hyperplanes, and since it is $(k+1)$-dimensional, we must have $m = n-k$. □

Proposition 10 *Primitivity is generic.*

Proof. Since $n+1$ independent points determine a sphere in E^n, any additional points are redundant. □

Indeed in discrete geometry literature Delone tessellations are known as Delone triangulations. In addition to “most” lattices, many other important Delone sets are primitive.

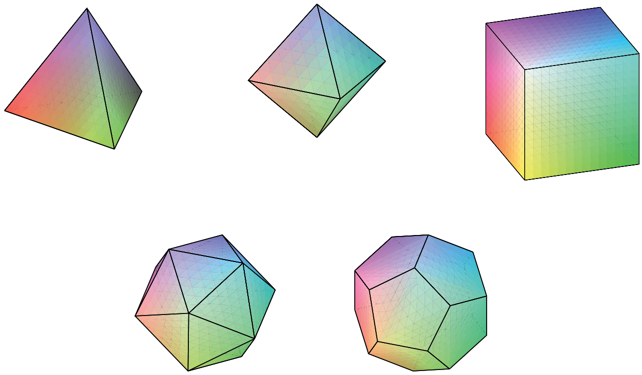

FIG. 5.10 – Examples of combinatorial duality of regular polyhedra. The tetrahedron is auto-dual. The cube and octahedron are combinatorially dual. The icosahedron and dodecahedron are also combinatorially dual.

5.3 Duality

We discussed dual lattices in Chapter 3. Here we introduce dual polytopes and dual tilings, for which duality has a different meaning.

Definition: combinatorially dual convex polytopes Two convex polytopes are said to be combinatorially dual if there is an inclusion-reversing bijection between the k-faces of one and the $(n-k)$-faces of the other.

For example, the cube and the regular octahedron are combinatorially dual, while the combinatorial dual of a tetrahedron is again a tetrahedron (see figure 5.10).

Definition: orthogonally dual polytopes Two combinatorially dual polytopes P and P' are said to be orthogonally dual if the corresponding k and $(n-k)$-faces are orthogonal.

Notice that we restrict these definitions to convex polytopes.

Duality for tilings is defined in an analogous way.

Definition: combinatorial and orthogonally dual tilings Two tilings by convex prototiles are combinatorially dual if there is an inclusion-reversing bijection between the k-faces of one and the $(n-k)$-faces of the other. When the corresponding k and $(n-k)$-faces are mutually orthogonal, the duality is said to be orthogonal.

Now we can formulate the duality relation between Voronoï and Delone tilings.

Proposition 11 *The tilings Δ_Λ and $\mathcal{T}_\Lambda$ are orthogonally dual.*

Proof. This is an immediate consequence of Proposition 6. We select a nested sequence of k-faces

$$D(o) \supset \mathbf{f} \supset \mathbf{f}_{n-2} \supset \cdots \supset \mathbf{f}_0 = v. \tag{5.3}$$

To construct an inclusion-reversing bijection ψ between $\mathcal{T}_\Lambda$ and Δ_Λ, we first set $\psi(D(o)) = o$. Let $D(p_1)$ be the Voronoï cell that shares $\mathbf{f}$ with $D(o)$. Then

$$op_1 = \bigcap_{v \in \mathbf{f}} \Delta(v) \tag{5.4}$$

and hence op_1 is an edge of Δ_Λ, so we set $\psi(\mathbf{f}) = op_1$. Next we set

$$\psi(\mathbf{f}_{n-2}) = \text{convex hull } \{o, p_1, \ldots, p_m\}, \tag{5.5}$$

where $D(p_2), \ldots, D(p_m)$ are the cells, in addition to $D(o)$ and $D(p_1)$, to which $\mathbf{f}_{n-2}$ belongs; this polygon is a 2-face of Δ_Λ. We continue in this way, taking for $\psi(\mathbf{f}_k)$ the $(n-k)$-face of Δ_Λ that is the convex hull of the points of Λ whose Voronoï cells share $\mathbf{f}_k$. Finally, the vertex v is associated to $\Delta(v)$. □

5.4 Voronoï and Delone cells of point lattices

5.4.1 Voronoï cells

When a Delone set Λ is a regular system of points (point lattice), its Voronoï tilings $\mathcal{V}_\Lambda$ is monohedral and we can speak of "the" Voronoï cell of the set. Thus by the Voronoï cell of a point lattice we will mean the Voronoï cell of the origin, $D(o)$. In this section we will discuss some of the fundamental properties of Voronoï cells of point lattices.

Since point lattices are orbits of translation groups, their Voronoï cells are parallelotopes. Since the Voronoï cell is the closure of a fundamental region for the translation subgroup of the symmetry group of the lattice, the volume of the Voronoï cell is equal to the volume of a lattice unit cell.

The Voronoï cell of a lattice is invariant under the lattice's point symmetry group.

Proposition 12 *The point symmetry group of a lattice L with fixed point o is also the symmetry group of the Voronoï cell $D(o)$; the full symmetry group of L is the symmetry group of the Voronoï tiling.*

Proof. This follows immediately from the definition of $D(o)$. □

Proposition 13 *$D(o)$ and its facets are centrosymmetric.*

Proof. Every lattice point is a center of symmetry for the lattice; thus $D(o)$ is centrosymmetric by construction. The midpoint between any pair of lattice points is also a center of symmetry for L; in particular if $\vec{f}$ is a facet vector, then $\frac{1}{2}\vec{f}$ is a center of symmetry for L. Thus it is the center of symmetry of $D(o) \cup D(f)$ and of $D(o) \cap D(f)$, and hence $\frac{1}{2}\vec{f}$ is the center of symmetry for the facet $\mathbf{f}$. □

Note: The k-faces of $D(o)$, $2 \leq k \leq n-2$, need not be centrosymmetric; for example, there are lattices in E^4 whose Voronoï cells have triangular or pentagonal 2-faces.

Since every point of $D(o)$ is a representative of a coset of L in $\mathbb{R}^n$, we can reformulate the definition of $D(o)$ in the following way.

Proposition 14 *The Voronoï cell of a lattice in E^n is the set of vectors $\vec{x} \in E^n$ of minimal norm in their L-coset $\vec{x} + L$:*

$$D_L = \{\vec{x} \in E^n | N(\vec{x}) \leq N(\vec{x} - \vec{\ell}),\ \forall \vec{\ell} \in L\}. \tag{5.6}$$

The interior points are unique in their coset but two or more boundary points may belong to the same coset: for example, if x is a point on the boundary $\partial D(o)$ of the Voronoï cell, then so is $-x$ and these points are congruent modulo L. This point x belongs to at least one intersection $D(o) \cap D(\vec{f})$, and translation by $-\vec{f}$ carries that intersection, and with it x, to $D(-\vec{f}) \cap D(o)$,

5.4.2 Delone polytopes

As in the case of general Delone sets, the tiles of the Delone tessellation induced by a lattice are convex polytopes whose vertices are the lattice points lying on the boundaries of empty spheres and the Delone and Voronoï tessellations are dual.

In general the Delone tiling has several prototiles. However, when $n = 2$, not only is the tiling monohedral, it is *isohedral*, i.e. the tiles form an orbit of the symmetry group of the tiling.

Proposition 15 *The Delone tiling associated to a lattice L in E^2 is isohedral.*

Proof. Since the midpoints of the edges of the Voronoï cell of L in E^2 are centers of symmetry for L, any pair of adjacent vertices can be interchanged by inversion in the center of the edge joining them. Thus all the vertices of the Voronoï cell are equivalent under the symmetry group of the lattice, from which it follows that the Delone cells corresponding to the vertices of $D(o)$ are equivalent too. □

5.4.3 Primitive lattices

Primitive Voronoï cells have received the most attention in the context of both lattices and quadratic forms. This is mainly due to the fact that the primitivity is generic. The relation to quadratic forms will be discussed in the next chapter. Here we describe several simple properties of primitive lattices.

Applying the definition of primitivity of Delone sets (see 5.2.1) to the lattice we get the following obvious statement:

Proposition 16 *A lattice L is primitive if and only if all its Delone cells are simplices.*

When $D(o)$ is primitive, exactly $n + k - 1$ Voronoï cells of the Voronoï tessellation share a given k-face, $k = 0, \ldots, n - 1$.

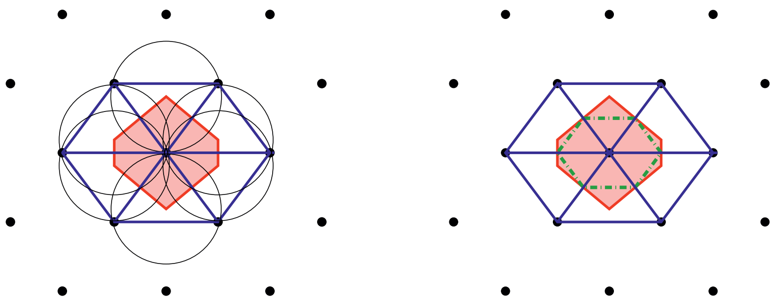

FIG. 5.11 – Illustration to Proposition 17. Left: Primitive two-dimensional lattice with its Voronoï cell whose vertices are situated in the centers of holes. Right: The same lattice with the Voronoï cell (shaded region with boundary), the dual to the Voronoï cell (dash line), and the Delone corona.

Definition: Delone corona The set of Delone cells that share the vertex o (the origin) is called the Delone corona of the lattice L.

Proposition 17 *If L is primitive, then the Delone corona of L is a scaled copy of the polytope dual to the Voronoï cell $D(o)$.*

This proposition is illustrated in figure 5.11.

We denote $\mathcal{V}_L$ the set of vertices of the Voronoï tiling of L. It is easy to check that $\mathcal{V}_L$ is a Delone set. The minimum distance between vertices of $D(o)$ can be taken as r_0, whereas R_0 can be chosen to be the length of the longest vertex vector of $D(o)$. Recall that when $D(o)$ is primitive, exactly $(n-k+1)$ Voronoï cells of the Voronoï tessellation share a given k-face, $k = 0, \ldots, n-1$.

For each k, $0 \leq k \leq n$, the set of k-faces of a lattice Voronoï tessellation belongs to a finite number of orbits of the translation group of the lattice; in general, each Voronoï cell contains several elements of each orbit. Let $\mathbf{f}_k$ be a k-face of $D(o)$ and let $\{\vec{c}_m\}$, $1 \leq m \leq n-k$, be the set of vectors corresponding to the centers of the other $n-k$ Voronoï cells which share this k-face with $D(o)$. Each translation $-\vec{c}_m$ transforms $D(c_m)$ into $D(o)$ and therefore $\mathbf{f}_k$ into $\mathbf{f}_k - \vec{c}_m$, another k-face of the Voronoï cell $D(o)$. Conversely, if $\mathbf{f}'_k$ is a k-face of $D(o)$, where $\mathbf{f}'_k + \vec{t} = \mathbf{f}_k$ for some $\vec{t} \in L$, then $\mathbf{f}_k$ is a k-face of $D(t)$. Thus we have

Proposition 18 *Each k-face of a primitive Voronoï cell $D(o)$ is equivalent, under translations of L, to exactly $n-k$ other k-faces of $D(o)$.*

This means that the number of k-faces of a primitive Voronoï cell should be a multiple of $n-k+1$. In fact for $0 \leq k < n-1$, it should be proportional to $2(n-k+1)$ (see proposition 29).

The set $\mathcal{V}_L$ can be decomposed into L-orbits. Selecting one Delone cell from each orbit, we have the closure of a fundamental region of L, and so the volume of the union of these Delone cells must be equal to the volume of the lattice introduced in (3.2) as $\mathrm{vol}(L) = |\det(\ell_i)|$.

Moreover the set $C(o) = \cup_{v \in V(o)} \Delta(v)$ is the union of $n+1$ fundamental domains; hence the value of the invariant vol(L) for all primitive lattices is equal to $(n+1)$:

Proposition 19 *When L is primitive,*

$$\mathrm{vol}(L) = \mathrm{vol}C(o)/\mathrm{vol}(D) = n+1. \tag{5.7}$$

Any vertex of the Voronoï cell of a primitive lattice L belongs to exactly n facets of that cell; since the corresponding facet vectors are linearly independent, these vectors form a basis of E^n though they may generate only a sub-lattice L'. But there are many primitive lattices for which this set of vectors is a basis. For example, this is the case for the primitive lattices in E^2, E^3, and E^4.

Definition: principal primitive A primitive lattice, and its Voronoï cell, is said to be principal primitive if for each vertex of the cell, the facet vectors of the n facets meeting at this vertex form a basis of the lattice.

The Delone cells of principal primitive lattices are simplices whose edges issuing from 0 are the edges of a unit cell for L. Thus all these simplices have the same volume, vol(simplex$(x_0, \dots, x_n)$) $= \det(L)/n!$.

Proposition 20 *A principal primitive Voronoï cell has* $(n+1)!$ *vertices.*

Proof. When all Delone cells have the same volume vol$\Delta(v)$, denoting the number of vertices of the Voronoï cell V by $N_0(V)$, we have

$$\frac{N_0(V)}{n+1} = \frac{\det(L)}{\mathrm{vol}\Delta(v)} = n!. \tag{5.8}$$

Corollary 4 *A principal primitive Voronoï cell has* $(n+1)!\ n/2$ *edges.*

Proof. Exactly n edges of the cell meet at each vertex, and each edge has two vertices. □

Taking into account that the Euler characteristic for a n-dimensional polytope is $1-(-1)^n$ and it is expressed as an alternative sum of the numbers of k-faces of an n-polytope, $N_k(n)$, namely $\sum_{0 \le k \le n-1} (-1)^k N_k(n) = 1-(-1)^n$, we can find immediately the number of faces for 3-dimensional principal primitive polytopes. The table 5.1 gives values of $N_k(n)$ for $n = 2, 3, 4$ for principal primitive polytopes. Note, that for $n = 2, 3, 4$ all primitive polytopes are principal. Additional topological restrictions on the numbers of k-faces for primitive higher dimensional polytopes will be discussed in the next chapter (see section 6.4).

5.5 Classification of corona vectors

In the geometry (and the algebra) of lattices, one is interested in the set of vectors that are (relatively) short. Historically, the vectors of minimum length have received the most attention. Here we consider three sets of "short"

TAB. 5.1 – Values of the number of k-faces, $N_k(n)$ for n-dimensional primitive polytopes for $n = 2, 3, 4$.

n	$N_0(n)$	$N_1(n)$	$N_2(n)$	$N_3(n)$
2	6	6		
3	24	36	14	
4	120	240	150	30

vectors, all defined in terms of the Voronoï cell of the lattice. We begin with the largest of these sets, the corona vectors of the lattice.

5.5.1 Corona vectors for lattices

The corona of a tile T in a tiling is defined in section 5.1. When T is the Voronoï cell $D(o)$ of a lattice then every tile in the corona is associated to a lattice vector.

Definition: corona vector The corona vectors of a lattice L are the vectors $\vec{c}$ from o to the centers c of the cells comprising the corona of the Voronoï cell $D(o)$.

Proposition 21 *A lattice vector $\vec{c} \in L$ is a corona vector if and only if $\frac{1}{2}\vec{c} \in \partial D(o)$.*

Proof. Let $\vec{c}$ be a corona vector. Let $I(o, c) = D(o) \cap D(c)$. Then $I(o, c) \neq \emptyset$, and it is convex because $D(o)$ and $D(c)$ are convex. The midpoint $\frac{1}{2}\vec{c}$ is a center of symmetry of the lattice that interchanges $D(o)$ and $D(c)$ and hence stabilizes $I(o, c)$, and again by convexity, $\frac{1}{2}\vec{c} \in I(o, c)$. The converse is immediate by the definition of the corona vector. □

Corollary 5 *$\vec{c} \in L$ is a corona vector if and only if $\frac{1}{2}\vec{c}$ is the center of symmetry of the nonempty intersection $D(o) \cap D(c)$.*

Corollary 6 *If a k-face of $D(o)$ does not contain a center of symmetry, then any tile that shares that k-face also shares one of higher dimension.*

We denote the set of corona vectors of L by $\mathcal{C}$.

Proposition 22 *The number of corona vectors is even.*

Proof. Since $D(o)$ is centrosymmetric, $\frac{1}{2}\vec{c} \in \partial D(o) \leftrightarrow -\frac{1}{2}\vec{c} \in \partial D(o)$. □

Theorem 7 *$\vec{c}$ is a corona vector of L if and only if it is a vector of minimal norm in its $L/2L$ coset.*

Proof. By definition, $\frac{1}{2}\vec{c}$ belongs to the Voronoï cell of o and so, by Proposition 21,

$$\vec{c} \in \mathcal{C} \leftrightarrow N\left(\frac{1}{2}\vec{c}\right) \leq N(\vec{c}/2 - \vec{\ell}), \quad \forall \vec{\ell} \in L. \tag{5.9}$$

Thus

$$\vec{c} \in \mathcal{C} \leftrightarrow N(\vec{c}) \leq N(\vec{c} - 2\vec{\ell}), \quad \forall \vec{\ell} \in L, \tag{5.10}$$

i.e. $\vec{c}$ is a vector of minimal norm in its $L/2L$ coset. □

Note that $\vec{c}$ and $\vec{c}\,'$ are two vectors of the same length and $\vec{c} - \vec{c}\,' \in 2L$ if and only if $\frac{1}{2}(\vec{c} - \vec{c}\,') \in L$. In this case, $\vec{c}\,'$ is the image of $\vec{c}$ through the center of symmetry $\frac{1}{2}(\vec{c} - \vec{c}\,')$. With this observation it is easy to prove (Theorem 8 below) that if $\pm\vec{c}$ are the only vectors of minimal length in their $L/2L$ coset, then they are facet vectors.

The corona vectors of a lattice are of special interest because they encode many of its properties.

Proposition 23 *A corona vector is the shortest lattice vector in its mL coset for all integers $m \geq 3$.*

Proof. If $\vec{c} \in \mathcal{C}$ and $\vec{x} \neq 0$, then $N(\vec{c} + 2\vec{x}) - N(\vec{c}) \geq 0$, so $(\vec{c}, \vec{x}) + N(\vec{x}) \geq 0$. Then

$$\begin{aligned} N(\vec{c} + m\vec{x}) - N(\vec{c}) &= 2m(\vec{c}, \vec{x}) + m^2 N(\vec{x}) \\ = m(2(\vec{c}, \vec{x}) + mN(\vec{x})) &= m(2((\vec{c}, \vec{x}) + N(\vec{x})) + (m-2)N(\vec{x})) \end{aligned} \tag{5.11}$$

which is positive for $m > 2$. Thus $N(\vec{c}) < N(\vec{c} + m\vec{x})$. □

Note that when $m > 2$, $\vec{c}$ and $-\vec{c}$ do not belong to the same mL coset.

Proposition 24 *The set $\mathcal{C}$ is the set of vertices (except o) of the Delone corona of o.*

Proof. The Delone corona of o is $\cup_{v \in V(o)} \Delta(v)$. For each such v, the vertices of $\Delta(v)$ are the centers of the Voronoï cells that meet at v, and thus by definition the vertices of $\Delta(v)$ are corona vectors. Conversely, every corona vector is a vertex of some $\Delta(v)$, $v \in V(o)$. □

5.5.2 The subsets $\mathcal{S}$ and $\mathcal{F}$ of the set $\mathcal{C}$ of corona vectors

We distinguish now two important subsets of the set $\mathcal{C}$ of corona vectors of a lattice L.

- The set $\mathcal{S}$ of vectors of minimal norm s in L, i.e. the set of shortest vectors.
- The set $\mathcal{F}$ of facet vectors of the Voronoï cell.

We have already noted the simple criterion for determining whether a lattice vector is an element of $\mathcal{F}$.

Theorem 8 *(Voronoï). The following conditions on $\vec{c} \in \mathcal{C}$ are equivalent:*

i) $\pm\vec{c}$ are the facet vectors;

ii) $\pm\vec{c}$ are the shortest vectors in their $L/2L$ coset;

iii) $(\vec{c}, \vec{v}) < N(\vec{v})\ \forall \vec{v} \in L,\ \vec{v} \neq 0,\ \vec{v} \neq \vec{c}$;

iv) the closed ball $B_{c/2}(|\frac{1}{2}\vec{c}|)$ *contains no points of* L *other than* o *and* c.

Proof.

ii) $\Rightarrow$ i). Assume $\pm\vec{c}$ are the shortest vectors in their $L/2L$ coset. Let $\mathbf{f}$ be a facet containing $\frac{1}{2}\vec{c}$. Then the image of $\frac{1}{2}\vec{c}$ through the center of symmetry $\frac{1}{2}\vec{f}$ is $\frac{1}{2}\vec{c}\,' = \vec{f} - \frac{1}{2}\vec{c}$, and hence $N(\vec{c}\,') = N(\vec{c})$ and $\vec{c}\,' = 2\vec{f} - \vec{c}$. Thus our hypothesis implies $\vec{c}\,' = \vec{c}$ or $\vec{c}\,' = -\vec{c}$. If $\vec{c}\,' = \vec{c}$, $\frac{1}{2}\vec{c}$ was fixed by this symmetry and hence $\vec{c} = \vec{f}$. The case $\vec{c}\,' = -\vec{c}$ is impossible, since in that case $\frac{1}{2}\vec{c}$ and $-\frac{1}{2}\vec{c}$ would lie in the same facet of $D(o)$. Thus $\vec{c}$ is a facet vector, and the same argument works for $-\vec{c}$.

i) $\Rightarrow$ ii). Conversely, assume that $\vec{c}$ is a facet vector. Then $\frac{1}{2}\vec{c}$ is the center of a facet and so is closer to $\vec{c}$ and to o than to any other points of L. That is,

$$\forall \vec{\ell} \in L, \vec{\ell} \neq 0, \quad N\left(\frac{1}{2}\vec{c}\right) < N\left(\frac{1}{2}\vec{c} - \vec{\ell}\right). \tag{5.12}$$

Again the same argument works for $-\vec{c}$, so $\pm\vec{c}$ are the shortest vectors in their $2L$ coset.

ii) $\Rightarrow$ iii). This is equivalent to condition $N(\vec{c}) < N(\vec{c} - 2\vec{v})$, $\forall \vec{v} \in L$, $\vec{v} \neq 0, \neq \vec{c}$.

i) $\Rightarrow$ iv). If $\vec{c}/2$ is the center of a facet, then it is equidistant from o and c and all other points of L are farther away. But any lattice point w in $B_{c/2}(|\frac{1}{2}\vec{c}|)$ would be at least as close, a contradiction. The converse is obvious. $\square$

Corollary 7 $\mathcal{S} \subseteq \mathcal{F} \subseteq \mathcal{C}$.

Proof. By the definition of a facet vector, $\mathcal{F} \subseteq \mathcal{C}$. It is also obvious that $\mathcal{S} \subseteq \mathcal{C}$, since $\mathcal{S}$ vectors have minimal norm in L and hence also in the $L/2L$ cosets to which they belong. To show that $\mathcal{S} \subseteq \mathcal{F}$, we prove that no two $\mathcal{S}$ vectors $\vec{s}_1, \vec{s}_2$, $\vec{s}_1 \neq \pm\vec{s}_2$, can belong to the same $L/2L$ coset. Let θ be the angle between $\vec{s}_1$ and $\vec{s}_2$; we may assume $0 < \theta < \pi$. If $\vec{s}_1 = \vec{s}_2 + 2\vec{y}$ for some $\vec{y} \in L$, we have $(\vec{s}_1 - \vec{s}_2) = \vec{y} \in L$, and

$$N\left(\frac{\vec{s}_1 - \vec{s}_2}{2}\right) = \frac{s}{2}(1 - \cos\theta) < s, \tag{5.13}$$

where s is the norm of the vectors in $\mathcal{S}$. This is a contradiction. Thus $\mathcal{S} \subseteq \mathcal{F}$. $\square$

There is exactly one planar lattice for which $\mathcal{S} = \mathcal{F} = \mathcal{C}$: the hexagonal lattice, whose Voronoï cell is a regular hexagon. Surprisingly, there are no examples in any higher dimension.

We next study some key properties of $\mathcal{F}$. The next two propositions are due to Minkowski [78].

Proposition 25 *(Minkowski).* $2n \leq |\mathcal{F}| \leq 2(2^n - 1)$.

Proof. The lower bound is implied by the centrosymmetry of $D(o)$. The 2^n cosets of $2L$ in L have as coset representatives the vectors $\{(\epsilon_1, \ldots, \epsilon_n)\}$ where $\epsilon_i \in \{0, 1\}$. Since if $\vec{f} \in \mathcal{F}$ the only other facet vector in its $L/2L$ coset is $-\vec{f}$, the maximum number of face vectors is twice the number of cosets, excluding of cause the 0-coset. □

Proposition 26 *(Minkowski).* $2(2^n - 1) \leq |\mathcal{C}| \leq 3^n - 1$.

Proof. The lower bound follows from the fact that every $L/2L$ coset contains at least two corona vectors. The upper bound is a corollary of Proposition 23 since there are 3^n cosets of $3L$ in L, one of which is represented by 0. □

The upper bound is attained in every dimension by the cubic lattice, whose Voronoï cell is the unit n-cube. To calculate the number of corona vectors for cubic lattices the notion of k-vector is useful.

Definition: k-vector. A vector $\vec{c} \in \mathcal{C}$ is a k-vector if $\frac{1}{2}\vec{c}$ lies in the interior of a k-face of $D(o)$, that is if it lies in the intersection of exactly $n-k$ independent facets.

For cubic lattices, the vector $\vec{c}$ is a k-vector if and only if $\epsilon_i = \pm 1$ for exactly $n-k$ values of i and is equal to 0 for all the others. Thus, since we do not include the 0-coset, the number of corona vectors for a n-dimensional cubic lattice is

$$|\mathcal{C}| = \sum_{k=0}^{n} 2^k \binom{n}{n-k} - 1 = 3^n - 1. \tag{5.14}$$

If L is primitive then $\mathcal{F} = \mathcal{C}$ and L has exactly $2(2^n - 1) < 3^n - 1$ corona vectors $(n \geq 2)$.

Taking into account that $|\mathcal{F}|$ is maximal if and only if $\mathcal{F}$ contains a representative of every $L/2L$ coset (except 0), we get

Proposition 27 $|\mathcal{F}|$ *is maximal if and only if* $\mathcal{F} = \mathcal{C}$.

Lattices with maximal $|\mathcal{F}|$ are not necessarily primitive: if $D(o)$ has "few" vertices then some of them will be an intersection of more than n facets (this occurs first when $n = 4$, see example in subsection 6.4.1). However, if $|\mathcal{F}|$ is maximal and the number of facets at each vertex of $D(o)$ is minimal, then L is primitive. More precisely,

Proposition 28 L *is primitive if and only if* $|\mathcal{F}|$ *is maximal and exactly* n *facets of* $D(o)$ *meet at each vertex.*

Proof. Suppose that L is primitive. Then every Delone cell is a simplex. Since at least n facets of $D(o)$ must meet at every vertex, all of the vertices of the Delone cell, except o, correspond to facet vectors, so $|\mathcal{F}|$ is maximal. Conversely, if $|\mathcal{F}|$ is maximal and exactly n facets meet at each vertex, then every corona vector is a facet vector and hence the Delone cells are simplices. $\square$

Denoting the number of k-faces of the Voronoï cell of an n-dimensional primitive lattice by $N_k(n)$, we have:

Proposition 29 *For a primitive lattice, $N_k(n)$ is a multiple of $2(n+1-k)$ for $0 \leq k < n-1$.*

Proof. Proposition 18 shows that this number is a multiple of $n+1-k$. Consider a k-face $\mathbf{f}_k$ and its image $\mathbf{f}'_k = -\mathbf{f}_k$ through the origin. If $\mathbf{f}_k$ and $\mathbf{f}'_k$ belong to the same translation orbit, there would be a translation $-\vec{c}$ carrying $\mathbf{f}_k$ into $\mathbf{f}'_k$. Then $\frac{1}{2}\vec{c} \in \mathbf{f}_k$. So $\vec{c} \in \mathcal{C}$, but $\vec{c} \notin \mathcal{F}$ since $k < n-1$. This is impossible, since $\mathcal{C} = \mathcal{F}$. So $\mathbf{f}_k$ and $\mathbf{f}'_k$ belong to two distinct translation orbits. Thus when a Voronoï cell is primitive, the k-faces belong to an even number of translation orbits, each containing $(n+1-k)$ k-faces of the cell. $\square$

Proposition 30 *In any lattice, the vectors of norm less than $2s$ are facet vectors, where s is the minimal norm of the lattice.*

Proof. Let $N(\vec{v}_1) < 2s$; we will show that there is no $\vec{v}_2$ in the same $L/2L$ coset with norm $N(\vec{v}_2) < 2s$. Assume $N(\vec{v}_2) \leq 2s$ and $\vec{v}_1 - \vec{v}_2 = \vec{y}$, $\vec{y} \in L$. Then

$$N(\vec{v}_1 - \vec{v}_2) = N(2\vec{y}) = 4N(\vec{y}) \geq 4s, \tag{5.15}$$

so

$$4s \leq N(\vec{v}_1 - \vec{v}_2) = N(\vec{v}_1) + N(\vec{v}_2) - 1(\vec{v}_1, \vec{v}_2) \leq 4s - 1(\vec{v}_1.\vec{v}_2). \tag{5.16}$$

Choosing $\vec{v}_2$ so that $(\vec{v}_1, \vec{v}_2) > 0$ - that is replacing $\vec{v}_2$ by $-\vec{v}_2$ if necessary - we have a contradiction. Thus $\vec{v}_1$ is a facet vector. $\square$

Corollary 8 *A vector of norm $2s$ is a corona vector.*

Proof. It follows from the proof of the preceding proposition that no vectors of norm $2s$ can be in the same $L/2L$ coset as a shorter vector. $\square$

Corollary 9 *The vectors of norm $2s$ in the same $L/2L$ coset are pairwise orthogonal.*

Proof. Let $N(\vec{v}_1) = 2s$; we will show that if there is a $\vec{v}_2$ in the same $2L$ coset with norm $N(\vec{v}_2) = 2s$ and $\vec{v}_2 \neq \vec{v}_1$, then $(\vec{v}_1, \vec{v}_2) = 0$. Let $\vec{v}_1 - \vec{v}_2 = 2\vec{y}$, $\vec{y} \in L$. Then again (5.15) and (5.16) takes place. Since if $(\vec{v}_1, \vec{v}_2) \neq 0$ we can replace $\vec{v}_2$ by $-\vec{v}_2$ if necessary and to assure that $(\vec{v}_1, \vec{v}_2) > 0$, we must have $(\vec{v}_1, \vec{v}_2) = 0$. $\square$

The following criterium, due to Venkov, allows us to distinguish the facet vectors among the vectors of norm $2s$.

Proposition 31 *A vector of norm $2s$ is a facet vector if and only if it is not a sum of two orthogonal vectors of $\mathcal{S}$.*

Proof. Let $\vec{s}_1, \vec{s}_2 \in \mathcal{S}$ where $(\vec{s}_1, \vec{s}_2) = 0$. Then the four vectors $\pm(\vec{s}_2 \pm \vec{s}_2)$ all have norm $2s$ and belong to the same $L/2L$ coset, so they cannot be facet vectors. Conversely, if $N(\pm\vec{c}_i) = 2s$ but $\pm\vec{c}_i \notin \mathcal{F}$ then there are vectors $\pm\vec{c}_j$ orthogonal to $\pm\vec{c}_i$ and in the same $2L$ coset. Then the four lattice vectors $\pm\frac{1}{2}(\vec{c}_1 \pm \vec{c}_2)$ are elements of $\mathcal{S}$ and form two orthogonal opposite pairs, and $\vec{c}_i = \frac{1}{2}(\vec{c}_i + \vec{c}_j) + \frac{1}{2}(\vec{c}_i - \vec{c}_j)$. □

The following obvious remark is also very useful:

Proposition 32 *$\mathcal{F}$ generates L.*

Proof. Since the Voronoï tesselation is facet-to-facet, we can pass from any cell, say $D(o)$ to any other, say $D(x)$, by a path that does not intersect the boundary of any face. This path defines a sequence of facet vectors from o to x. □

The set $\mathcal{S}$ of shortest vectors may not generate L, even if it spans the whole E^n. Also note that a generating set need not include a basis. For example, the integers 2 and 3 generate $\mathbb{Z}$ but neither 2 or 3 does. When $n > 9$, there exist lattices in E^n generated by $\mathcal{S}$ which have no basis in that set. The first example, in 11 dimensions, was found by Conway and Sloane [36].

5.5.3 A lattice without a basis of minimal vectors

Conway and Sloane have proved in [36] that the 11-dimensional lattice with Gram matrix

$$\begin{bmatrix}
60 & 5 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\
5 & 60 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\
5 & 5 & 60 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\
5 & 5 & 5 & 60 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\
5 & 5 & 5 & 5 & 60 & 5 & -12 & -12 & -12 & -12 & -7 \\
5 & 5 & 5 & 5 & 5 & 60 & -12 & -12 & -12 & -12 & -7 \\
-12 & -12 & -12 & -12 & -12 & -12 & 60 & -1 & -1 & -1 & -13 \\
-12 & -12 & -12 & -12 & -12 & -12 & -1 & 60 & -1 & -1 & -13 \\
-12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & 60 & -1 & -13 \\
-12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & -1 & 60 & -13 \\
-7 & -7 & -7 & -7 & -7 & -7 & -13 & -13 & -13 & -13 & 96
\end{bmatrix} \tag{5.17}$$

has minimal norm 60, is generated by its 24 minimal vectors, but no set of 11 minimal vectors forms a basis.

We want just to use this example to illustrate relations between facet vectors and shortest vectors of the lattice. We note that for the lattice (5.17) all lattice vectors with norm less than 120 are facet vectors. In particular, the basis in which the Gram matrix is written is formed by facet vectors. Numerical calculations made by Engel (private communication) show that there are 2974 facet vectors. The maximal norm for facet vectors is 168. The minimal norm of lattice vectors which are not facet vectors is 122. There are 20 lattice vectors with norm 122 which are not facet vectors.

We do not touch here the question of existence of a basis of facet vectors conjectured by Voronoï and discussed later on several occasions [66, 52, 53].