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On differences of two squares

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Abstract: The arithmetic function $\rho(n)$ counts the number of ways to write a positive integer n as a difference of two squares. Its average size is described by the Dirichlet summatory function $\sum_{n\le x}\rho(n)$, and in particular by the error term $R(x)$ in the corresponding asymptotics. This article provides a sharp lower bound as well as two mean-square results for $R(x)$, which illustrates the close connection between $\rho(n)$ and the number-of-divisors function $d(n)$.

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1 Introduction

In this note we shall be concerned with the arithmetic function

$$\rho(n) := \#\{(u,v)\in \mathbb{Z}^+\times\mathbb{Z} :\ u^2-v^2=n\ \} \qquad (n\in\mathbb{Z}^+)\,. \tag{1}$$

It is well known that this is closely related to the number-of-divisors function $d(n)$. In fact, it is easy to see (cf. [10]) that

$$\rho(n)=\begin{cases} d(n) & \text{for } n \text{ odd,}\\ d(\frac{n}{4}) & \text{if } 4|n,\\ 0 & \text{else.}\end{cases} \tag{2}$$

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In other words,

$$\rho(n) = d(n) - 2d(\frac{n}{2}) + 2d(\frac{n}{4}), \tag{3}$$

if we define $d(w) = 0$ for $w \notin \mathbb{Z}^+$. The average size of $d(n)$ is usually described by the formula

$$D(x) = \sum_{1 \le n \le x} d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x) \tag{4}$$

where x is large, γ denotes the Euler-Mascheroni constant, and the main term is the sum of the residues of $\zeta^2(s)\frac{x^s}{s}$ at $s = 0$ and $s = 1$. On the error term $\Delta(x)$ a wealth of deep results have been established: See the monographs of A. Ivić [5], E.C. Titchmarsh [19], E. Krätzel [8, 9], and the recent survey article by Ivić, Krätzel and the authors [6]. In view of (3) and (4), it is natural to study the remainder term $R(x)$ defined by

$$\sum_{1 \le n \le x} \rho(n) = \frac{x}{2}(\log x + 2\gamma - 1) + \frac{1}{4} + R(x). \tag{5}$$

Obviously,

$$R(x) = \Delta(x) - 2\Delta(\frac{x}{2}) + 2\Delta(\frac{x}{4}). \tag{6}$$

Evidently, every O-estimate for $\Delta(x)$ trivially implies a corresponding upper bound for $R(x)$. In particular,

$$R(x) = O\left(x^{131/416}(\log x)^{26957/8320}\right) \tag{7}$$

is immediate from M. Huxley's hitherto strongest result on the Dirichlet divisor problem [4]. Because of the alternating sign in (6), it is less straightforward to transfer lower estimates from $\Delta(x)$ to $R(x)$. Nevertheless, M. Kühleitner [10, 11] managed to adapt the methods due to J.L. Hafner [2, 3], resp., K. Corrádi and I. Kátai [1], to derive (for certain constants $C, C' > 0$)

$$R(x) = \Omega_+\left((x \log x)^{1/4}(\log\log x)^{(3+4\log 4)/4} \exp(-C\sqrt{\log\log\log x}\,)\right) \tag{8}$$

and

$$R(x) = \Omega_-\left(x^{1/4} \exp\left(C'(\log\log x)^{1/4}(\log\log\log x)^{-3/4}\right)\right). \tag{9}$$

2 Statement of results

Our first aim will be to show how a recent and deep estimate of K. Soundararajan [18] can be used to deduce a lower bound for $R(x)$ which is slightly sharper, apart from the lack of information concerning the sign.

Theorem 2.1. *For large x, the error term $R(x)$ defined by (5) satisfies*

$$R(x) = \Omega\left(x^{1/4}\,\mathbf{L}^*(x)\right),$$

where

$$\mathbf{L}^*(x) := (\log x)^{1/4} (\log\log x)^{(3/4)(2^{4/3}-1)} (\log\log\log x)^{-5/8} .$$

More generally, for any function $\mathbf{L}(x)$ *of the shape*

$$\mathbf{L}(x) = \prod_{k=1}^{K} (\log_k x)^{\alpha_k} ,$$

where α_k *are arbitrary real constants, and* $\log_k$ *denotes the* k*-fold iterated logarithm, each of the lower estimates*

$$\Delta(x) = \Omega\left(x^\alpha\, \mathbf{L}(x)\right) , \qquad R(x) = \Omega\left(x^\alpha\, \mathbf{L}(x)\right) ,$$

where $\frac{1}{4} \le \alpha < \frac{1}{2}$*, implies the other one.*

It is a common conjecture in problems of this kind that (up to logarithmic factors) $x^{1/4}$ should give just the "true" order of magnitude of the error term involved. In fact, $R(x) \ll x^{1/4}$ in mean-square, as will be immediate from the following much more precise asymptotics.

Theorem 2.2. *For large* X*,*

$$\int_1^X R^2(x)\,\mathrm{d}x = C_\rho\, X^{3/2} + O\left(X \log^4 X\right) ,$$

where the constant C_ρ *is given by*

$$C_\rho = \frac{1}{21\pi^2}\left(15 - 9\sqrt{2}\right) \frac{\zeta^4\left(\frac{3}{2}\right)}{\zeta(3)} = 0,424738\dots .$$

In the classic case of the divisor problem, a mean-square asymptotics for $\Delta(x)$, of the same accuracy, has been established by E. Preissmann [17]. He improved upon a previous result by K.C. Tong [20] who had an error term of $O(X \log^5 X)$.

It is natural to try to determine a localized form of the last theorem, i.e., to ask the following question: How small can an interval $[X-L, X+L]$ be such that

$$\int_{X-L}^{X+L} R^2(x)\,\mathrm{d}x \sim C_\rho\, \left((X+L)^{3/2} - (X-L)^{3/2}\right) \tag{10}$$

remains true? From Theorem 2.2 it is immediate that (10) holds provided that $L = L(X)$ satisfies

$$\lim_{X\to\infty} \frac{X^{1/2}\log^4 X}{L(X)} = 0 .$$

By an argument more specialized for the short interval case, we are able to obtain the following refinement.

Theorem 2.3. *The asymptotics* (10) *holds true for any* $L = L(X) < X - 1$ *which satisfies*

$$\lim_{X\to\infty} \frac{X^{1/2}\,\log^3 X}{L(X)} = 0 . \tag{11}$$

3 Proof of Theorem 2.1

There are various ways to infer this result on the basis of Soundararajan's estimate [18]

$$\Delta(x) = \Omega\left(x^{1/4}\,\mathbf{L}^*(x)\right) . \tag{12}$$

One alternative would be to start from formulae (22), (23) below and to mimic Soundararajan's argument. We prefer an elementary reasoning which will have the advantage to yield the more general statement of Theorem 2.1. This is a variant of what Y.-K. Lau and K.M. Tsang [12] used when considering the mean-square of the Riemann zeta-function along the critical line. It is convenient to consider $R^*(x) := -\frac{1}{2}R(4x)$ instead of $R(x)$. We take for granted that $\Delta(x) = \Omega\left(x^\alpha\,\mathbf{L}(x)\right)$ and assume that, for $\mathbf{L}(x)$ and α as specified in Theorem 2.1, and every $\varepsilon_0 > 0$

$$|R^*(x)| \le \varepsilon_0\, x^\alpha\, \mathbf{L}(x)\,, \tag{13}$$

for all $x \ge x_0(\varepsilon_0)$. We proceed to deduce a contradiction. By (6),

$$R^*(x) = -\Delta(x) + \Delta(2x) - \tfrac{1}{2}\Delta(4x)\,. \tag{14}$$

Now let $a := \frac{1}{2}(1+i)$, $b := \frac{1}{2}(1-i)$, then

$$a+b=1,\ \ ab=\frac{1}{2},\ \ |a| = |b| = 2^{-1/2}\,. \tag{15}$$

We further put

$$W(x) := -\Delta(x) + a\,\Delta(2x)\,, \tag{16}$$

then (14) reads

$$R^*(x) = W(x) - b\,W(2x)\,. \tag{17}$$

With $J := [\frac{\log x}{\log 2}] + 1$ we iterate (16) J times to obtain

$$\Delta(x) = a\,\Delta(2x) - W(x) = \cdots = a^J \Delta(2^J x) - \sum_{j=0}^{J-1} a^j\, W(2^j x)\,. \tag{18}$$

Accordingly, iterating (17) J times (with x replaced by y), we get

$$W(y) = R^*(y) + b\,W(2y) = \cdots = b^J\, W(2^J y) + \sum_{m=0}^{J-1} b^m\, R^*(2^m y)\,. \tag{19}$$

Using (19) in (18), we arrive at

$$\Delta(x) = a^J \Delta(2^J x) - \sum_{j=0}^{J-1} a^j b^J W(2^{j+J} x) - \sum_{j=0}^{J-1}\sum_{m=0}^{J-1} a^j b^m R^*(2^{j+m} x)\,. \tag{20}$$

We use only the crude bound $\Delta(t) \ll t^{1/3}\log t$, which by (16) also implies that $W(t) \ll t^{1/3}\log t$. Recalling (15), we see that

$$a^J \Delta(2^J x) \ll 2^{-J/6} x^{1/3}\log x \ll x^{1/6}\log x$$

and

$$\sum_{j=0}^{J-1} a^j b^J W(2^{j+J}x) \ll \sum_{j=0}^{J-1} 2^{-(j+J)/6} x^{1/3} \log x \ll x^{1/6} \log x \,.$$

To bound the double sum in (20), we use (13), assuming that $x \geq x_0(\varepsilon_0)$. For $j, m < J$, obviously $\mathbf{L}(2^{j+m}\, x) \ll \mathbf{L}(x)$. Hence, recalling (15) again,

$$\sum_{j=0}^{J-1}\sum_{m=0}^{J-1} a^j b^m R^*(2^{j+m}x) \ll \sum_{j=0}^{J-1}\sum_{m=0}^{J-1} 2^{-(j+m)/2}\, \varepsilon_0 \left(2^{j+m}x\right)^{\alpha} \mathbf{L}(x) \ll \varepsilon_0 x^{\alpha}\, \mathbf{L}(x)\,.$$

Using the last three estimates in (20), we obtain that

$$\Delta(x) \ll \varepsilon_0 x^{\alpha}\, \mathbf{L}(x)$$

for x sufficiently large. Since ε_0 can be chosen arbitrarily small, this contradicts the Ω-bound for $\Delta(x)$ assumed in Theorem 2.1. The other direction of the conclusion is immediate by (6) and hence the result follows. □

4 Proof of Theorem 2.2

When considering the mean-square of $R(x)$, we will need a sharp Voronoï type approximation. For $\Delta(x)$, this is provided by the well-known expression

$$S_\Delta(x, M) := \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{1\leq n\leq M} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right)\,, \tag{21}$$

where x and M are large real parameters. The following precise estimate has been established by T. Meurman [14].

Lemma 4.1 (T. Meurman [14, Lemma 3])**.** *For $x \geq 1$ and $x \ll M \ll x^A$, where $A \geq 1$ is an arbitrary constant,*

$$\Delta(x) - S_\Delta(x, M) \ll \begin{cases} x^{-1/4} & \textit{if } \ \|x\| \geq x^{5/2}M^{-1/2}\,, \\ x^{\varepsilon} & \textit{always}, \end{cases}$$

where $\|x\|$ denotes the distance from the nearest integer, and $\varepsilon > 0$ is arbitrary.

Our strategy will be to approximate $R(x)$ by an analogous sum

$$S_\rho(x, M) := \frac{x^{1/4}}{\pi} \sum_{1\leq n\leq M} \frac{\rho(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)\,. \tag{22}$$

This is connected with the divisor problem by the following identity.

Lemma 4.2. *For arbitrary $x, M \ge 1$,*

$$S_\rho(x,4M) = S_\Delta(x,M) - 2S_\Delta\left(\frac{x}{2},2M\right) + 2S_\Delta\left(\frac{x}{4},4M\right).$$

Proof (of Lemma 4.2)**.** Using (2), we easily conclude that

$$S_\rho(x,4M) = \frac{x^{1/4}}{\pi}\sum_{\substack{1\le n\le 4M\\ n\equiv 1 \mod 2}}\frac{d(n)}{n^{3/4}}\cos\left(2\pi\sqrt{nx}-\frac{\pi}{4}\right) + \frac{x^{1/4}}{\pi}\sum_{1\le m\le M}\frac{d(m)}{(4m)^{3/4}}\cos\left(4\pi\sqrt{mx}-\frac{\pi}{4}\right).$$

Since the last term clearly equals $\frac{1}{2}S_\Delta(x,M)$, we obtain

$$S_\rho(x,4M) - \tfrac{1}{2}S_\Delta(x,M) = \frac{\sqrt{2}}{\pi}\left(\frac{x}{4}\right)^{1/4}\sum_{1\le n\le 4M}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{n\frac{x}{4}}-\frac{\pi}{4}\right) - \frac{x^{1/4}}{\pi}\sum_{1\le m\le 2M}\frac{d(2m)}{(2m)^{3/4}}\cos\left(4\pi\sqrt{m\frac{x}{2}}-\frac{\pi}{4}\right).$$

Here the first term on the right hand side is equal to $2S_\Delta(\frac{x}{4},4M)$. In the last sum we use the identity $d(2m) = 2d(m) - d(\frac{m}{2})$, to infer that

$$\begin{aligned} S_\rho(x,4M) - \tfrac{1}{2}S_\Delta(x,M) - 2S_\Delta\left(\frac{x}{4},4M\right) &= \\ &= -\frac{\sqrt{2}}{\pi}\left(\frac{x}{2}\right)^{1/4}\sum_{1\le m\le 2M}\frac{d(m)}{m^{3/4}}\cos\left(4\pi\sqrt{m\frac{x}{2}}-\frac{\pi}{4}\right) \\ &\quad + \frac{x^{1/4}}{\pi}\sum_{1\le k\le M}\frac{d(k)}{(4k)^{3/4}}\cos\left(4\pi\sqrt{kx}-\frac{\pi}{4}\right) \\ &= -2S_\Delta\left(\frac{x}{2},2M\right) + \tfrac{1}{2}S_\Delta(x,M). \end{aligned}$$

This is just the assertion of Lemma 4.2. □

Combining (6) with Lemmas 4.1 and 4.2 yields the following approximation.

Lemma 4.3. *For $x \ge 1$ and $x \ll M \ll x^A$, with $A \ge 1$ an arbitrary constant, the expression $S_\rho(x,4M)$ defined by (22) satisfies*

$$R(x) = S_\rho(x,4M) + \mathcal{E}(x,M), \tag{23}$$

where

$$\mathcal{E}(x,M) \ll \begin{cases} x^{-1/4} & if \min\left(\|x\|, \left\|\frac{x}{2}\right\|, \left\|\frac{x}{4}\right\|\right) \ge x^{5/2}(4M)^{-1/2}, \\ x^{\varepsilon} & always. \end{cases}$$

We shall make reference to this Lemma in the concluding remarks at the end of the paper.

For the present purpose of proving Theorem 2.2, we proceed to integrate $R^2(x)$ over an interval $[Y, 2Y]$, Y sufficiently large, using (23) with $M = Y^7$. We first claim that

$$\int_Y^{2Y} \mathcal{E}^2(x, Y^7)\,\mathrm{d}x \ll Y^{1/2}\,. \tag{24}$$

In fact, the set

$$\mathcal{M}(Y) = \{x \in [Y, 2Y] : \ \min\left(\|x\|\,, \left\|\frac{x}{2}\right\|, \left\|\frac{x}{4}\right\|\right) < x^{5/2}(4Y^7)^{-1/2}\ \}$$

has a measure of $O(1)$, hence

$$\int_{\mathcal{M}(Y)} \mathcal{E}^2(x, Y^7)\,\mathrm{d}x \ll Y^{2\varepsilon}\,.$$

On the other hand,

$$\int_{[Y,2Y]\setminus\mathcal{M}(Y)} \mathcal{E}^2(x, Y^7)\,\mathrm{d}x \ll \int_Y^{2Y} x^{-1/2}\,\mathrm{d}x \ll Y^{1/2}\,,$$

thus (24) is true. To evaluate the corresponding integral over $S_\rho^2(x, 4Y^7)$, we first set, for $u \geq \sqrt{Y}$,

$$G(Y, u) := \int_{\sqrt{Y}}^{u} \left(\sum_{1\leq n\leq 4Y^7} \frac{\rho(n)}{n^{3/4}} \cos\left(2\pi\sqrt{n}\,t - \frac{\pi}{4}\right)\right)^2 \mathrm{d}t\,. \tag{25}$$

Using the definition (22), we get

$$\begin{aligned}\int_Y^{2Y} S_\rho^2(x, 4Y^7)\,\mathrm{d}x &= \frac{2}{\pi^2}\int_{\sqrt{Y}}^{\sqrt{2Y}} u^2\frac{\partial G}{\partial u}(Y, u)\,\mathrm{d}u \\ &= \frac{4}{\pi^2} Y G(Y, \sqrt{2Y}) - \frac{4}{\pi^2}\int_{\sqrt{Y}}^{\sqrt{2Y}} u\,G(Y, u)\,\mathrm{d}u \end{aligned} \tag{26}$$

by the change of variable $x = u^2$ and an integration by parts. To evaluate $G(Y, u)$, we employ the well-known result due to H.L. Montgomery and R.C. Vaughan [15].

Lemma 4.4 (Montgomery and Vaughan [15, Corollary 2])**.** *For an arbitrary finite index set $\mathcal{J}$, let $(a_j)_{j\in\mathcal{J}}$ be a complex sequence and let $(\lambda_j)_{j\in\mathcal{J}}$ be a sequence of pairwise distinct reals. Write*

$$\delta_j := \min_{k\in\mathcal{J}, k\neq j} |\lambda_k - \lambda_j|\ .$$

Then, for arbitrary real T_0 and $T > 0$,

$$\int_{T_0}^{T_0+T} \left|\sum_{j\in\mathcal{J}} a_j \exp(i\lambda_j t)\right|^2 \mathrm{d}t = T\sum_{j\in\mathcal{J}} |a_j|^2 + O\left(\sum_{j\in\mathcal{J}} \frac{|a_j|^2}{\delta_j}\right)\,,$$

where the O-constant is absolute.

To apply this to (25), we use the identity $\cos(\alpha) = \frac{1}{2}(\exp(i\alpha) + \exp(-i\alpha))$. Further, we choose $\mathcal{J}$ as the set of all nonzero integers of modulus $\leq 4Y^7$, and, for all $j \in \mathcal{J}$,

$$a_j = \frac{1}{2}\frac{\rho(|j|)}{|j|^{3/4}} \exp\left(-\operatorname{sgn}(j)\frac{\pi}{4}i\right) , \quad \lambda_j = 2\pi \operatorname{sgn}(j)\sqrt{|j|} .$$

It is clear that $\delta_j \asymp |j|^{-1/2}$, thus for the error term it follows that

$$\sum_{j\in\mathcal{J}} \frac{|a_j|^2}{\delta_j} \ll \sum_{0<|j|\leq 4Y^7} \frac{d^2(|j|)}{|j|} \ll (\log Y)^4$$

(cf. (2) and A. Ivić [5, p. 137, f. (5.25)]). Hence the application of Lemma 4.4 to (25) overall yields

$$\begin{aligned} G(Y,u) &= \tfrac{1}{2}\left(u - \sqrt{Y}\right) \sum_{1\leq n\leq 4Y^7} \frac{\rho^2(n)}{n^{3/2}} + O\left(\log^4 Y\right) \\ &= \frac{3\pi^2}{2} C_\rho \left(u - \sqrt{Y}\right) + O\left(\log^4 Y\right) \end{aligned} \tag{27}$$

for $u \leq \sqrt{2Y}$, because

$$\sum_{n>4Y^7} \frac{\rho^2(n)}{n^{3/2}} \ll Y^{-7/2+\varepsilon}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\rho^2(n)}{n^{3/2}} &= \sum_{\substack{n=1 \\ n\equiv 1 \mod 2}}^{\infty} \frac{d^2(n)}{n^{3/2}} + \sum_{m=1}^{\infty} \frac{d^2(m)}{(4m)^{3/2}} \\ &= \left(\frac{(1-2^{-3/2})^3}{1+2^{-3/2}} + \frac{1}{8}\right) \frac{\zeta^4\left(\frac{3}{2}\right)}{\zeta(3)} \\ &= \frac{1}{7}\left(15 - 9\sqrt{2}\right) \frac{\zeta^4\left(\frac{3}{2}\right)}{\zeta(3)} = 3\pi^2 C_\rho , \end{aligned} \tag{28}$$

where C_ρ is the constant defined in Theorem 2.2. Using (27) in (26), we immediately get

$$\int_Y^{2Y} S_\rho^2(x, 4Y^7)\,\mathrm{d}x = C_\rho\left((2Y)^{3/2} - Y^{3/2}\right) + O\left(Y \log^4 Y\right) .$$

Applying this formula with $Y = \frac{X}{2}, \frac{X}{4}, \frac{X}{8}, \ldots$ and summing up, we arrive at

$$\int_1^X S_\rho^2(x, 4Y^7)\,\mathrm{d}x = C_\rho X^{3/2} + O\left(X \log^4 X\right) .$$

Finally, appealing to (23), (24), and Cauchy's inequality, we complete the proof of Theorem 2.2. □

5 Proof of Theorem 2.3

We may suppose that

$$X^{1/2}\log^3 X < L = L(X) \le X^{1/2}\log^5 X\,, \tag{29}$$

else the result follows by Theorem 2.2. In a way, our argument will be simpler than the proof of Theorem 2.2, because it will avoid the Hilbert type inequality of Montgomery and Vaughan (Lemma 4.4), and in fact be similar to the method used in Nowak [16]. For $x \in [X - L(X), X + L(X)]$, we infer from formulas (22), (23) that

$$R(x) = \frac{x^{1/4}}{\pi} \sum_{1\le n\le X} \frac{\rho(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(X^{\varepsilon}\right) \qquad (\varepsilon > 0)\,. \tag{30}$$

From this it is immediate that

$$R(x) = \frac{x^{1/4}}{\pi} S_1(x) + O\left(X^{1/4}\,|S_2(x)|\right) + O\left(X^{\varepsilon}\right)\,, \tag{31}$$

with

$$\begin{aligned} S_1(x) &:= \sum_{1\le n\le M} \frac{\rho(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right) \\ &= \frac{1}{2} \sum_{0<|m|\le M} \frac{\rho(|m|)}{|m|^{3/4}}\, e\left(\operatorname{sgn}(m)\left(\sqrt{|m|x} - \frac{1}{8}\right)\right), \\ S_2(x) &:= \sum_{M<n\le X} \frac{\rho(n)}{n^{3/4}} e(\sqrt{nx})\,, \end{aligned}$$

where $e(w) = e^{2\pi i w}$ as usual, and $M < X$ is another large real parameter, independent of X. We shall use the simple fact that, for real functions F, G defined on an interval I (with an obvious brief notation),

$$\int_I (F+G)^2 = \int_I F^2 + O\left(\left(\int_I F^2\right)^{1/2} \left(\int_I G^2\right)^{1/2}\right) + O\left(\int_I G^2\right)\,. \tag{32}$$

We first estimate the mean-square of $X^{1/4}\,|S_2(x)|$, claiming that

$$\int_{X-L}^{X+L} \left|X^{1/4}\, S_2(x)\right|^2 \mathrm{d}x \ll X\log^3 X + LX^{1/2}M^{-1/6+\varepsilon} \qquad (\varepsilon > 0)\,. \tag{33}$$

To prove this, we employ a device based on the Fejér kernel

$$\varphi(w) := \left(\frac{\sin(\pi w)}{\pi w}\right)^2 .$$

By Jordan's inequality, $\varphi(w) \ge 4/\pi^2$ for $|w| \le \frac{1}{2}$, and the Fourier transform has the particularly simple shape

$$\widehat{\varphi}(y) = \int_{\mathbb{R}} \varphi(w) e(wy)\,\mathrm{d}w = \max(0, 1 - |y|)\,.$$

By the change of variable $x = (\sqrt{X}+u)^2$ (which implies that $|u| \le L/\sqrt{X}$), we thus conclude that

$$\int_{X-L}^{X+L} \left|X^{1/4}\, S_2(x)\right|^2 \mathrm{d}x \ll X \int_{-L/\sqrt{X}}^{L/\sqrt{X}} \left|S_2\left((\sqrt{X}+u)^2\right)\right|^2 \mathrm{d}u$$

$$= 2LX^{1/2} \int_{-\frac12}^{\frac12} \left|S_2\left((\sqrt{X}+2(L/\sqrt{X})w)^2\right)\right|^2 \mathrm{d}w \le$$

$$\le \tfrac12 \pi^2 LX^{1/2} \int_{\mathbb{R}} \left|S_2\left(\left(\sqrt{X}+2(L/\sqrt{X})w\right)^2\right)\right|^2 \varphi(w)\,\mathrm{d}w =$$

$$= \tfrac12 \pi^2 LX^{1/2} \sum_{M<m,n\le X} \frac{\rho(m)\rho(n)}{(mn)^{3/4}}\, e(\sqrt{X}(\sqrt{m}-\sqrt{n}))\, \widehat{\varphi}\left(2(L/\sqrt{X})(\sqrt{m}-\sqrt{n})\right) \ll$$

$$\ll LX^{1/2} \sum_{\substack{M<m,n\le X\\ |\sqrt{m}-\sqrt{n}|<\sqrt{X}/(2L)}} \frac{d(m)d(n)}{(mn)^{3/4}} \ll LX^{1/2} \sum_{M<m\le X} \frac{d(m)}{m^{3/2}} \sum_{n\in\mathbb{N}_*:\ \left|\sqrt{n}-\sqrt{m}\right|<\sqrt{X}/(2L)} d(n)$$

$$\le LX^{1/2} \sum_{M<m\le X} \frac{d(m)}{m^{3/2}} \left(D\left(\left(\sqrt{m}+\frac{\sqrt{X}}{2L}\right)^2\right) - D\left(\left(\sqrt{m}-\frac{\sqrt{X}}{2L}\right)^2\right)\right). \qquad (34)$$

By (4), with the simple bound $\Delta(x) \ll x^{1/3}\log x$, it follows that, for arbitrary large reals $Y_1 \asymp Y \asymp Y_2$,

$$D(Y_1) - D(Y_2) \ll |Y_1 - Y_2| \log Y + Y^{1/3} \log Y .$$

Therefore, the last line in (34) is

$$\ll LX^{1/2} \sum_{M<m\le X} \frac{d(m)}{m^{3/2}} \left(\frac{\sqrt{m}\sqrt{X}}{L}\log m + m^{1/3}\log m\right)$$

$$\ll X \sum_{1\le m\le X} \frac{d(m)}{m} \log m + LX^{1/2} \sum_{m>M} m^{-7/6+\varepsilon} .$$

But this is just $\ll$ the right hand side of (33), as asserted. Looking back at (31), we proceed to evaluate the mean-square of $\dfrac{x^{1/4}}{\pi} S_1(x)$. Let $S_1(x)^2 = S_3 + S_4(x)$ with

$$S_3 := \frac12 \sum_{1\le n\le M} \frac{\rho^2(n)}{n^{3/2}},$$

$$S_4(x) := \frac14 \sum_{\substack{1\le |m|,|n|\le M\\ m+n\ne 0}} \frac{\rho(|m|)\rho(|n|)}{|mn|^{3/4}} \times$$

$$\times e\left(\left(\operatorname{sgn}(m)\sqrt{|m|} + \operatorname{sgn}(n)\sqrt{|n|}\right)\sqrt{x} - \frac18(\operatorname{sgn}(m) + \operatorname{sgn}(n))\right).$$

Using the elementary fact that, for an arbitrary nonzero real number A,

$$\int_{X-L}^{X+L} x^{1/2} e\left(A\sqrt{x}\right)\,\mathrm{d}x = 2\int_{\sqrt{X-L}}^{\sqrt{X+L}} t^2\, e(At)\,\mathrm{d}t \ll \frac{X}{|A|},$$

we obtain

$$\int_{X-L}^{X+L} x^{1/2} S_4(x)\,\mathrm{d}x \ll X \sum_{\substack{1\le|m|,|n|\le M\\ m+n\ne 0}} \frac{\rho(|m|)\rho(|n|)\,|mn|^{-3/4}}{|\operatorname{sgn}(m)\sqrt{|m|}+\operatorname{sgn}(n)\sqrt{|n|}|} \ll B(M)\,X\,. \tag{35}$$

Here $B(M)$ denotes a positive bound depending only on M. On the other hand,

$$\begin{aligned}\int_{X-L}^{X+L}\left(\frac{x^{1/4}}{\pi}\right)^2 S_3\,\mathrm{d}x &= \frac{1}{3\pi^2}\left((X+L)^{3/2}-(X-L)^{3/2}\right)\sum_{1\le n\le M}\frac{\rho^2(n)}{n^{3/2}}\\ &= \frac{1}{3\pi^2}\left((X+L)^{3/2}-(X-L)^{3/2}\right)\sum_{n=1}^{\infty}\frac{\rho^2(n)}{n^{3/2}} + O\left(LX^{1/2}M^{-1/2+\varepsilon}\right)\\ &= C_\rho\left((X+L)^{3/2}-(X-L)^{3/2}\right)+O\left(LX^{1/2}M^{-1/2+\varepsilon}\right)\end{aligned} \tag{36}$$

appealing again to (28). Combining (35) and (36), we arrive at

$$\begin{aligned}\int_{X-L}^{X+L}\left(\frac{x^{1/4}}{\pi}S_1(x)\right)^2\mathrm{d}x &= C_\rho\left((X+L)^{3/2}-(X-L)^{3/2}\right)\\ &\quad +O\left(LX^{1/2}M^{-1/2+\varepsilon}\right)+O\left(B(M)X\right).\end{aligned}$$

Using this along with (33) in (31), and applying (32), we conclude that

$$\begin{gathered}\int_{X-L}^{X+L} R^2(x)\,\mathrm{d}x = C_\rho\left((X+L)^{3/2}-(X-L)^{3/2}\right)\\ +O\left(LX^{1/2}M^{-1/2+\varepsilon}\right)+O\left(B(M)X\right)\\ +O\left(\left(L^{1/2}X^{1/4}+B^{1/2}(M)X^{1/2}\right)\left(X^{1/2}\log^{3/2}X+L^{1/2}X^{1/4}M^{-1/12+\varepsilon}+L^{1/2}X^{\varepsilon}\right)\right)\\ +O\left(X\log^3 X+LX^{1/2}M^{-1/6+\varepsilon}+LX^{2\varepsilon}\right).\end{gathered}$$

Hence, recalling the condition (11) and the fact that $(X+L)^{3/2}-(X-L)^{3/2}\asymp LX^{1/2}$,

$$\limsup_{X\to\infty}\left|\frac{\int_{X-L}^{X+L} R^2(x)\,\mathrm{d}x}{(X+L)^{3/2}-(X-L)^{3/2}}-C_\rho\right| \ll M^{-1/12+\varepsilon}\,.$$

Since M can be chosen arbitrarily large, this completes the proof of Theorem 2.3.

6 Concluding remarks

The results established show a very close analogy between the error terms $R(x)$ and $\Delta(x)$. This fact is explained by the great similarity of the Voronoï type approximations provided by Lemmas 4.1 and 4.3, respectively. On the basis of Lemma 4.3, one could deduce more results on $R(x)$ which have direct counterparts in the theory of $\Delta(x)$. For instance, there is a series of papers concerned with asymptotics for higher power moments of $\Delta(x)$, including also the discussion of the short interval case. See D.R. Heath-Brown

[7], K.-M. Tsang [21], Y.-K. Lau and K.-M. Tsang [13], and W. Zhai [22–24]. Starting from Lemma 4.3 and carrying over the techniques employed in the papers cited, it is straightforward to deduce asymptotics

$$\int_1^X R^k(x)\,\mathrm{d}x \sim C_\rho^{(k)}\, X^{1+k/4}\,,$$
$$\int_{X-L}^{X+L} R^k(x)\,\mathrm{d}x \sim C_\rho^{(k)}\,\left((X+L)^{1+k/4} - (X-L)^{1+k/4}\right)\,,$$

for a certain range of integers $k > 2$, even with fairly precise error terms.

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